# Achieving Seventh-Order Amplitude Accuracy in Leapfrog Integrations

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# Sources of uncertainty in simulations of weather and climate

- initial conditions
- boundary conditions
- internal chaotic variability
- external forcing
- model error
  - dynamical assumptions
  - parameterisations of sub-gridscale processes
  - numerical approximations
    - discrete spatial grid or truncated spectral expansion
    - discrete time stepping

# Time-stepping methods

- Weather and climate models are essentially solving the ODE $dx/dt = f(x)$, where $x$ is a large state vector containing the values of all the variables at all the grid-points and f is a given nonlinear function
- Many current models use the "leapfrog" second-order centred discretisation in time, $x_{n+1} = x_{n-1} + 2\,\Delta t\, f\,(x_n)$, together with a stabilising filter that reduces the accuracy to first-order
- The large $O(\Delta t)$ numerical errors in this scheme reduce the accuracy of weather forecasts and climate predictions
- There is a need to devise better schemes, analyse their theoretical properties, implement them in a hierarchy of models, and test their performance and ability to reduce errors

# Time-stepping methods

weather and climate models

computational fluid dynamics

increasing accuracy

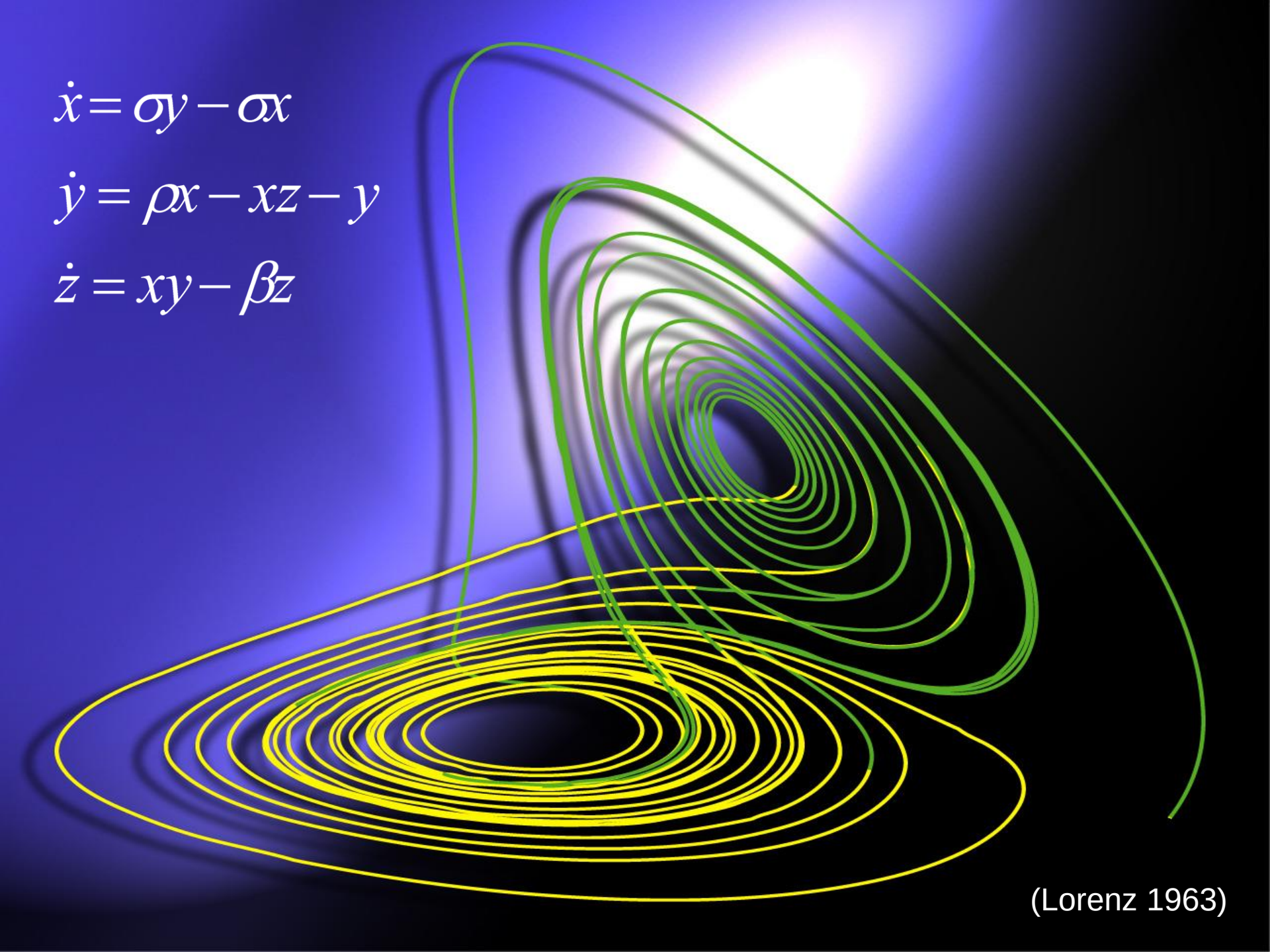
| Method | Order | Formula | Storage factor | Efficiency factor | Amplitude error | Phase error | Maximum $\omega\Delta t$ |
|---|---|---|---|---|---|---|---|
| Forward (Adams–Bashforth) | 1 | $\phi^{n+1} = \phi^n + hF(\phi^n)$ | 2 | 0 | $1 + \frac{p^2}{2}$ | $1 - \frac{p^2}{3}$ | 0 |
| Backward (Adams–Moulton) | 1 | $\phi^{n+1} = \phi^n + hF(\phi^{n+1})$ | Implicit | $\infty$ | $1 - \frac{p^2}{2}$ | $1 - \frac{p^2}{3}$ | $\infty$ |
| Matsuno | 1 | $\phi^* = \phi^n + hF(\phi^n)$<br>$\phi^{n+1} = \phi^n + hF(\phi^*)$ | 2 | .5 | $1 - \frac{p^2}{2}$ | $1 + \frac{2}{3}p^2$ | 1 |
| Asselin-filtered leapfrog | 1 | $\phi^{n+1} = \overline{\phi^{n-1}} + 2hF(\phi^n)$<br>$\overline{\phi^n} = \phi^n + \gamma(\overline{\phi^{n-1}} - 2\phi^n + \phi^{n+1})$ | 3 | <1 | $1 - \frac{\gamma}{2(1-\gamma)}p^2$ | $1 + \frac{1+2\gamma}{6(1-\gamma)}p^2$ | <1 |
| Leapfrog | 2 | $\phi^{n+1} = \phi^{n-1} + 2hF(\phi^n)$ | 2 | 1 | 1 | $1 + \frac{p^2}{6}$ | 1 |
| Runge–Kutta (Williamson/Huen) | 2 | $q_1 = hF(\phi^n), \quad \phi_1 = \phi^n + q_1$<br>$q_2 = hF(\phi_1) - q_1, \quad \phi^{n+1} = \phi_1 + q_2/2$ | 2 | 0 | $1 + \frac{p^4}{8}$ | $1 + \frac{p^2}{6}$ | 0 |
| Adams–Bashforth | 2 | $\phi^{n+1} = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$ | 3 | 0 | $1 + \frac{p^4}{4}$ | $1 + \frac{5}{12}p^2$ | 0 |
| Adams–Moulton (Trapizoidal) | 2 | $\phi^{n+1} = \phi^n + \frac{h}{2}[F(\phi^{n+1}) + F(\phi^n)]$ | Implicit | $\infty$ | 1 | $1 - \frac{p^2}{12}$ | $\infty$ |
| Leapfrog, then Adams–Bashforth (Magazenkov) | 2 | $\phi^n = \phi^{n-2} + 2hF(\phi^{n-1})$<br>$\phi^{n+1} = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$ | 3 | .67 | $1 - \frac{p^4}{4}$ | $1 + \frac{p^2}{6}$ | .67 |
| Leapfrog-Trapizoidal (Kurihara) | 2 | $\phi^* = \phi^{n-1} + 2hF(\phi^n)$<br>$\phi^{n+1} = \phi^n + \frac{h}{2}[F(\phi^n) + F(\phi^*)]$ | 3 | .71 | $1 - \frac{p^4}{4}$ | $1 - \frac{p^2}{12}$ | 1.41 |
| Young's method A | 2 | $\phi_1 = \phi^n + hF(\phi^n)/2$<br>$\phi_2 = \phi_1 + hF(\phi_1)/2$<br>$\phi^{n+1} = \phi^n + hF(\phi_2)$ | 3 | 0 | $1 + \frac{p^6}{128}$ | $1 + \frac{p^2}{24}$ | 0 |
| Runge–Kutta (Williamson) | 3 | $q_1 = hF(\phi^n), \quad \phi_1 = \phi^n + q_1/3,$<br>$q_2 = hF(\phi_1) - 5q_1/9, \quad \phi_2 = \phi^1 + 15q_2/16$<br>$q_3 = hF(\phi_2) - 153q_2/128, \quad \phi^{n+1} = \phi_2 + 8q_3/15$ | 2 | .58 | $1 - \frac{p^4}{24}$ | $1 + \frac{p^4}{30}$ | 1.73 |
| ABM predictor-corrector | 3 | $\phi^* = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$<br>$\phi^{n+1} = \phi^* + \frac{5h}{12}[F(\phi^*) - 2F(\phi^n) + F(\phi^{n-1})]$ | 4 | .60 | $1 - \frac{19}{144}p^4$ | $1 + \frac{1243}{8640}p^4$ | 1.20 |
| Adams–Moulton | 3 | $\phi^{n+1} = \phi^n + \frac{h}{12}[5F(\phi^{n+1}) + 8F(\phi^n) - F(\phi^{n-1})]$ | Implicit | 0 | $1 + \frac{p^4}{24}$ | $1 - \frac{11}{720}p^4$ | 0 |
| Adams–Bashforth | 3 | $\phi^{n+1} = \phi^n + \frac{h}{12}[23F(\phi^n) - 16F(\phi^{n+1}) + 5F(\phi^{n-2})]$ | 4 | .72 | $1 - \frac{3}{8}p^4$ | $1 + \frac{289}{720}p^4$ | 0.72 |
| Runge–Kutta (Classical) | 4 | $k_0 = hF(\phi^n)$<br>$k_1 = hF(\phi^n + k_0/2)$<br>$k_2 = hF(\phi^n + k_1/2)$<br>$k_3 = hF(\phi^n + k_2)$<br>$\phi^{n+1} = \phi^n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$ | 3* | .70 | $1 - \frac{p^6}{144}$ | $1 - \frac{p^4}{120}$ | 2.82 |
| ABM predictor-corrector | 4 | $\phi^* = \phi^n + \frac{h}{12}[23F(\phi^n) - 16F(\phi^{n-1}) + 5F(\phi^{n-2})]$<br>$\phi^{n+1} = \phi^* + \frac{3h}{8}[F(\phi^*) - 3F(\phi^n) + 3F(\phi^{n-1}) - F(\phi^{n-2})]$ | 5 | .59 | $1 - \frac{265}{1536}p^6$ | $1 - \frac{329}{2880}p^4$ | 1.18 |
| Adams–Moulton | 4 | $\phi^{n+1} = \phi^n + \frac{h}{24}[9F(\phi^{n+1}) + 19F(\phi^n) - 5F(\phi^{n-1}) + F(\phi^{n-2})]$ | Implicit | 0 | $1 + \frac{p^6}{48}$ | $1 + \frac{19}{720}p^4$ | 0 |
| Adams–Bashforth | 4 | $\phi^{n+1} = \phi^n + \frac{h}{24}[55F(\phi^n) - 59F(\phi^{n-1}) + 37F(\phi^{n-2}) - 9F(\phi^{n-3})]$ | 5 | .43 | $1 - \frac{13}{24}p^6$ | $1 - \frac{251}{720}p^4$ | .43 |

