Multiple Tasks Kinematics Using Weighted Pseudo-Inverse for
Kinematically Redundant Manipulators

Jonghoon Park  Youngjin Choi  Wan Kyun Chung  Younig Youn
Post Doctoral Graduate Student Professor Professor

Robotics and Bio-Mechatronics (RNB) Lab., Department of Mechanical Engineering
Pohang University of Science and Technology (POSTECH), Pohang, 790-784, Republic of Korea
E-mail: {coolcat,yjchoi,wkchung,youm}@postech.ac.kr

Abstract— This paper proposes a method to accommodate multiple tasks for redundancy utilization, which is based on a specific weighted pseudo-inverse. The proposed method also has task priority imposition property as those conventional task priority based method. In order to deal with general situations of task specification, the so-called semi-definitely weighted pseudo-inverse is devised.

1 Introduction

Joint motion of a kinematically redundant manipulator is not kinematically determined by task motion only, since the joint has more degrees-of-freedom than the task dimension. To resolve such a kinematic indeterminacy from the viewpoint of tasks, one may desire to execute additional tasks. This approach has already been well-developed in redundant kinematics, and it is called the task priority based method (TPBM) [1], [2], [3]. The TPBM has the common feature that it can execute a primary task exactly, and a secondary task is executed as exact as possible.\(^1\)

Meanwhile, it is well-known that usage of weighted pseudoinverses can change behaviors of motion for redundant manipulators. For example, the weight matrix was chosen to reflect a dynamic behavior [5], [6], to distribute joint motion in a desirable way [7], to establish gauge-invariance of solution [8], etc. As a matter of fact, choice of a specific weight matrix amounts to choice of a specific coordinate transformation. Then it may be possible to choose a set of coordinate axes which is favorable in the sense of multiple tasks execution. In this article, we will show that a simple weighted pseudo-inverse solution can guide this choice of the coordinate axes and simultaneously realize the basic idea of conventional TPBM. In other words, using an appropriate weighted pseudo-inverse of Jacobian matrix, multiple tasks can be executed as exactly as possible in the sense of task priority. Though the work can be regarded as a reformulation of the TPBM using a weighted pseudo-inverse, it gives new insight on multiple task kinematics, and results in favorable and efficient formulation.

This paper is organized in the following manner. First, various situations occurring in the context of multiple tasks kinematics of redundant manipulators are disclosed using geometrical inspection in Sec. 2. Sec. 3 summarizes main results on the weighted pseudo-inverse solution to the multiple tasks kinematics. Starting from the bordered Gramian formulation of the TPBM, there follows a naturally weighted pseudo-inverse formulation. To include kinematically indeterminate case, we introduce the so-called semi-definitely weighted pseudoinverse, that is the weighted pseudo-inverse using a special positive semi-definite weight matrix. Sec. 4 provides geometrical basis on the multiple task kinematics, e.g. the joint space decomposition according to multiple task kinematics. Also, geometrical meaning of the proposed weighted pseudo-inverse is illustrated with interpretation of the chosen weight matrix. Finally, in order for efficient formulation which can be extended to dynamic control, the proposed weighted pseudo-inverse is formulated using the recently developed kinematically decoupled joint space decomposition (KD-JSD) in Sec. 5. The semi-definitely weighted KD-JSD is also introduced.

2 Kinematics of Multiple Tasks

For specific discussion, let us denote the kinematics of a redundant manipulator by

\[ \dot{p} = J(q)\dot{q} \]  

(1)

where the Jacobian matrix \( J(q) \in \mathbb{R}^{m \times n} \) relates task velocity \( p \in \mathbb{R}^m \) and joint velocity \( \dot{q} \in \mathbb{R}^n \) (\( n > m \)). In particular, assume that we are given an additional \( l \)-dimensional task defined by

\[ \dot{h} = H(q)\dot{q} \in \mathbb{R}^l, \]  

(2)

where \( H(q) \in \mathbb{R}^{l \times n} \) is called the secondary Jacobian for the additional task velocity \( \dot{h} \in \mathbb{R}^l \).

