

The eigenvalues of an  $n \times n$  complex matrix  $A$  are the roots of the  $n^{\text{th}}$  degree polynomial  $\det(A - \lambda I) = 0$ . Further the positive square roots of the eigenvalues of the positive semidefinite matrix  $A^*A$  are defined as singular values of  $A$ . Our main interest is in matrix inequalities especially in those involving eigenvalues, singular values, determinants or norms. These are difficult to compute. It is however often useful to know the approximate location of the eigenvalues/singular values. For example, when  $A$  is nonsingular, the ratio of the largest to the smallest singular values - called condition number, is useful in determining whether the equation  $Ax = b$  is ill-conditioned or not. Also when  $A$  is Hermitian positive semidefinite, singular values are themselves the eigenvalues of  $A$ . In theory of stability of solutions to differential equations bounds for real parts of eigenvalues and other eigenvalue inequalities (when we have time delay(s)) play an important role [14,29,37]. Also  $\sum_{k=0}^{\infty} \beta_k T^k$  converges if and only if all eigenvalues of  $T$  lie inside the circle of convergence of scalar series  $\sum_{k=0}^{\infty} \beta_k z^k$ . Furthermore to obtain matrix inequalities, given certain pieces of information about  $A$  is an important problem. For example, when we know diagonal elements of  $A$  and sum of absolute values of off-diagonal elements of each row we get Gerschgorin theorem [32].

We will first obtain results for non-negative matrices. We recall that a real matrix  $A$  is totally positive (nonnegative) if all minors of  $A$  are positive (nonnegative). Such matrices have become increasingly important in approximation theory [25] and

have many interesting properties. For example, a totally positive (nonnegative) matrix has distinct positive eigenvalues (nonnegative and may not be distinct). Also if  $A = (a_{ij})$  is a totally nonnegative  $n \times n$  matrix, then  $\det A \leq \prod_{i=1}^n a_{ii}$ . These and other related matrices (e.g. oscillatory matrices) are areas of active research [1,8,12,15,16,23,24,25,30,39,40,46,49].

We will investigate these matrices, for new matrix inequalities and will study their properties and factorizations. We will employ theory of majorization, properties of Schur concave/convex functions and determinantal inequalities [7,42] in our study.

Next we will study (0,1) normal matrices [34]. Here row and column sum vectors are same. We will find new inequalities involving spectral radius, eigenvalues and determinants for such matrices. A very important subclass of (0,1) normal matrices is the set of adjacency matrices of simple undirected graphs. Such matrices are symmetric and non-negative. Thus they have real eigenvalues. Furthermore, the largest eigenvalue is nonnegative and is called index. We will find bounds for index, other eigenvalues and spread (maximum distance between eigenvalues) [4,9,10,11,13,17,18,26,38,47,55]. These results give bounds for chromatic number[48]. We also expect to obtain improved eigenvalue bounds for regular and bipartite graphs [13,58].

As noted in [3] an interesting quantity in Hückel theory is the sum of the energies of all the electrons in a molecule, the so-

called total  $\pi$ -electron energy  $E_\pi$ . For a molecule with  $n = 2k$  atoms, the total  $\pi$  – electron energy can be shown to be  $E_\pi = 2 \sum_{i=1}^k \lambda_i$ , where  $\lambda_i$ ,  $i=1,2,\dots,k$  are the top  $k$  eigenvalues of the adjacency matrix of the graph, because of the symmetry of the spectrum we can write  $E_\pi = \sum_{i=1}^k |\lambda_i|$ . Motivated by this we define the energy of a graph as  $\sum_{i=1}^k |\lambda_i|$ , where  $\lambda_1, \dots, \lambda_n$  are eigenvalues of the adjacency matrix. We shall also consider this sum in our study [2,4,45 and references their in].

Also, when  $G$  is a graph of  $n$  vertices, its Laplacian matrix is the  $n \times n$  matrix  $L(G) = D(G) - A(G)$ , where  $A(G)$  is the familiar  $(0,1)$  adjacency matrix and  $D(G)$  is the diagonal matrix of vertex degrees. Such matrices arise in the study of numerical solution of Wave equation of the type  $Z_{xx} + Z_{yy} = -kZ/c^2$  [19, 27, 43, 44, 50,54]. We note that the second smallest eigenvalue of Laplacian is positive if and only if the graph is connected. This eigenvalue known as algebraic connectivity will also be investigated.