(Durran 1991)

$$\dot{x} = \sigma y - \sigma x$$

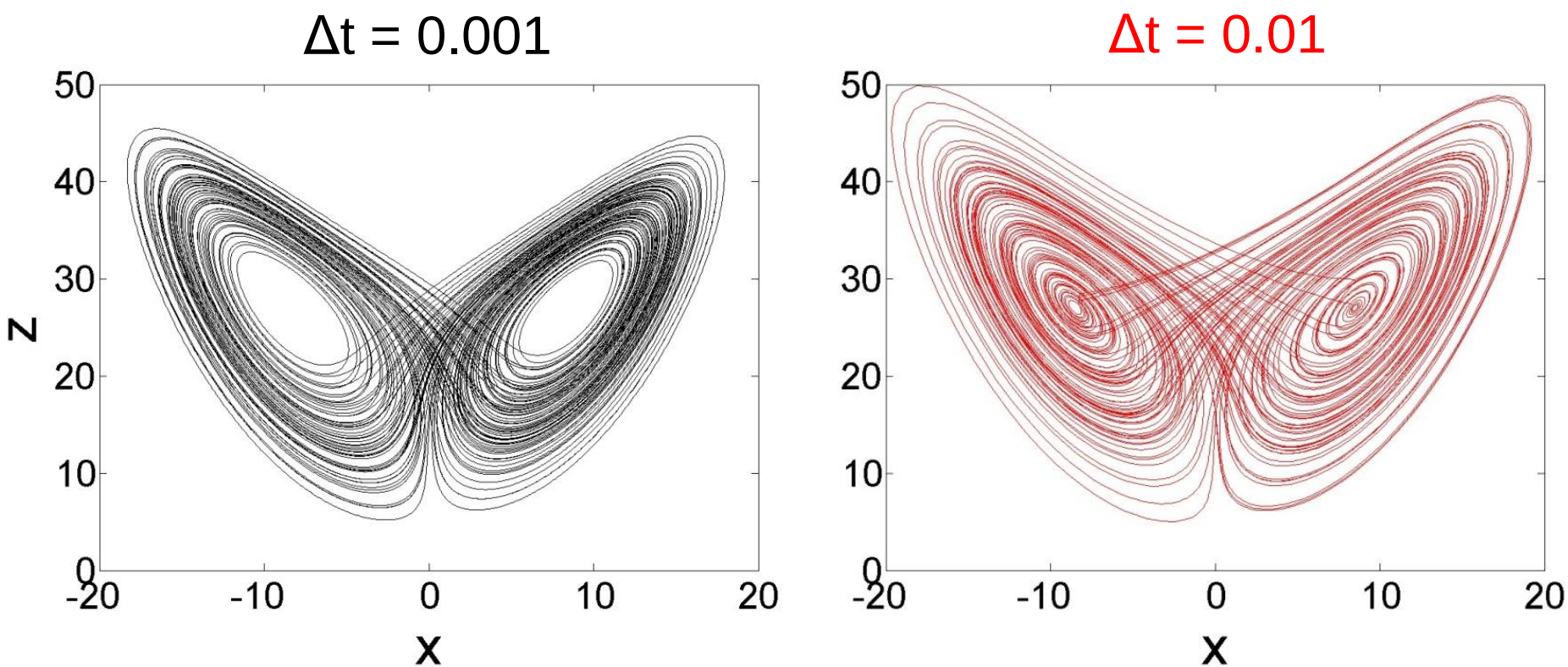
$$\dot{y} = \rho x - xz - y$$

$$\dot{z} = xy - \beta z$$



(Lorenz 1963)

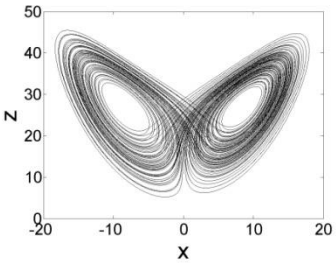
# Impact of different time steps on the ‘climate’ of the Lorenz attractor

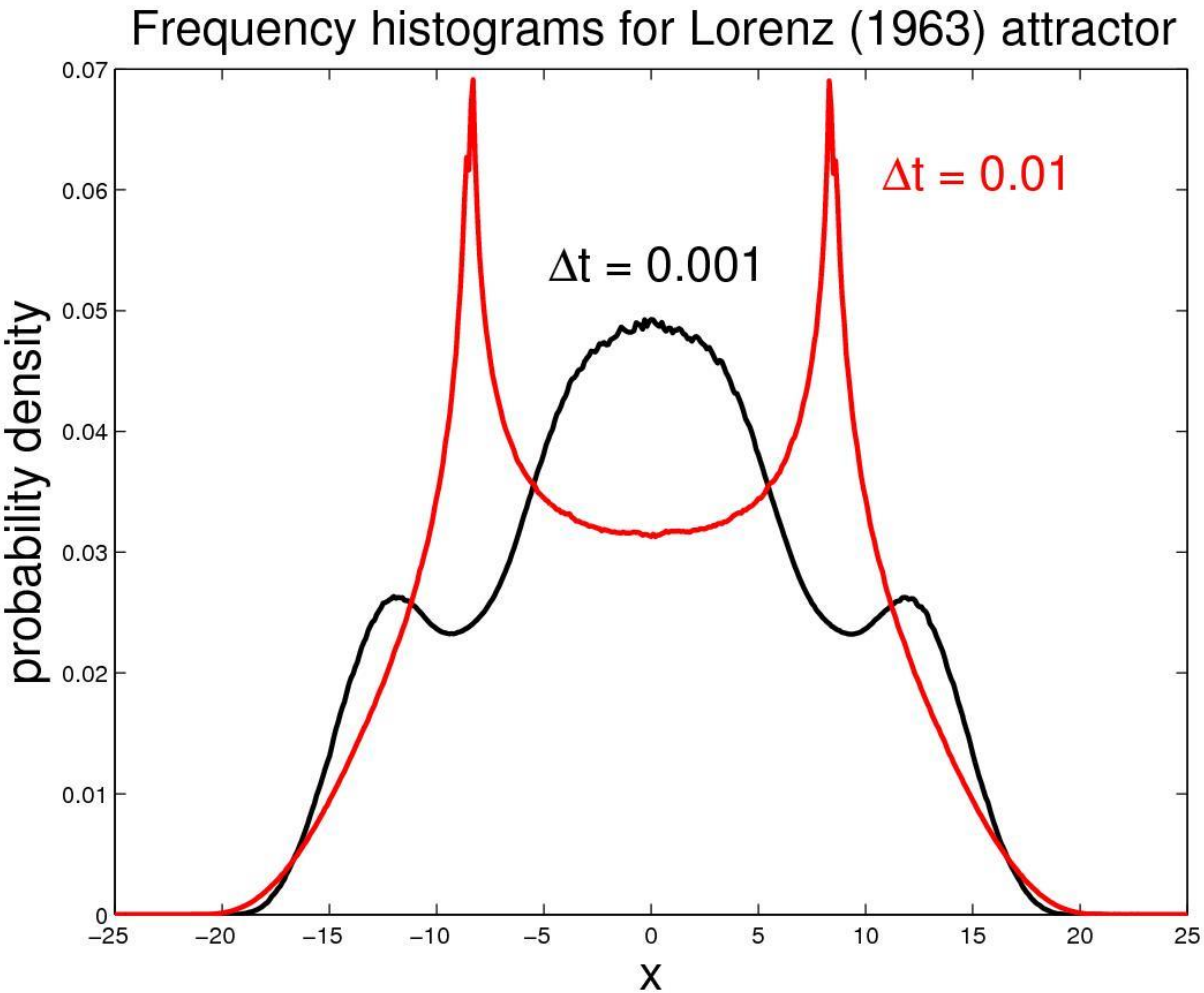


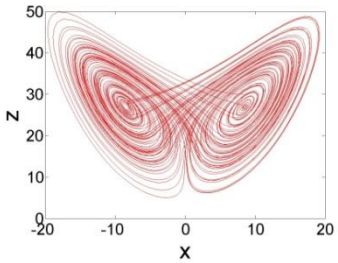

Using the explicit Euler forward scheme

(Williams 2016)