Note that the dimension \( l \) does not have to be equal to the redundancy \( r(= n - m) \) in general. In this case, the multiple task kinematics falls into one of three situations according to the dimension of the secondary task: kinematically determinate, kinematically indeterminate, and kinematically overspecified according to whether \( l = r \), \( l < r \), and \( l > r \), respectively. Given a primary task \( \dot{p}^\star \) and a secondary task \( \dot{h}^\star \), the representative illustration of each situation is shown in Fig. 1, where the joint velocity \( \dot{q}^\star \) is a possible solution in light of the motivation of the TPBM. Let us denote the null space by \( \text{Ker}\{\cdot\} \) and the range space

\(^1\)Chiaverini’s algorithm [3] may result in error in a secondary task [4], even if the secondary task can be executed exactly using the other algorithms [1], [2]. However, the secondary task error can be made small using closed-loop kinematic control.
by \( \text{Im}\{\cdot\} \). As shown in Fig. 1 (a), the solution to the TPBM is uniquely determined in the kinematically determinate case, whereas in the indeterminate case there are infinitely many solutions, shown in Fig. 1 (b), the differences of which belong to the null space of \( J_{\text{AUG}} = [J \ H] \). Contrary to the previous cases, in the over-specified case there does not exist any exact solution which accomplishes the given two velocities, since the two range spaces has some common subspace, i.e., \( Q_{ph} = \text{Im}\{J^T\} \cap \text{Im}\{H^T\} \neq \emptyset \). According to the task priorities, the secondary solution should be sacrificed by the amount of the components belonging to the common subspace, and the solution will be the one denoted by \( \tilde{q}^* \) in Fig. 1 (c).

3 Weighted Pseudoinverse Solution to Multiple Task Kinematics

As a matter of fact, the original TPBM was formulated by solving the least square problem of the possible secondary task error of (2), i.e., \( H \tilde{q} - \tilde{h} \) under the constraint of the primary task kinematics (1). In this section, we reformulate the least square problem associated with multiple tasks execution using a weighted pseudo-inverse solution.

3.1 Bordered Grammian formulation

The problem can be solved by applying Lagrangian multiplier optimization theorem. The theorem states that the optimal solution is obtained where the gradients of the Lagrangian

\[
\mathcal{L}(\tilde{q}, \lambda) = \frac{1}{2} (H \tilde{q} - \tilde{h})^T (H \tilde{q} - \tilde{h}) + \lambda^T (J \tilde{q} - \tilde{p})
\]

with respect to \( \tilde{q} \) and \( \lambda \in \mathbb{R}^m \) become all zero. Therefore one gets

\[
H^T H \tilde{q} + J^T \lambda = H^T \tilde{h} \\
J \tilde{q} = \tilde{p}.
\]

The matrix version of the equations is

\[
\begin{bmatrix}
H^T & J^T \\
J & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{q} \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
H^T \tilde{h} \\
\tilde{p}
\end{bmatrix},
\]

where the coefficient matrix has the form called the bordered Grammian matrix and denoted here by

\[
G = \begin{bmatrix}
H^T & J^T \\
J & 0
\end{bmatrix} \in \mathbb{R}^{(n+m) \times (n+m)}.
\]

Hence the solution to the least square problem can be obtained by solving the linear equation (5).

The solution exists if and only if \( H^T \tilde{h} \in \text{Im}\{J^T\} \) and \( \tilde{p} \in \text{Im}\{J\} \). The first always holds, and the second holds if the kinematics (4) is consistent. Then the solutions are given of the form

\[
\begin{bmatrix}
\tilde{q} \\
\lambda
\end{bmatrix} = G^+ \begin{bmatrix} H^T \tilde{h} \\
\tilde{p}
\end{bmatrix} + \{I - G^+ G\} z.
\]

To facilitate further discussions we want to describe the general properties of the bordered Grammian matrix (6), which we owe to [9]. By defining

\[
W = H^T H + J^T J \in \mathbb{R}^{n \times n} \\
Y = J W^+ J^T \in \mathbb{R}^{m \times m},
\]

we have the following properties:
Using (8) and (9) the final solution is given by
\[ G = \begin{bmatrix} W^+ & 0 \\ 0 & Y^+ Y^T \end{bmatrix}. \] (9)