For general non-negative matrices, we shall study the Schur complement and generalized Schur complement, to obtain bounds for the largest eigenvalue [5, 41, 61].

We will use perturbation theory for eigenvalues [6, 56] to find bounds for extreme eigenvalues/singular values. We will also study bounds for eigenvalues of complex or real matrices with real eigenvalues[35,51,52,53,59]. More generally we will obtain matrix inequalities for non-normal matrices. First we shall find bounds for  $\|AA^* - A^*A\|$  and then obtain new

eigenvalue bounds involving trace of  $A$ , its powers,  $A^*A$ , bounds for  $\|AA^*-A^*A\|$  and/or its rank [20, 21, 22, 28, 31, 36, 57, 60]. We will obtain bounds for linear combinations of eigenvalues and magnitude of eigenvalues [59,60]. We shall also consider positive definite matrices and obtain bounds for smallest eigenvalue and condition number.

Furthermore, we shall obtain new matrix inequalities for singular values, real singular values (eigenvalues of the Hermitian matrix  $\frac{1}{2} (A+A^*)$ ) and imaginary singular values (eigenvalues of

$\frac{1}{2i} (A-A^*)$ ) and determinant [33,59]. Bounds for real and imaginary parts of eigenvalues/singular values will follow. We will also study easy to compute bounds involving row sums and diagonal elements. We will also employ MATLAB to run computer experiments on our bounds and compare them with known bounds.

## **REFERENCES**

- [1] T. Ando, Totally Positive Matrices, Linear Algebra and its Applications, 90: 165-219 (1987).
- [2] R. Balakrishanan, The energy of a graph, Linear Algebra and its Applications, 387: 287- 295 (2004).
- [3] R.B. Bapat, On some matrices associated with a graph, preprint.
- [4] R.B. Bapat, Graphs and matrices, Hindustan Book Agency Delhi (2010).
- [5] A. Berman and R.J. Plemmons, Nonnegative Matrices in Mathematical Sciences, Academic Press(1979).
- [6] R. Bhatia, Perturbation Bounds for Matrix Eigenvalues, Longman, (1987).
- [7] R. Bhatia, Matrix Analysis, Springer Verlag(1997).
- [8] R. Bhatia, Positive Definite Matrices, Hindustan Book Agency (2007).
- [9] N. Biggs, Algebraic Graph Theory, Cambridge University Press (1993).
- [10] R.A. Brualdi, Matrices, Eigenvalues and Directed Graphs, Linear and Multilinear Algebra, 11: 143-165 (1982).
- [11] R.A. Brualdi, Spectra of Digraphs, Linear Algebra and its Applications, 432: 2181-2213 (2010).
- [12] R.A. Brualdi and S. Kirkland, Totally Nonnegative  $(0,1)$ , Matrices, Linear Algebra and its Applications, 432: 1650-1662 (2010).

- [13] A. Bhattacharya, et al., On the first Eigenvalue of Bipartite graphs, *The Electronic Journal of Combinatorics*, 15: 1-23 (2008).
- [14] S.A. Campbell, Delay Independent Stability for Additive Neural Networks, *Differential Equations and Dynamical Systems*, 9: 115-138 (2001).
- [15] R. Canto et al., Characterizations of rectangular totally and strictly positive matrices, *Linear Algebra and its Applications*, 432: 2623-2633 (2010).
- [16] C.W. Cryer, Some Properties of totally positive matrices, 15: 1-25 (1976).
- [17] D.M. Cvetrovic, *Spectra of Graphs*, Academic Press(1979).
- [18] D.M. Cvetrovic and P. Rowlinson, The largest eigenvalue of a Graph, *Linear and Multilinear Algebra*, 28: 3-33 (1990).
- [19] K. Ch. Das, An improved upper bound for Laplacian Graph Eigenvalues, *Linear Algebra and its Applications*, 368: 269-278 (2003).
- [20] N.A. Derzko and A.M. Pfaffer, Bounds for the Spectral Radius of a Matrix, *Math. Computation*, 99: 62-67 (1965).
- [21] P.J. Eberlein, On Measures of Nonnormality of Matrices, *Amer. Math. Monthly*: 995-996 (1965).
- [22] L. Elsner and M.H.C. Paardekooper, On Measures of Nonnormality of Matrices, *Linear Algebra and its Applications*, 92: 107-124 (1987).
- [23] S.M. Fallet and C.R. Johnson, Hadmard Duals, Retractability and Oppenheim's inequality, preprint.