# Impact of different time steps on the ‘climate’ of the Lorenz attractor

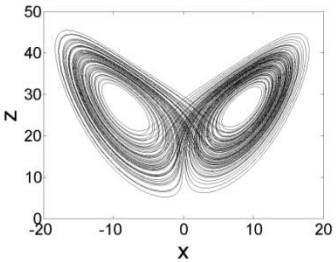


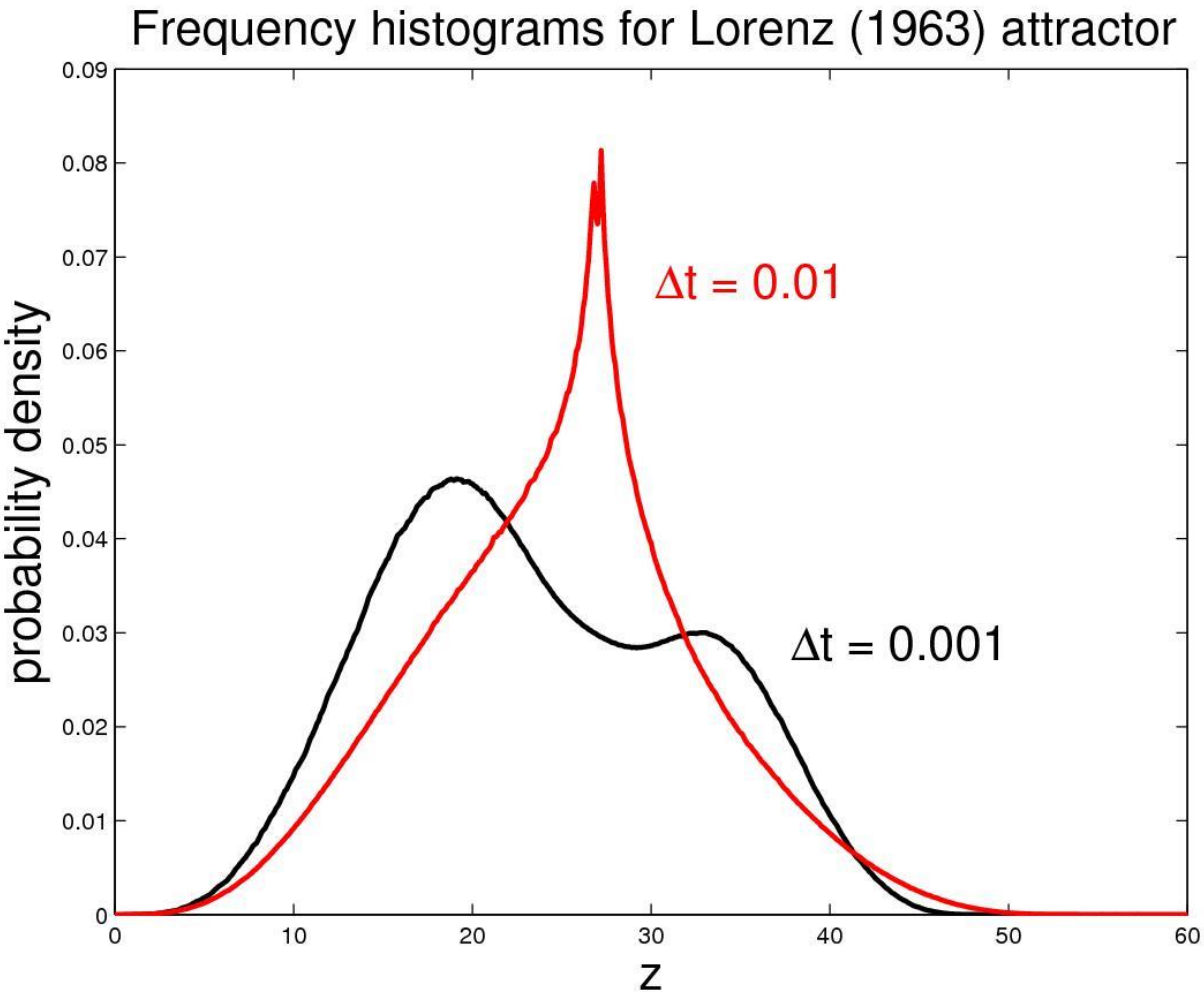



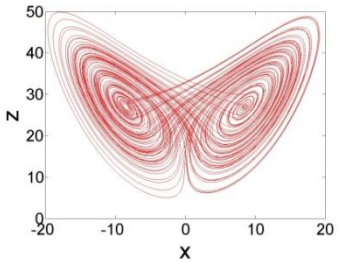




(Williams 2016)

# Impact of different time steps on the ‘climate’ of the Lorenz attractor



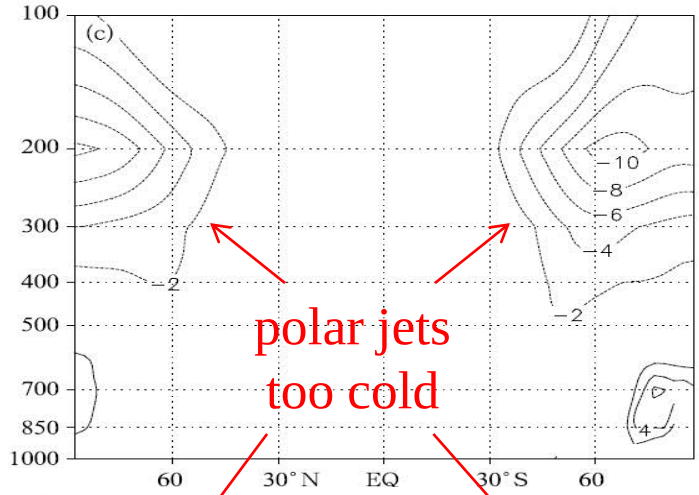




(Williams 2016)

# Impact of time stepping in weather and climate prediction

annual-mean zonal-mean temperature error (°C) in CAM relative to ERA40

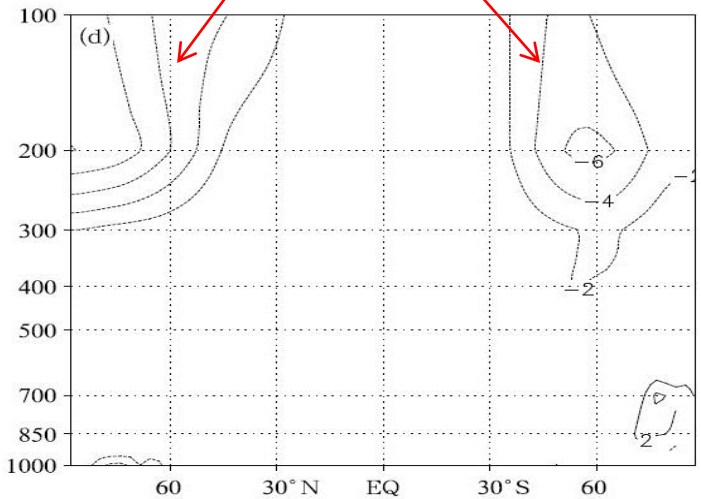
pressure (hPa)




polar jets too cold

First-order time-stepping scheme

pressure (hPa)

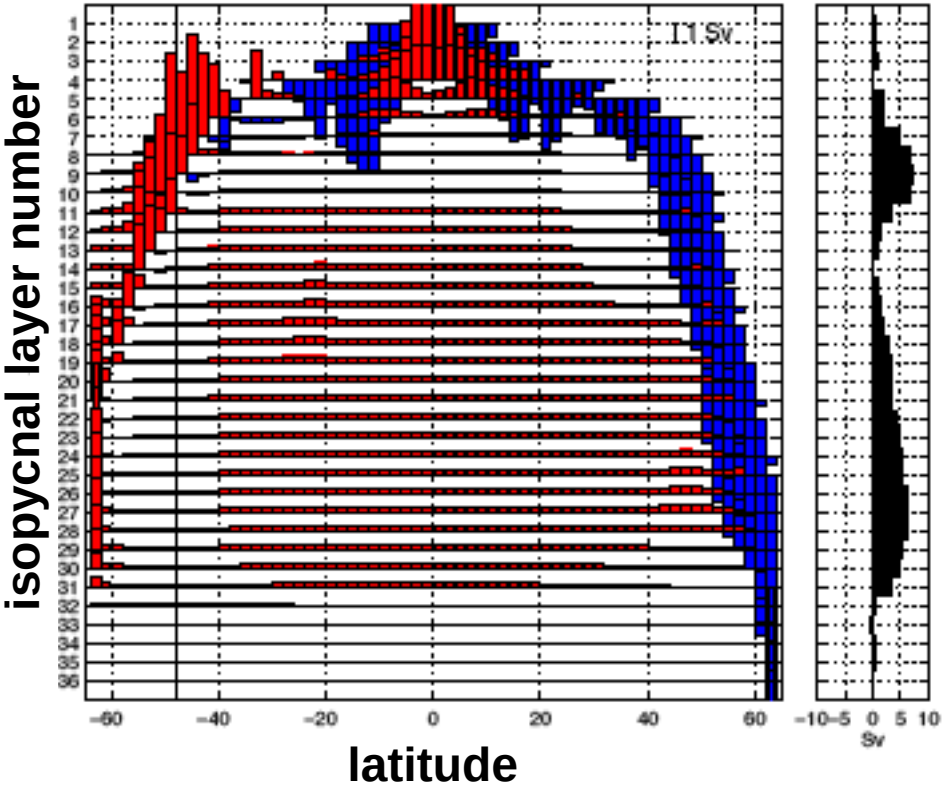



Second-order time-stepping scheme (with same $\Delta t$)

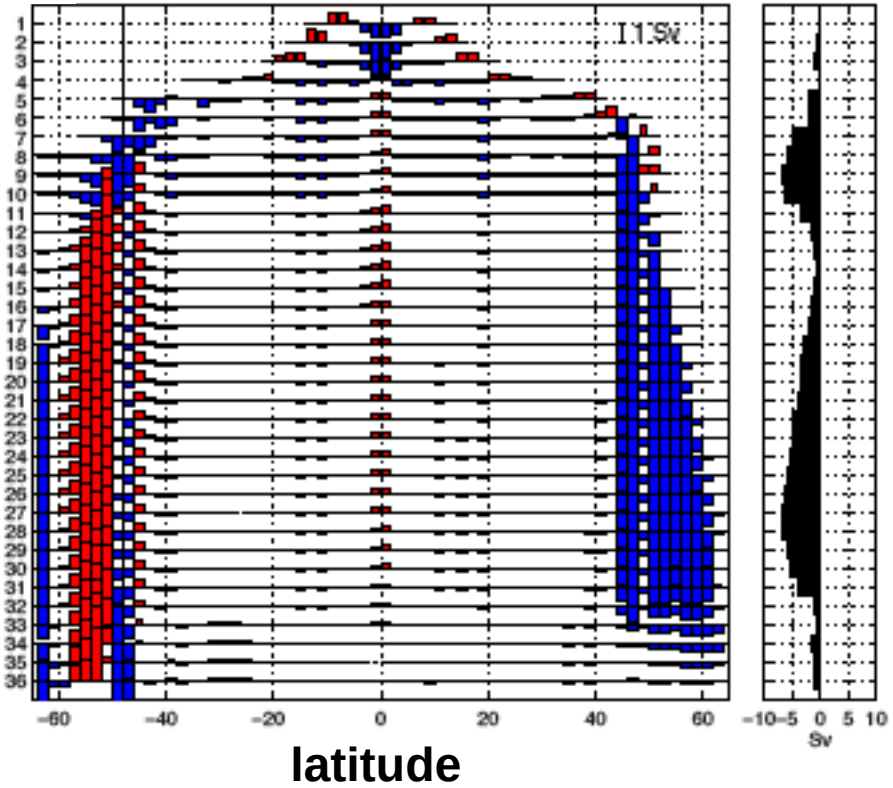
(Zhao & Zhong 2009)