Using (8) and (9) the final solution is given by
\[ \dot{q} = W^+ \left( J^T Y^+ \dot{p} + \left( I - J^T Y^+ J W^+ \right) H^T \dot{h} \right) + \left( I - W^+ W \right) z \]
\[ \lambda = Y^+ J W^+ H^T \dot{h} + \left( I - Y^+ \right) \dot{p} + \left( I - J J^T \right) \zeta. \] (10)

**Lemma 1:** Given primary task and secondary task velocity \( \dot{p} \) and \( \dot{h} \), the joint velocity \( \dot{q} \) given by (10) has natural task priority imposition. Furthermore, if \( \mathcal{Q}_{ph} \) is empty, then the secondary task is also executed exactly, i.e., \( H \dot{q} = H H^+ \dot{h} \).

**Proof:** Using (BG2) and (BG3), it is easy to show that the following equations hold
\[ J \dot{q}^* = J J^T \dot{p}^* \] (11a)
\[ H \dot{q}^* = H W^+ J^T Y^+ \dot{p}^* + H \left( I - W^+ J^T Y^+ J \right) W^+ H^T \dot{h} \] (11b)

which shows that the lower priority is put on the secondary task in general. Furthermore, if the condition \( \mathcal{Q}_{ph} = \emptyset \), denoted by (NSBG0), is met, the conditions below follow (9):
\[ \text{rank}(W) = \text{rank}(J) + \text{rank}(H) \] (NSBG1)
\[ Y = J W^+ J^T = J J^+ \] (NSBG2)
\[ H W^+ H^T = H H^+ \] (NSBG3)
\[ J W^+ H^T = 0. \] (NSBG4)

Using the properties of (NSBG3) and (NSBG4), the secondary task residual error (11b) reduces to \( H \dot{q}^* = H H^+ \dot{h} \). \( \square \)

### 3.2 Semi-definitely weighted pseudoinverse
Recognize that the coefficient matrix of \( \dot{p} \) in the bordered Grammian solution (10) has a similar form as the usual weighted pseudoinverse. For example, if \( J \) has full rank, then \( Y^+ = Y^{-1} \) due to (BG1). Furthermore, if \( W \) has full-rank, \( W^+ = W^{-1} \). Then the equation reduces to
\[ \dot{q} = J W^+ \dot{p} + \left( I - J W^+ J \right) W^{-1} H^T \dot{h}. \]

where the weighted pseudoinverse, denoted by \( J W^+ \), is defined by
\[ J W^+ = W^{-1} J^T \left( J W^{-1} J T \right)^{-1}, W = W^T > 0. \] (12)

This observation is very useful in the sense that multiple tasks kinematics can be implemented simply by adopting the weighted pseudoinverse using the weight matrix \( W = J^T J + H^T H \). One problem arises if the full rank condition of the weight matrix \( W \) is not met, in particular as in the kinematically indeterminate case. We will show in this section that a generalized version of the weighted pseudoinverse can be defined even in the case where \( W \) does not have full rank.

A most natural candidate for generalization of the conventional weighted pseudoinverse (12) using a positive semi-definite weight matrix, is the one given by
\[ \dot{J} W^+ = \left( J W^+ J^T \right)^+, \quad W = W^T \geq 0 \] (13)

where the same notation \( \dot{J} W^+ \) is kept. The following proposition states that the matrix is indeed the weighted pseudoinverse.

**Proposition 1:** Given a positive semi-definite weight matrix \( W = J^T J + H^T H \), the matrix defined by (13) is the weighted-pseudoinverse of \( J \).

**Proof:** It suffices to show the matrix (13) satisfies all the properties of the weighted pseudoinverse (WP1), (WP2), (WP3) and (WP4), as shown in Appendix A. \( \square \)

Hence we call the matrix as the semi-definitely weighted pseudoinverse of \( J \). The semi-definitely weighted pseudoinverse defined above has the following property.

**Proposition 2:** The solution \( \dot{q} = J W^+ \dot{p} \) using the semi-definitely weighted pseudoinverse (13) is the minimum-length solution, i.e., \( \| \dot{q} \| \) is minimum, to minimizing \( \| H \dot{q} \| \) subject to the kinematic constraint (1). Also, the minimum value of \( \| H \dot{q} \| \) is
\[ \| H \dot{q} \|^2 = \dot{p}^T (Y^+ - I) \dot{p}. \]

**Proof:** By setting \( \dot{h} = 0 \), the solution (10) reduces to the following problem: minimize \( \dot{q}^T H^T H \dot{q} \) subject to \( \dot{p} = J \dot{q} \). The minimum length condition trivially holds. The minimum value can be easily obtained by substituting the solution. \( \square \)

### 4 Geometrical Analysis of Multiple Task Kinematics

#### 4.1 Joint space decomposition

Generally the joint space of a redundant manipulator governed by multiple tasks, say two tasks defined by (1) and (2), can be decomposed into four subspaces [10], i.e.,
\[ \dot{q} = \dot{q}_{ph} + \dot{q}_{pn} + \dot{q}_{nh} + \dot{q}_{nn} \] (14)

where
\[ \dot{q}_{nn} \in \mathcal{Q}_{nn} = \ker(J) \cap \ker(H) \]
\[ \dot{q}_{nh} \in \mathcal{Q}_{nh} = \ker(J) \cap \ker(H)^\perp \]
\[ \dot{q}_{pn} \in \mathcal{Q}_{pn} = \ker(J)^\perp \cap \ker(H) \]
\[ \dot{q}_{ph} \in \mathcal{Q}_{ph} = \ker(J)^\perp \cap \ker(H)^\perp, \]

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where the superscript $\perp$ denotes the orthogonal complement of a subspace.

It is of help to understand actual decomposition of the joint velocity to each subspace. The general solution (10) to multiple tasks kinematics can be rewritten, using the generalized notion of weighted pseudoinverse, as

$$\dot{q} = J^W+\dot{p} + \left\{I - J^W+J\right\} W^+H^T\dot{h} + \left\{I - W^+W\right\} z,$$

(15)

where $J^W+$ may denote semi-definitely weighted pseudoinverse or usual one depending on the dimensions. The situation can be classified at large by the condition whether $Q_{ph}$ exists or not, because the subspace indicates incompatibility between two tasks. If $Q_{ph} \neq \emptyset$, the solution can be decomposed of the form

$$\dot{q} = H^W+\dot{p} + \left\{I - H^W+H\right\} J^W+\dot{p} + \left\{I - J^W+J\right\} W^+H^T\dot{h} + \left\{I - W^+W\right\} z.$$

First noting that the last two terms in the right-hand side belongs to $\text{Ker}(J)$ and the last term $\dot{q}_{nn} = \left\{I - W^+W\right\} z \in Q_{nn}$, we can conclude that $\dot{q}_{nh} = \left\{I - J^W+J\right\} W^+H^T\dot{h}$ is the component belonging to $Q_{nh}$. Second, as the first two terms is decomposition of a joint velocity belonging to $\text{Im}(J^W+)$, there follows $\dot{q}_{pn} = \left\{I - H^W+H\right\} J^W+\dot{p} \in Q_{pn}$ and $\dot{q}_{ph} = H^W+HJ^W+\dot{p} \in Q_{ph}$. Note that the joint velocity $\dot{q}_{ph}$ induces the secondary task error due to task incompatibility.

Remark 1: If $Q_{ph} = \emptyset$, then the property (NSBG4) simplifies the solution (15) to

$$\dot{q} = W^+J^T\dot{p} + W^+H^T\dot{h} + \left\{I - W^+W\right\} z.$$

(16)

Therefore, we can conclude that $\dot{q}_{nn} = \left\{I - W^+W\right\} z \in Q_{nn}$, $\dot{q}_{nh} = W^+H^T\dot{h} \in Q_{nh}$, and $\dot{q}_{pn} = W^+J^T\dot{p} \in Q_{pn}$.

It can be shown that each subspace dimension is given by

$$\dim Q_{nn} = n - \gamma$$
$$\dim Q_{nh} = -l + \gamma + \dim \text{Ker}(H^T)$$
$$\dim Q_{pn} = -m + \gamma + \dim \text{Ker}(J^T)$$

$$\dim Q_{ph} = m + l - \gamma - \dim \text{Ker}(J^T) - \dim \text{Ker}(H^T).$$

where $\gamma = \text{rank}(J)$. Noting that $\dim \text{Ker}(J^T) + \dim \text{Ker}(H) = m$, we have $\dim \text{Ker}(J^T) = m - \alpha$, where $\alpha = \text{rank}(J)$. Similarly, $\dim \text{Ker}(H^T) = l - \beta$, where $\beta = \text{rank}(H)$.