# Impact of time stepping in the MICOM ocean GCM

**physical diapycnal volume fluxes**




**numerical diapycnal volume fluxes**




(van Leeuwen, Leeuwenburg, Drijfhout & Katsman; unpublished)

- “In the weather and climate prediction community, when thinking in terms of model predictability, there is a tendency to associate model error with the physical parameterizations. In this paper, it is shown that time truncation error can be a substantial part of the total forecast error” (Teixeira et al. 2007)

- The sensitivity of the skill of medium-range weather forecasts to the time-stepping method is about the same as to the physics parameterizations (Amezcua 2012)

- “Climate simulations are sensitive not only to physical parameterizations of subgrid-scale processes but also to the numerical methodology employed” (Pfeffer et al. 1992)

- “Many published conclusions on parameter sensitivity, calibrated values and associated uncertainty may be questionable due to numerical artifacts introduced by unreliable time stepping schemes” (Kavetski & Clark 2010)

- “In general, much less concern is given to the temporal accuracy than the spatial accuracy of GCMs” (Thrastarson & Cho 2011)

- Reducing the time step “leads to a statistically significant (at the 5% confidence level) reduction in the number of cyclones over the Northern Hemisphere extratropics” (Jung et al. 2012)

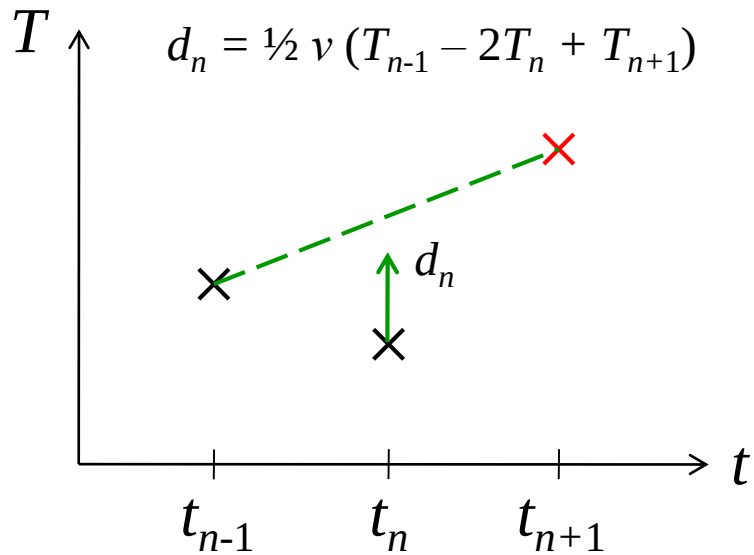
- “Interestingly, both the improvement in near-surface winds in the tropical Pacific and the meridional mean circulation in the tropics found when going from T159 to T511 can also be achieved if the coarser-resolution, T159 model is run with the same, shorter time step used by the T511 model (i.e., 15 min). This suggests that the improvements seen in near-surface tropical winds when going from T159 to T511 are primarily due to the shorter time step required to attain stability rather than increased horizontal resolution.” (Jung et al. 2012)

# Leapfrog with Robert–Asselin filter

LF+RA
(Robert 1966, Asselin 1972)

$$\frac{\mathrm{d}\,T}{\mathrm{d}\,t} = f\,(T\,)$$

$$d_n = \tfrac{1}{2}\,\nu\,(T_{n\text{-}1} - 2T_n + T_{n+1})$$



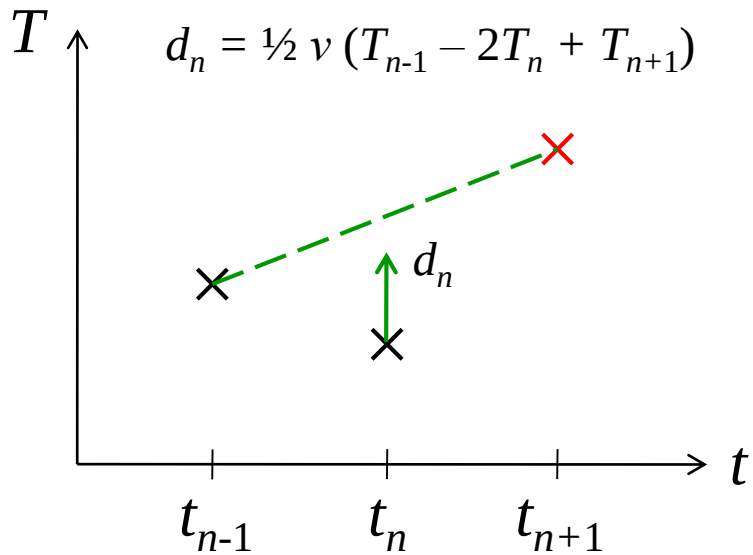

- use leapfrog to calculate $T_{n+1} = T_{n\text{-}1} + 2\,\Delta t\,f\,(T_n)$
- RA filter nudges $T_n = T_n + d_n$
- reduces curvature <u>but does not conserve mean</u>
- amplitude accuracy is 1st order

# Leapfrog with Robert–Asselin filter

- Widely used in current numerical models
  - atmosphere: ECHAM, MAECHAM, MM5, CAM, MESO-NH, HIRLAM, KMCM, LIMA, SPEEDY, IGCM, PUMA, COSMO, FSU-GSM, FSU-NRSM, NCEP-GFS, NCEP-RSM, NSEAM, NOGAPS, RAMS, CCSR/NIES-AGCM
  - ocean: OPA, ORCA, NEMO, HadOM3, DieCAST, TIMCOM, GFDL-MOM, POM, MICOM, HYCOM, POSEIDON, NCOM, ICON, OFES, SOM
  - coupled: HiGEM (oce), COAMPS (atm), PlaSim (atm), ECHO (atm), MIROC (atm), FOAM (oce), NCAR-CCSM (atm), BCM (oce), NCEP-CFS (atm/oce), QESM (oce), CHIME (oce), FORTE (atm)
  - others: GTM, ADCIRC, QUAGMIRE, MORALS, SAM, ARPS, CASL, CReSS, JTGCM, ECOMSED, UKMO-LEM, MPI-REMO
- Asselin (1972) has received over 450 citations
- Has many problems
  - "*The Robert–Asselin filter has proved immensely popular, and has been widely used for over 20 years. However, it is not the last word…*" (Lynch 1991)
  - "*Replacement of the Asselin time filter… can be a feasible way to improve the ability of climate models*" (Zhao & Zhong 2009)
  - "*The Robert–Asselin filter can produce slewing frequency as well as the well-known damping and phase errors*" (Thrastarson & Cho 2011)

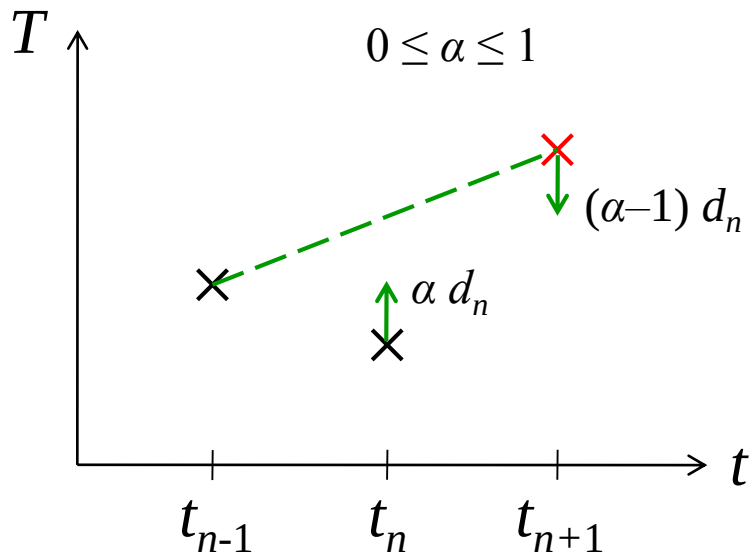
# A proposed improvement

LF+RA
(Robert 1966, Asselin 1972)




- use leapfrog to calculate $T_{n+1}$
- RA filter nudges $T_n$
- reduces curvature <u>but does not conserve mean</u>
- amplitude accuracy is 1st order

LF+RAW
(Williams 2009, 2011)




- use leapfrog to calculate $T_{n+1}$
- RAW filter nudges $T_n$ <u>and</u> $T_{n+1}$
- reduces curvature <u>and conserves mean</u> (for α=½)
- amplitude accuracy is 3rd order

# Simple test integration

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\omega Y$$
$$\frac{\mathrm{d}Y}{\mathrm{d}t} = +\omega X$$