4.2 Geometrical interpretation of the weight matrix

Let us assume that the dimension of the secondary task is equal to the degrees of redundancy. Consider the following conventional TPBM algorithms by [2] and [3].

$$\dot{q} = J^+\dot{p} + H^+\left\{\dot{h} - HJ^+\dot{p}\right\}$$

(17)

$$\dot{q} = J^+\dot{p} + \left\{I - J^+J\right\} H^+\dot{h},$$

(18)

where $H = H \left\{I - J^+J\right\}$.

Careful geometrical analysis of both methods is shown in Fig. 3. The figure shows geometrically that (17) can execute the secondary task exactly.

Indeed we can eliminate the unnecessary residual error of the method (18), shown in Fig. 3 (b), by taking a suitable weighted pseudoinverse instead of the pseudoinverse. The method (18) can be modified to

$$\dot{q} = J^W+\dot{p} + \left\{I - J^W+J\right\} H^+\dot{h},$$

(19)

with which the residual error is given by

$$\dot{h} - H\dot{q} = (I - HH^+)\dot{h} + HJ^W+\left(p - JH^+\dot{h}\right).$$

For an arbitrary weight matrix, the solution and its residual error is depicted in Fig. 4 (a). To nullify the residual error, the algebraic condition below has to be met

$$HW^{-1}J^T \equiv 0$$

(20)

which is corresponding to (NSBG4). Therefore, if $Q_{ph} = \text{Im}(J^T) \cap \text{Im}(J^T) = \emptyset$, the weight matrix given by

$$W = J^TJ + H^TH$$

(21)

eliminates the error, which is equal to the weight matrix (7a). The condition stated by (20) yields the weight matrix which rotates the range space of $J^T$ as shown in Fig. 4 (b).
5 Formulation using Kinematically Decoupled Joint Space Decomposition

In this section, we show that the weighted pseudoinverse solution to the multiple tasks kinematics can be implemented by the inverse or semi-definitely weighted inverse of a specific single matrix. The method is called the kinematically decoupled (KD) TPBM.

The Lagrange multiplier $\lambda \in \mathbb{R}^m$ in (3) can be eliminated exploiting a null space matrix $Z \in \mathbb{R}^{n \times n}$ which is a full row-rank matrix satisfying $JZ \equiv 0$. Then by multiplying $Z$ to (3) there follows

$$ZH^TH\ddot{q} = ZH^T\dot{h}.$$  \hspace{1cm} (22)

By augmenting the $r$-dimensional equation to the $m$-dimensional equality constraint (4), the solution to the problem is obtained by solving the square linear equation

$$\begin{bmatrix} J \\ ZH^T \end{bmatrix} \ddot{q} = \begin{bmatrix} \dot{p} \\ ZH^T \dot{h} \end{bmatrix}.$$  

In particular the equation has the similar form as the kinematically decoupled joint space decomposition (KD-JSD) \cite{11}, except that the weight matrix is positive semi-definite if $l < n$. We can alleviate the discrepancy by modifying the necessary condition (22) to the trivially equivalent one

$$Z \begin{bmatrix} J^TJ + H^TH \end{bmatrix} \ddot{q} = ZH^T\dot{h}$$

making use of the fact that $ZJ^T = 0$. Then by regarding $W = J^TJ + H^TH$ the total kinematic system corresponding to the TPBM can be written as the similar form as the KD-JSD as follows

$$\begin{bmatrix} J \\ ZW \end{bmatrix} \ddot{q} = \begin{bmatrix} \dot{p} \\ ZH^T \dot{h} \end{bmatrix}.$$  \hspace{1cm} (23)

In particular, we will choose the matrix $Z$ such that it satisfies \cite{6}

$$JZ^T = 0, \quad I - JW + J = ZZ^T.$$  \hspace{1cm} (24)

Especially note that the weight matrix is the same as the matrix $W$ of (21) in the weighted pseudo-inverse solution of the TPBM.