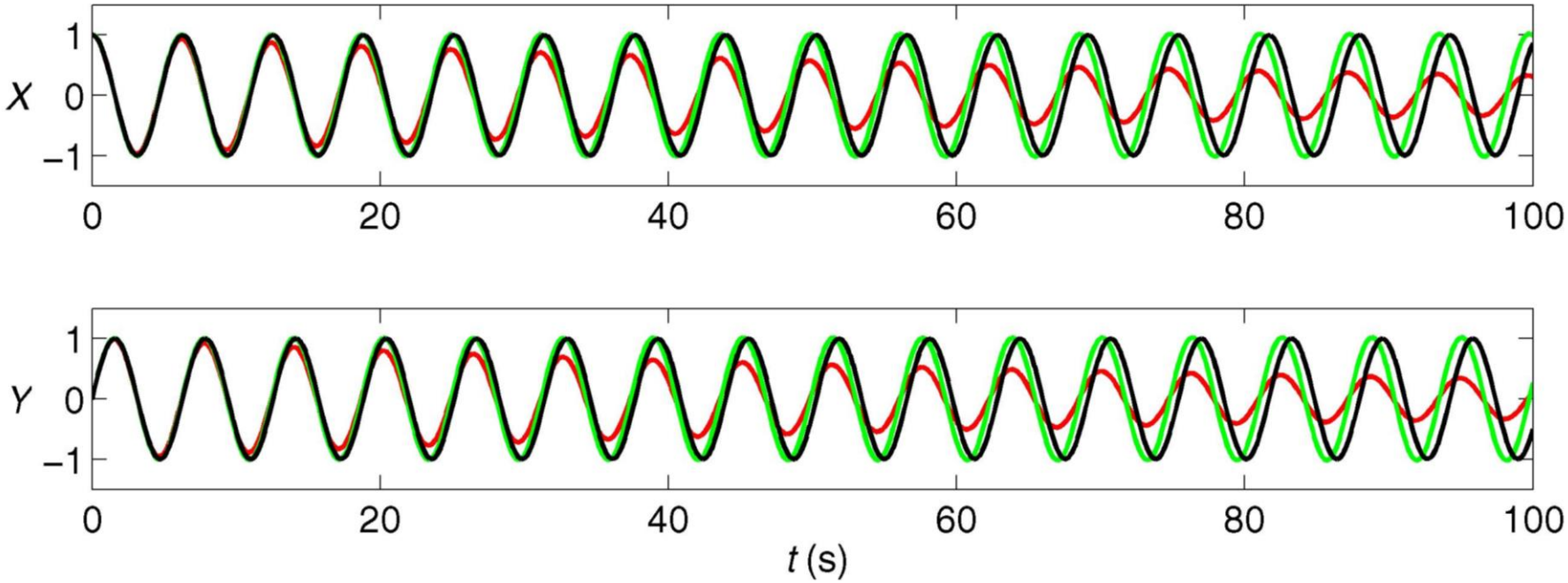
$$\omega = 1 \text{ rad s}^{-1}$$

**exact**

**LF+RA**

**LF+RAW$_{\alpha=1/2}$**

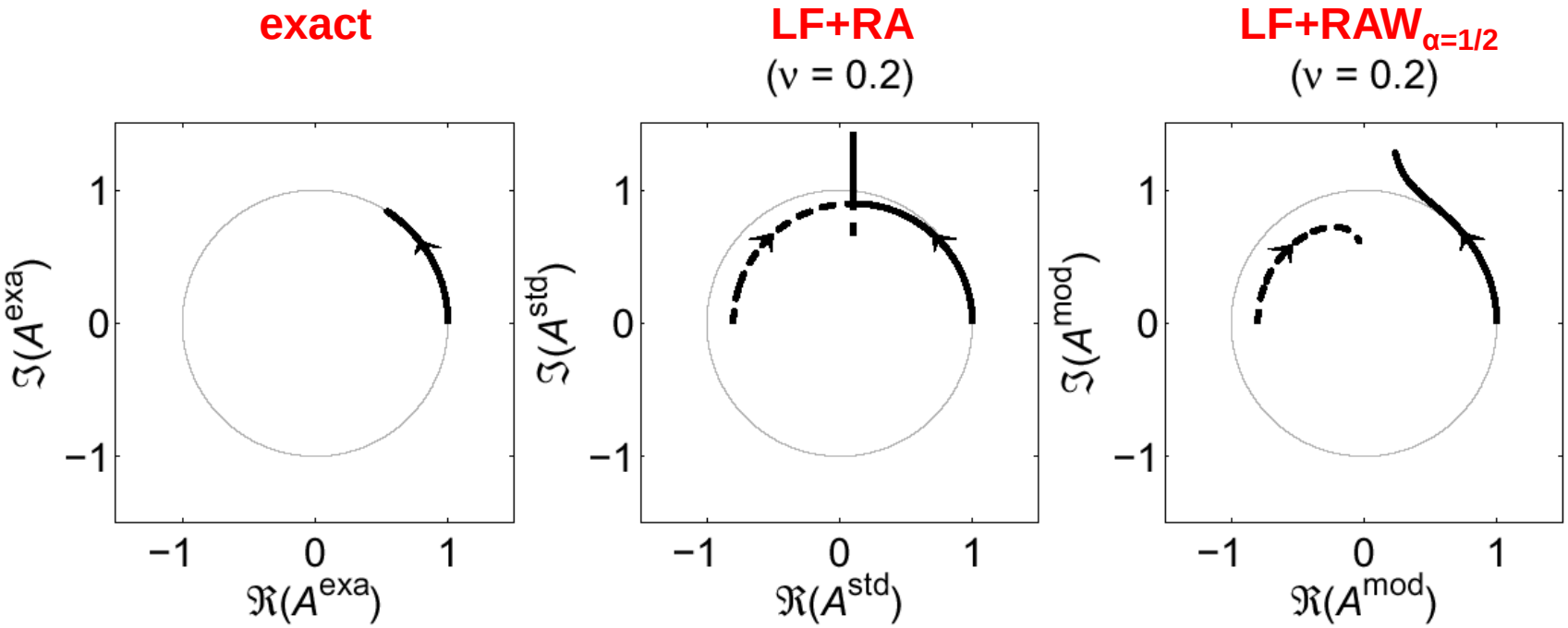
$$\left.\right\} \quad \Delta t = 0.2 \text{ s}, \quad \nu = 0.2$$




(Williams 2009)

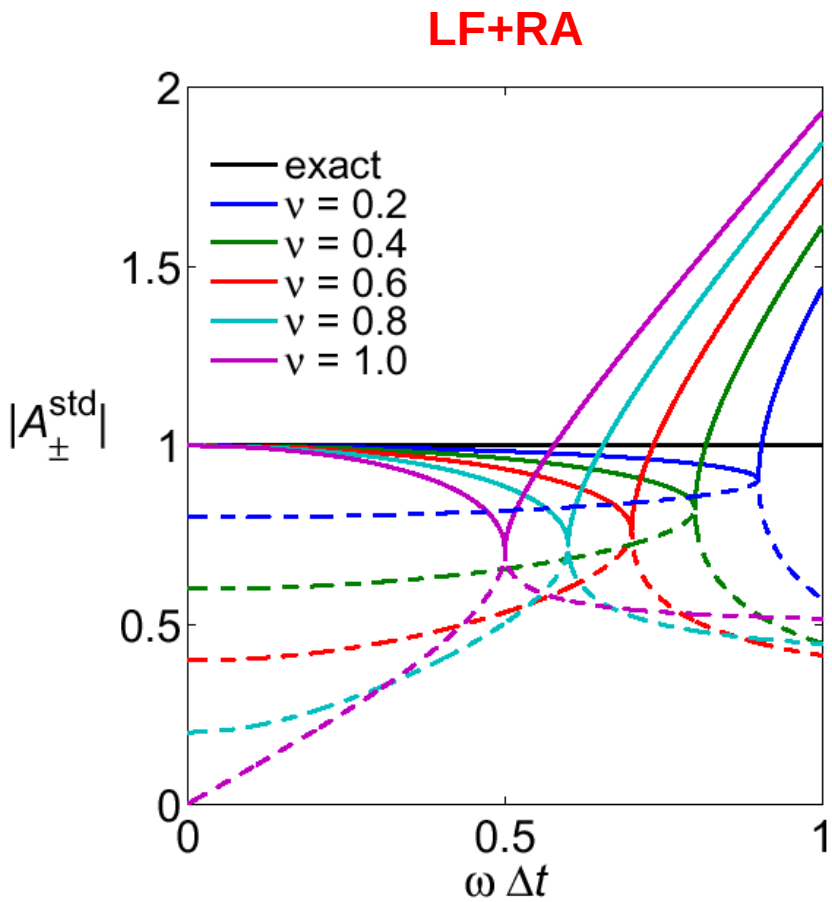
# Analysis: numerical amplification

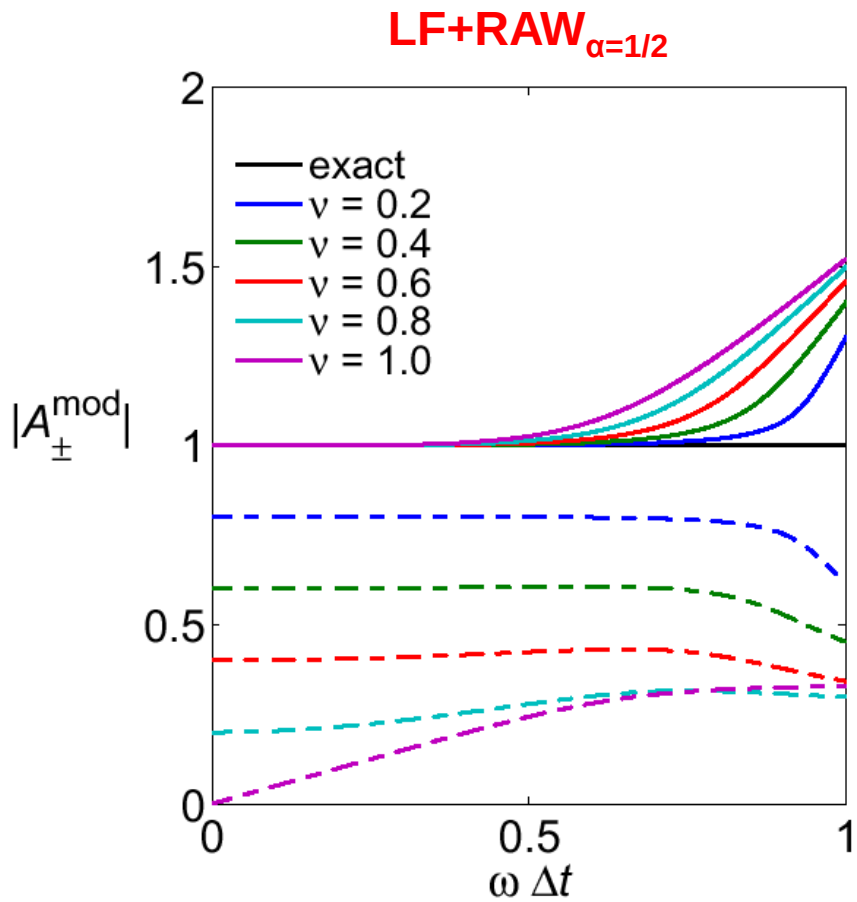
Let $\dot{F} = \mathrm{i}\omega F$ and $A = F(t+\Delta t) \,/\, F(t)$ and trace $A$ as $\omega\Delta t = 0 \rightarrow 1$:




(Williams 2009)