5.1 Positive-Definitely Weighted KD-TPBM

If $W$ has full rank $n$, then $W > 0$ and the solution to (23) is obtained as

$$\ddot{q} = \begin{bmatrix} J \\ ZW \end{bmatrix}^{-1} \begin{bmatrix} \dot{p} \\ ZH^T \dot{h} \end{bmatrix}.$$  \hspace{1cm} (25)

This equation is called the positive-definitely weighted KD-TPBM hereinafter, since the weight matrix $W$ is positive definite. We can show that this solution is equivalent to the weighted pseudoinverse solution

$$\ddot{q} = JW + \dot{p} + \left\{ I - JW + J \right\}W^{-1}H^T\dot{h}, \quad W > 0$$  \hspace{1cm} (26)

which is the special form of (15), using the properties of the KD-JSD. It was shown \cite{6} that

$$\begin{bmatrix} J \\ ZW \end{bmatrix}^{-1} = [JW + ZW^#], \quad W > 0$$  \hspace{1cm} (27)
for the weighted pseudo-inverse \( J^{W+} = W^{-1}J^T(JW^{-1}J^T) \)
and the so-called unsymmetric weighted generalized inverse
\( Z^{W#} = Z^T(ZWZ^T)^{-1} \). Then the solution can be compactly given by
\[
\dot{q} = J^{W+}\ddot{p} + Z^{W#}ZH^T\dot{h}.
\] (28)

For the complete proof of the KD-JSD, refer to [12]. The unsymmetric weighted generalized inverse \( Z^{W#} \) captures the asymmetry existing in the null space projection operator associated with the weighted pseudo-inverse \( J^{W+} \), as the following property holds
\[
Z^{W#}ZW = I - J^{W+}J.
\] (29)

Using this property, the positive-definitely weighted KD-TPBM (25) is equivalent to the positive definite weighted pseudoinverse solution (26).

5.2 Semi-Definitely Weighted KD-TPBM

Now, we are to formulate the semi-definitely weighted KD-TPBM employing only a single inversion of the matrix, just as the definitely weighted KD-TPBM can be formulated using the inverse of the matrix \( \begin{bmatrix} J \\ ZW \end{bmatrix} \) as in (25). However, since the weight matrix is not of full rank, the inversion is impossible. To generalize the conventional KD-JSD to the positive semi-definite case, firstly we generalize the definition of the unsymmetric weighted generalized inverse \( Z^{W#} \) as
\[
Z^{W#} = Z^T\left\{ZWZ^T\right\}^+, \quad W \geq 0.
\] (30)

Then one can show that the following property holds
\[
\left\{I - J^{W+}J\right\}W^+W = W^+WZ^{W#}ZW.
\] (31)

Using the property the general weighted pseudo-inverse solution (15) is written as
\[
\dot{q} = J^{W+}\ddot{p} + W^+WZ^{W#}ZW + H^T\dot{h} + \left\{I - W^+W\right\}z
\]
\[
= J^{W+}\ddot{p} + W^+WZ^{W#}ZH^T\dot{h} + \left\{I - W^+W\right\}z
\] (32)

using the property (BG2).

Next, consider the semi-definitely weighted pseudoinverse of the matrix \( \begin{bmatrix} J \\ ZW \end{bmatrix} \) for the semi-definite weight matrix \( W \). Using the definition (13) it can be shown that
\[
\begin{bmatrix} J \\ ZW \end{bmatrix}^{W+} = [J^{W+} W^+WZ^{W#}], \quad W \geq 0
\] (33)

Using the semi-definitely weighted pseudoinverse, (23) can be solved as
\[
\dot{q} = \begin{bmatrix} J \\ ZW \end{bmatrix}^{W+} \left( \begin{bmatrix} \dot{p} \\ ZH^T\dot{h} \end{bmatrix} \right) + \left\{I - \begin{bmatrix} J \\ ZW \end{bmatrix}^{W+} \begin{bmatrix} J \\ ZW \end{bmatrix}\right\}z
\]
\[
= J^{W+}\ddot{p} + W^+WZ^{W#}ZH^T\dot{h}
\]
\[
+ \left\{I - J^{W+}J - W^+WZ^{W#}ZW\right\}z.
\]