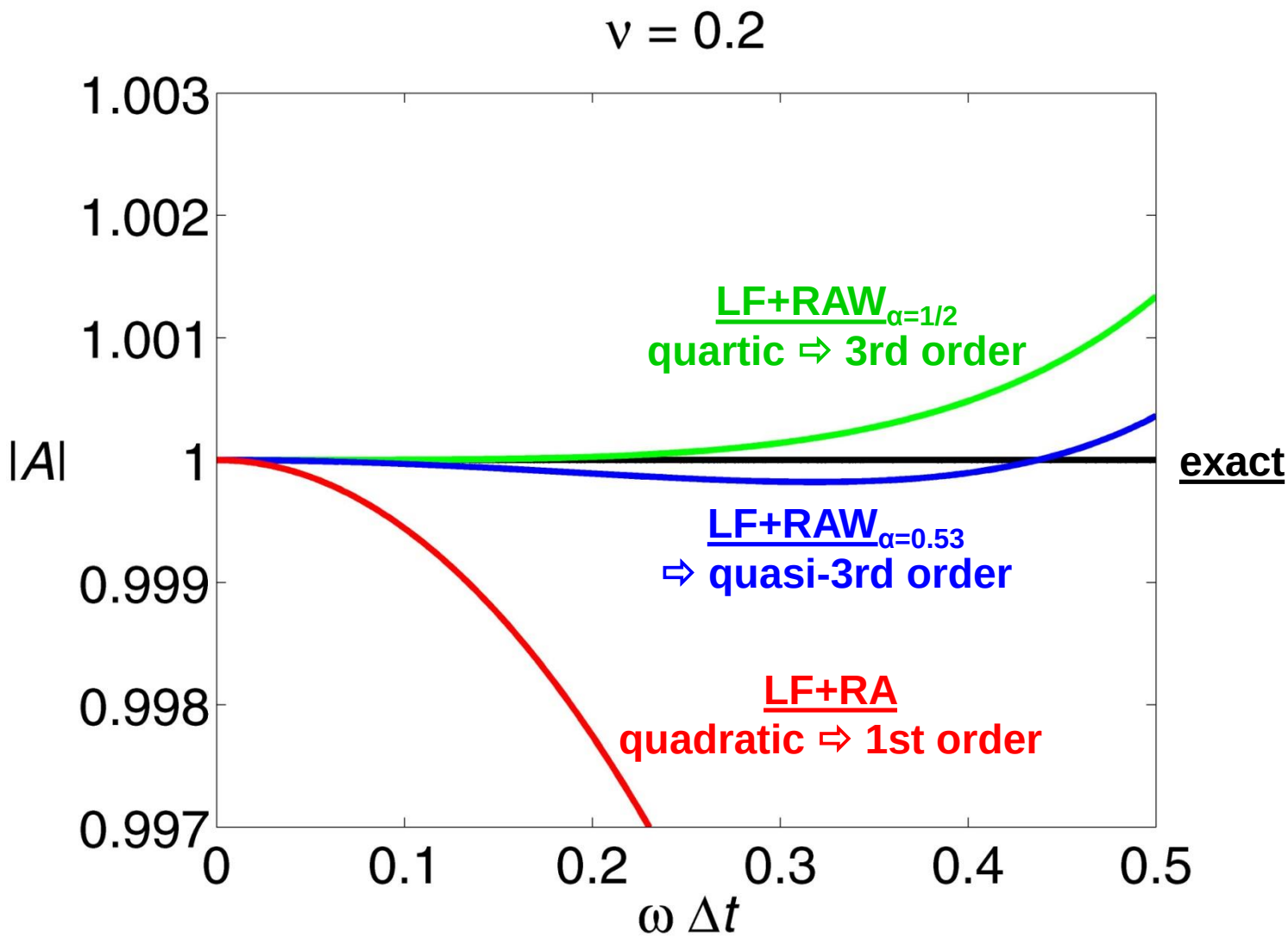
# Analysis: numerical amplification




(Williams 2009)

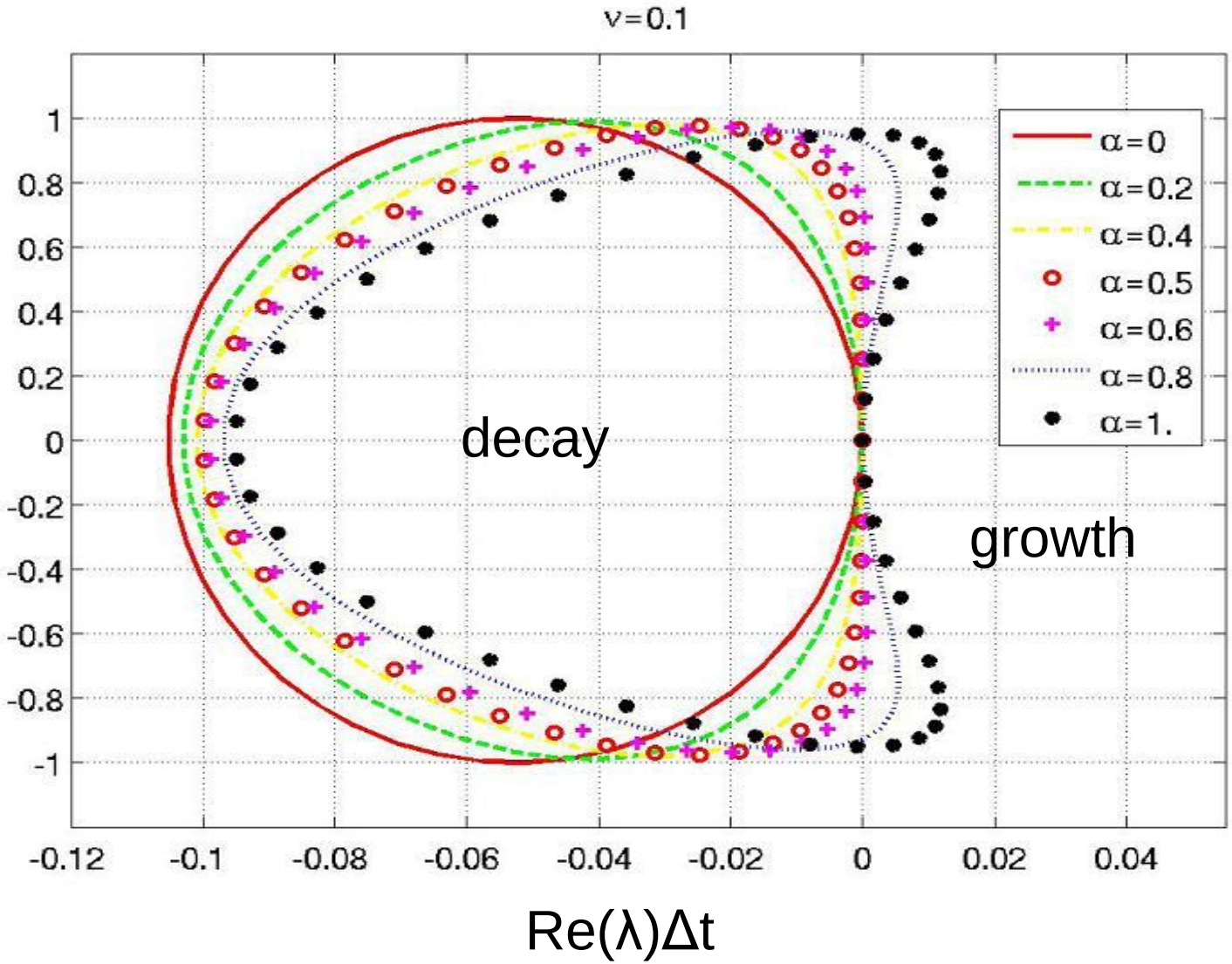
# Analysis: numerical amplification




(Williams 2009)

# Analysis: numerical stability

$\dot{F} = \lambda F$

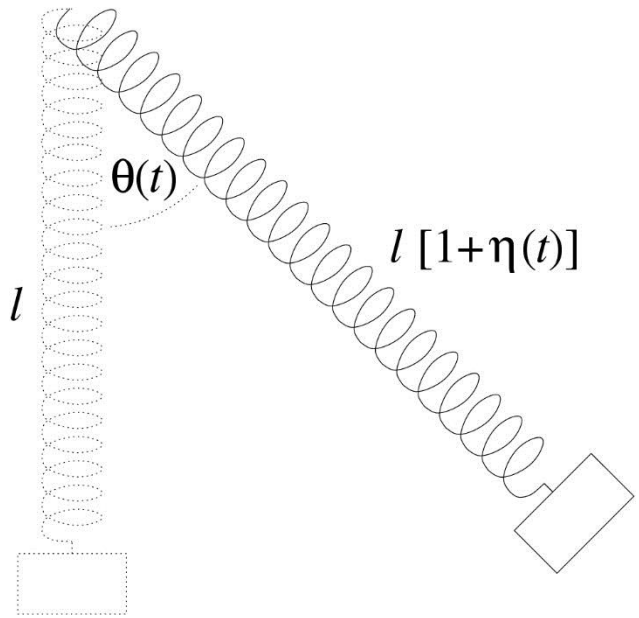


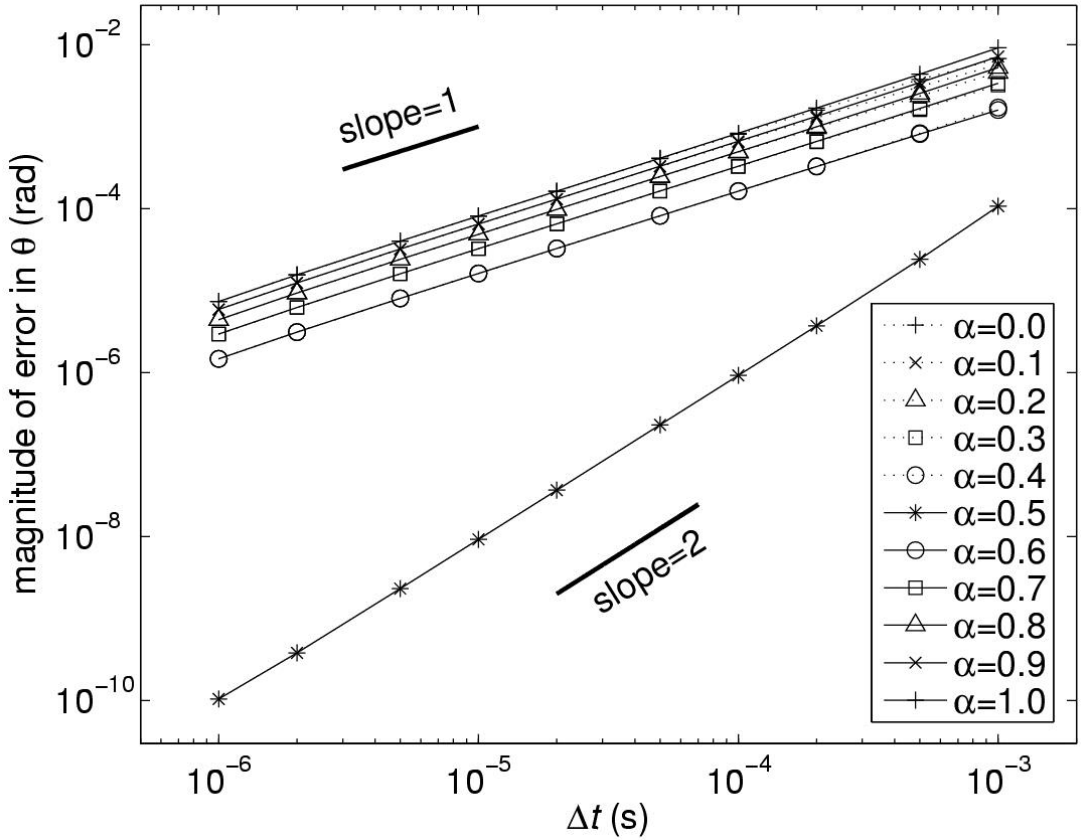

Im(λ)Δt

Re(λ)Δt

# Analysis: numerical convergence

Semi-implicit integrations of the elastic pendulum (or "swinging spring")




(Williams 2011)

# Implementation in existing code

```
! Compute tendency at this time step
tendency = f[x_this]

! Leapfrog step
x_next = x_last + tendency*2*delta_t

! Compute filter displacement
d = nu*(x_last - 2*x_this + x_next)/2

! Apply filter
x_this = x_this + d*alpha
x_next = x_next + d*(alpha-1)
```

(Williams 2011)

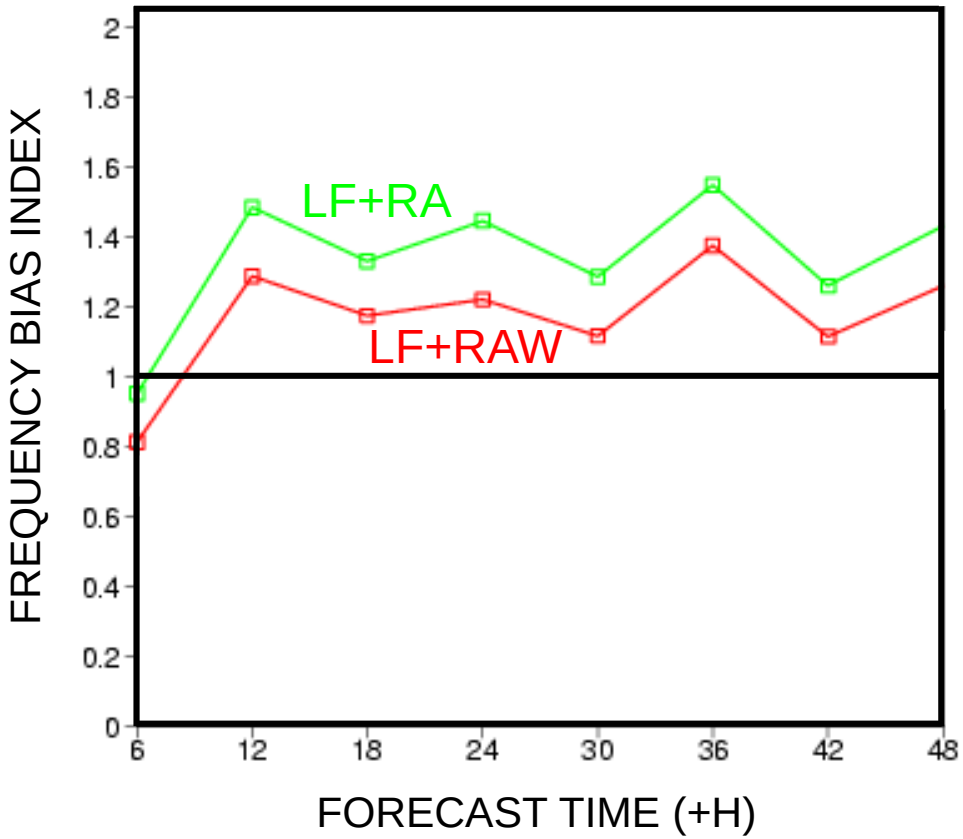
# Some recent implementations

The RAW-filtered leapfrog...

- is the default time-stepping method in the atmosphere of MIROC5, the latest version of the Model for Interdisciplinary Research On Climate (Watanabe et al. 2010)
- has been used in the regional climate model COSMO-CLM (CCLM) with $\alpha$=0.7, and “can lead to a significant improvement, especially for the simulated temperatures” (Wang et al. 2013)
- is the default time-stepping method in TIMCOM, the Talwan Multi-scale Community Ocean Model, and gives simulations that are in better agreement with observations (Young et al. 2014)
- has been implemented in an ice model, and improves the spin-up and conservation energetics of the physical processes (Ren & Leslie 2011)
- has been implemented in the SPEEDY atmosphere GCM, and significantly improves the skill of medium-range weather forecasts (Amezcua et al. 2011)
- has been found to perform well in various respects in semi-implicit integrations (Durran & Blossey 2012, Clancy & Pudykiewicz 2013)

# Implementation in COSMO-ME

- Statistical analysis: two 48h forecasts per day from 16 Dec 2008 to 18 Jan 2009
- The modified filter significantly improves precipitation forecasts (2-10 mm / 6h)




over-forecast

↑

unbiased

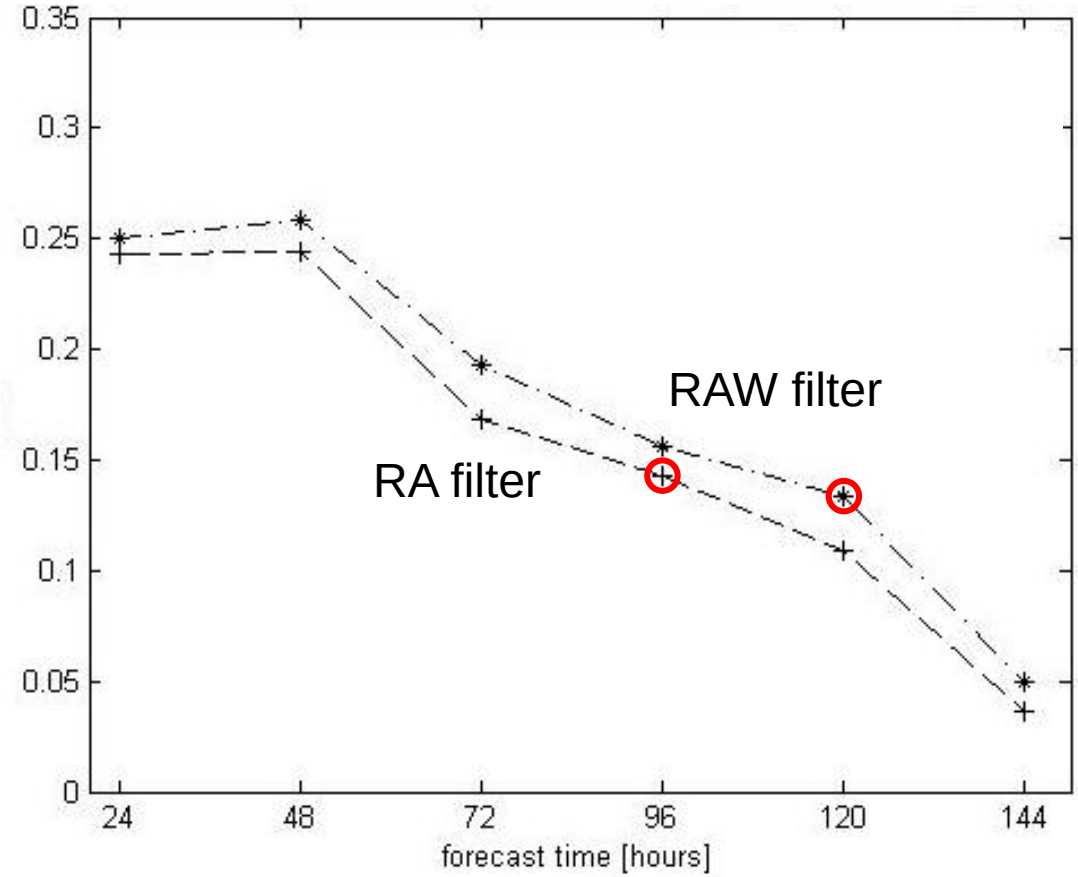
↓

under-forecast

Lucio Torrisi, CNMCA, Italy

# Implementation in SPEEDY

ACC for surface pressure in the tropics (25°S-25°N)




5-day forecasts made using the RAW filter have approximately the same skill as 4-day forecasts made using the RA filter

(Amezcua, Kalnay & Williams 2011)

# Some recent developments

- The stability of the Crank–Nicolson-Leapfrog method with the Robert–Asselin–Williams time filter has been interrogated rigorously by Nicholas Hurl, Bill Layton, Yong Li, and Catalin Trenchea (2014)

- A higher-order Robert–Asselin (hoRA) time filter has been developed by Yong Li and Catalin Trenchea (2014), which is almost as accurate, stable, and efficient as the third-order Adams–Bashforth (AB3) method but easier to implement

# Composite-tendency leapfrog with (1, -2, 1) filter

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \left\{ \begin{aligned} x_{n+1} &= \overline{\overline{x}}_{n-1} + 2\Delta t\left[\gamma f(\overline{x}_n) + (1-\gamma) f(x_n)\right] \\ \overline{\overline{x}}_n &= \overline{x}_n + \frac{\nu\alpha}{2}\left[\overline{\overline{x}}_{n-1} - 2\overline{x}_n + x_{n+1}\right] \\ \overline{x}_{n+1} &= x_{n+1} - \frac{\nu(1-\alpha)}{2}\left[\overline{\overline{x}}_{n-1} - 2\overline{x}_n + x_{n+1}\right] \end{aligned} \right.$$

leapfrog

filter

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i}\omega x \left\{ \begin{aligned} & & |A_\mathrm{P}| - 1 &= \frac{\nu(1-2\alpha)}{2(2-\nu)}(\omega\Delta t)^2 + \mathcal{O}[(\omega\Delta t)^4] \\ & \alpha = \frac{1}{2} & |A_\mathrm{P}| - 1 &= \frac{\nu(4\gamma - 3 + \nu - \nu\gamma)}{4(2-\nu)^2}(\omega\Delta t)^4 + \mathcal{O}[(\omega\Delta t)^6] \\ & \gamma = \frac{3-\nu}{4-\nu} & |A_\mathrm{P}| - 1 &= \frac{\nu}{4(4-\nu)(2-\nu)^2}(\omega\Delta t)^6 + \mathcal{O}[(\omega\Delta t)^8] \end{aligned} \right.$$