Thanks to the property (31) we have
\[
I - J^{W+}J - W^+WZ^{W#}ZW = \left\{I - J^{W+}J\right\}\left\{I - W^+W\right\} = I - W^+W.
\]

Therefore, the semi-definitely weighted KD-TPBM is given by
\[
\dot{q} = \begin{bmatrix} J \\ ZW \end{bmatrix}^{W+} \left( \begin{bmatrix} \dot{p} \\ ZH^T\dot{h} \end{bmatrix} \right) + \left\{I - W^+W\right\}z.
\] (34)

Lemma 2: The KD-TPBM (25) and (34), or each equivalent expressions given above, impose strict priorities among all three tasks \( \dot{p}, \dot{h}, \) and \( z \). A consistent primary task is always executed exactly. If \( Q_{ph} \) is empty, then the secondary task is also executed exactly.

Proof: The KD-TPBM (32) is equivalent to
\[
\dot{q} = J^{W+}\ddot{p} + \left\{I - J^{W+}J\right\}\left\{H^+\dot{h} + \left\{I - W^+W\right\}z\right\}.
\]

which discriminates the priorities among three tasks. Computing both tasks resulting from the solution
\[
J\dot{q} = JJ^{W+}\ddot{p} = YY^+\ddot{p} = JJ^+\ddot{p}
\]
\[
ZW\dot{q} \equiv ZWZW^{W#}ZH^T\dot{h} = ZH^T\dot{h}.
\]

The secondary task reduces to \( ZH^T\dot{h} = ZH^T\dot{h} \), and we can see that \( H\dot{q} - \dot{h} \in \text{Ker}\{ZH^T\} \). It suffices to show that \( ZH^T \) has full column-rank, if \( Q_{ph} \) is empty, which is indeed the case.

6 Concluding Remarks

In this paper, we formulated the kinematics accommodating multiple tasks in the sense of task priority using the weighted pseudo-inverse solution. The solution is more favorable than the conventional TPBM solutions from the analytic and geometric viewpoint. By formulating the solution using the KD-JSD we can efficiently cope with general situations of multiple task kinematics. However, the design of algorithm for multiple kinematics can not be complete without considering various singularity situations. As a matter of fact, a modified version of conventional damped least square inverse method proved capable of smoothing joint trajectory at or near such singularities [12], which is not listed in the current article due to space limitation.

Appendix

A Proof of the Semi-Definitely Weighted Pseudoinverse \( J^{W+} \)

A matrix \( X \) is called the \( W \)-weighted pseudoinverse of \( A \) if it satisfies
\begin{align*}
(WP1) \quad AXA &= A \\
(WP2) \quad XAX &= X \\
(WP3) \quad AX &\text{ is symmetric} \\
(WP4) \quad WXAX &= X
\end{align*}

We have to show that the semi-definitely weighted pseudoinverse \( J^{W+} \) defined by (13) for \( W \) of (21) satisfies
the same properties of the weighted pseudoinverse defined above. Recalling the definition of $Y$ in (7b), note that $J^{W+} = W^+ J^T Y^+$. Then (WP1) and (WP2) hold using the properties of (BG1), (BG2), and (BG3), because

$$JJ^{W+} = JW^+ J^T Y^+ J = YY^+ J = JJ^+ J = J,$$

$$J^{W+} J^{W+} = W^+ J^T Y^+ JW^+ J^T Y^+ = W^+ J^T Y^{++} + Y Y^+$$

$$= W^+ J^T Y^+ = J^{W+}.$$

Since $JJ^{W+} = YY^+$, (WP3) is satisfied. Finally since

$$WJ^{W+} = WW^+ J^T Y^+ J = J^T Y^+ J$$

it is symmetric, i.e.

$$\left(WJ^{W+}\right)^T = WJ^T Y^+ J = WJ^{W+} J,$$

which satisfies (WP4). It is worth commenting that this semi-definitely weighted pseudoinverse is valid even in the case where (NSBG0) does not hold, since only the generic properties of (BG1), (BG2), and (BG3) are used in establishing.

**References**