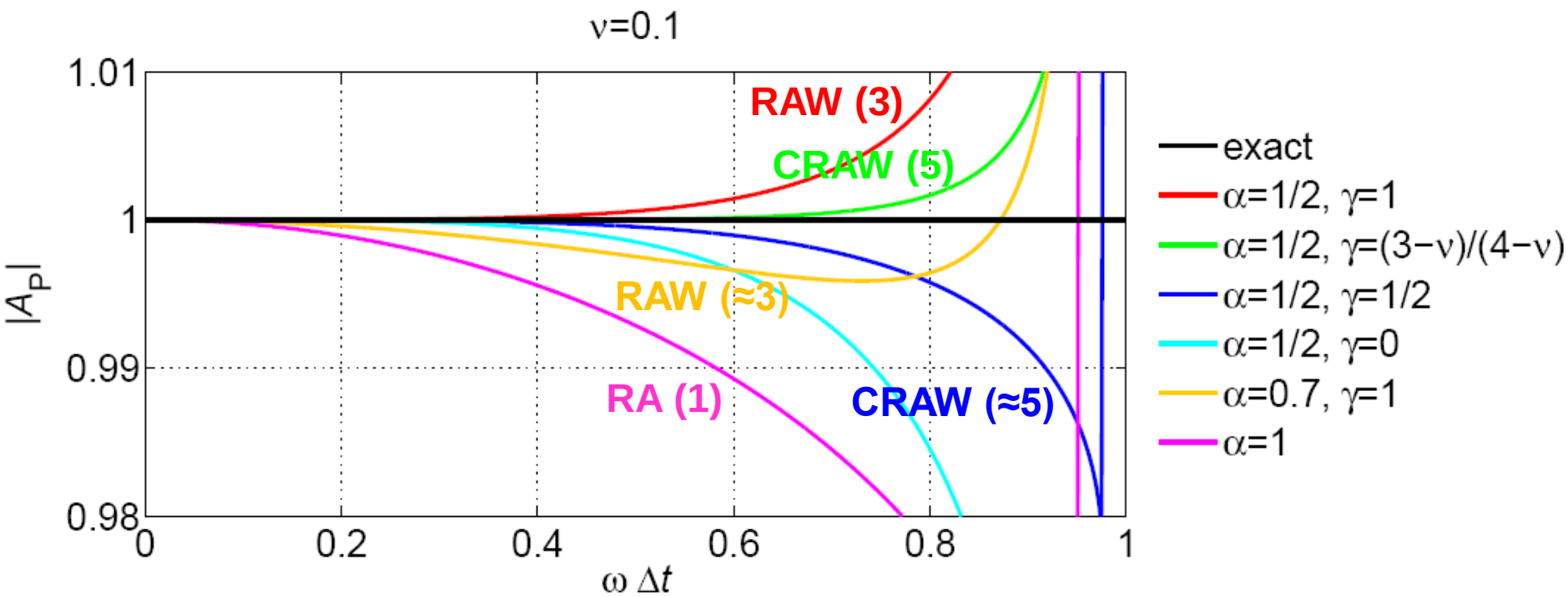
1st order

3rd order

5th order

(Williams 2013)

# Composite-tendency leapfrog with (1, -2, 1) filter




(Williams 2013)

# Composite-tendency leapfrog with (1, -4, 6, -4, 1) filter

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \left\{ \begin{array}{rcl} x_{n+1} = \overline{\overline{x}}_{n-1} + 2\Delta t\left[\gamma f(\overline{x}_n) + (1-\gamma) f(x_n)\right] & & \\ \overline{\overline{x}}_n & = & \overline{x}_n + \nu\alpha\left[\overline{\overline{x}}_{n-3} - 4\overline{\overline{x}}_{n-2} + 6\overline{\overline{x}}_{n-1} - 4\overline{x}_n + x_{n+1}\right] \\ \overline{x}_{n+1} & = & x_{n+1} - \nu(1-\alpha)\left[\overline{\overline{x}}_{n-3} - 4\overline{\overline{x}}_{n-2} + 6\overline{\overline{x}}_{n-1} - 4\overline{x}_n + x_{n+1}\right] \end{array} \right.$$

leapfrog

filter

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i}\omega x \left\{ \begin{array}{ll} & |A_{\mathrm{P}}| - 1 = -\dfrac{\nu(1-2\alpha)}{2(1-\nu-2\alpha\nu)}(\omega\Delta t)^4 + \mathcal{O}[(\omega\Delta t)^6] \\ \alpha = \dfrac{1}{2} & |A_{\mathrm{P}}| - 1 = \dfrac{\nu(5-8\gamma-9\nu+14\nu\gamma)}{8(1-2\nu)^2}(\omega\Delta t)^6 + \mathcal{O}[(\omega\Delta t)^8] \\ \gamma = \dfrac{5-9\nu}{2(4-7\nu)} & |A_{\mathrm{P}}| - 1 = -\dfrac{5\nu(4-13\nu+11\nu^2)}{32(1-2\nu)^2(4-7\nu)}(\omega\Delta t)^8 + \mathcal{O}[(\omega\Delta t)^{10}] \end{array} \right.$$

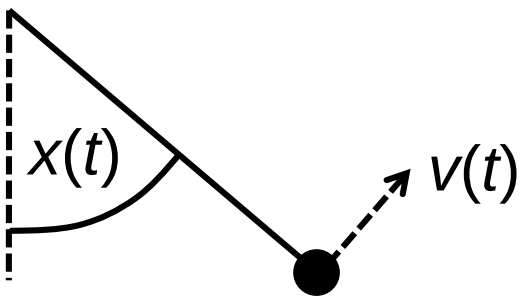
3rd order

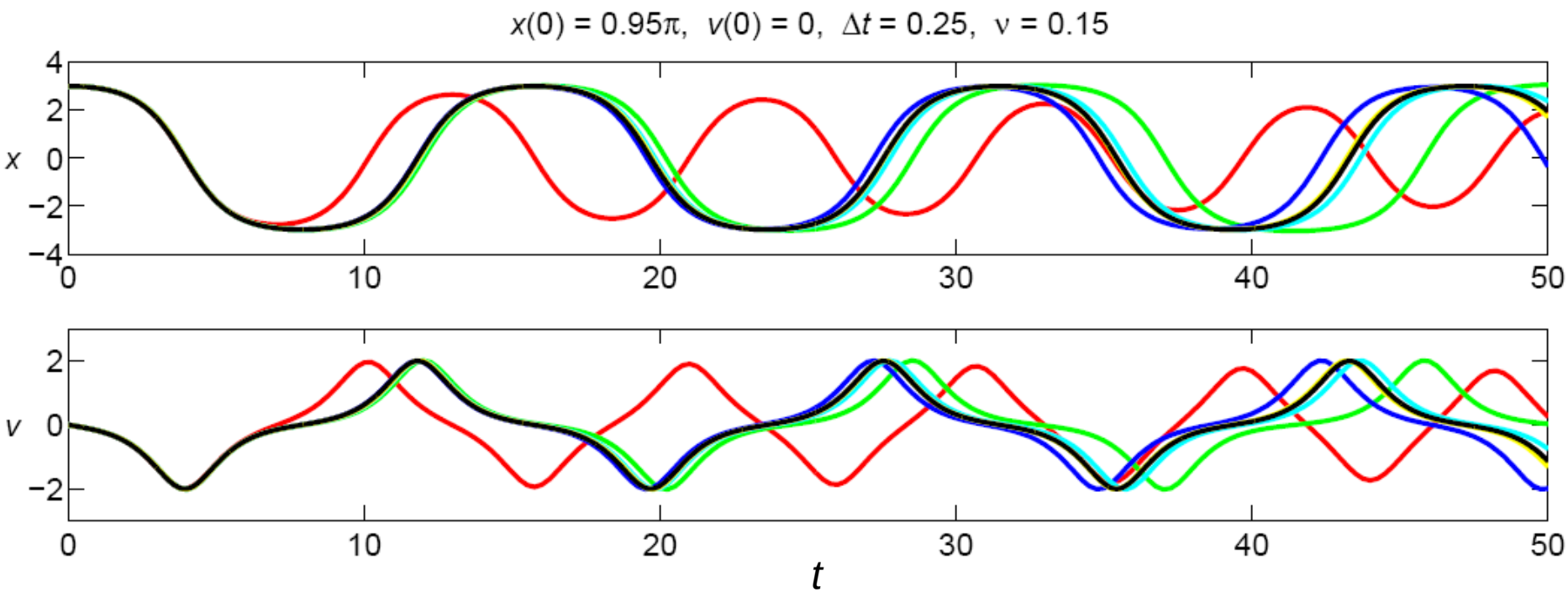
5th order

7th order

(Williams 2013)

# Nonlinear simple pendulum




(Williams 2013)

# Summary table

| filter | $\alpha$ | $\gamma$ | no. computational modes | no. function evaluations per time step | order | amplitude order of accuracy | phase order of accuracy | maximum $\omega\Delta t$ for stability ($\nu = 0.1$) | maximum $\omega\Delta t$ for accuracy ($\nu = 0.1$) |
|---|---|---|---|---|---|---|---|---|---|
| $(1, -2, 1)$ | 1 | – | 1 | 1 | 1 | 1 | 2 | 0.951 | 0.030 |
| | 1/2 | 1 | 1 | 1 | 2 | (3) | 2 | 0 | (0.448) |
| | 1/2 | $(3-\nu)/(4-\nu)$ | 2 | 2 | 2 | (5) | 2 | 0 | (0.712) |
| | 1/2 | 1/2 | 2 | 2 | 2 | 3 | 2 | 0.975 | 0.475 |
| | 1/2 | 0 | 2 | 1 | 2 | 3 | 2 | 0.832 | 0.333 |
| | 0 | 1 | 1 | 1 | 1 | (1) | 2 | 0 | (0.030) |
| $(1, -4, 6, -4, 1)$ | 1 | – | 3 | 1 | 2 | (3) | 2 | 0 | (0.228) |
| | 1/2 | 1 | 3 | 1 | 2 | 5 | 2 | 0.690 | 0.424 |
| | 1/2 | $(5-9\nu)/[2(4-7\nu)]$ | 4 | 2 | 2 | 7 | 2 | 0.616 | 0.586 |
| | 1/2 | 0 | 4 | 1 | 2 | (5) | 2 | 0 | (0.394) |
| | 0 | 1 | 3 | 1 | 2 | 3 | 2 | 0.777 | 0.240 |

(Williams 2013)

# Summary

- Time stepping is an important contributor to model error in today's weather and climate models
- The Robert–Asselin filter is widely used but is dissipative and reduces accuracy
- The RAW filter has approximately the same stability but much greater accuracy
- Implementation in an existing code is trivial and there is no extra computational cost
- 5th-order and even 7th-order amplitude accuracy may be achieved, by using a composite tendency and/or a more discriminating filter

# Further information

Williams PD (2013) Achieving seventh-order amplitude accuracy in leapfrog integrations. *Monthly Weather Review* **141**(9), 3037-3051.

Williams PD (2011) The RAW filter: An improvement to the Robert–Asselin filter in semi-implicit integrations. *Monthly Weather Review* **139**(6), 1996-2007.

Williams PD (2009) A proposed modification to the Robert–Asselin time filter. *Monthly Weather Review* **137**(8), 2538-2546.

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