- [24] J. Garloff, Majorization between the diagonal elements and the eigenvalues of an oscillating matrix, *Linear Algebra and its Applications*, 47: 181-184 (1982).
- [25] M. Gasca and J.M. Pena, Total Positivity and Neville Elimination, *Linear Algebra and its Applications*, 165: 25-44 (1992).
- [26] D.A. Gregory, et al. , The Spread of the spectrum of a graph, *Linear Algebra and its Applications*, 332 -334: 23-35 (2001).
- [27] W.H. Haemers, et al., On the sum of Laplacian Eigenvalues of graphs, *Linear Algebra and its Applications*, 432 : 2214-2221 (2010).
- [28] P. Henrici, Bounds for iterates, spectral variation and fields of values of non- normal Matrices, *Numer. Math.*, 4: 24-40 (1962).
- [29] J. Hofbauer and J.W.-H So, Diagonal dominance and Harmless Delays, *Proc. Amer Math. Soc.*, 128: 2675-2682 (2000).
- [30] O. Holtz and H. Schneider, Open Problems on GKK- Matrices, *Linear Algebra and its Applications*, 345: 263-267 (2002).
- [31] T. Huang, et al., Estimates for bounds of some numerical characters of Matrices, *Linear Algebra and its Applications*, 319: 137-145 (2000).
- [32] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press(1985).
- [33] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press(1991).
- [34] S.K. Jain and L.E. Snyder, Nonnegative Normal Matrices, *Linear Algebra and its Applications*, 182: 147-155 (1993).
- [35] S.T. Jensen and G.P.H. Styan, Some comments and a bibliography on the Laguerre – Samuelson inequality with extensions and applications

in Statistics and matrix theory, in Analytic and Geometric Inequalities and their Appl., H. M. Srivastava and T. M. Rassias eds., Kulwer Acad. Publ.(1999).

[36] R. Kress and H.D. Devries, On Nonnormal matrices, Linear Algebra and its Applications, 8: 109-120 (1974).

[37] R. Kumar, Harmless Delays in Delay Differential Equations and Dynamical Systems,14: 91-103 (2006).

[38] R. Kumar, Bounds for eigenvalues of a graph, Journal of Mathematical Inequalities, 4: 399-404 (2010).

[39] C.K. Li and R. Mathias, Interlacing inequalities for totally non-negative matrices, Linear Algebra and its Applications, 341: 35-44 (2002).

[40] X.P.Liu and S.M. Fallat, Multiplicative principal- Minor inequalities for a class of oscillatory matrices, JIPAM, 9: 1-18 (2008).

[41] L. Lu and M.K. Ng, Localization of Perron roots, Linear Algebra and its Applications, 392: 103-117 (2004).

[42] A.W.Marshall and I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, (1979).

[43] R. Merris, Laplacian Matrices of Graphs: A Survey, Linear Algebra and its Applications, 197: 143-176 (1994).

[44] B. Mohar, Laplace eigenvalues of graphs, A Survey, Discrete Math., 109: 171-183 (1992).

[45] B. Mohar, On the sum of  $k$  largest eigenvalues of graphs and symmetric Matrices, Journal of Combinatorial Theory, Series B, 99: 306-313 (2009).



- [46] M.G. Neubauer, An Inequality for positive definite matrices and application in Combinatorial Matrices, *Linear Algebra and its Applications*, 267: 163-174 (1997).
- [47] V. Nikiforov, Some Inequalities for the largest eigenvalue of a graph, *Combinatorics probability and Computing*, 11: 179-189 (2002).
- [48] V. Nikiforov, Chromatic number and spectral radius, *Linear Algebra and its Applications*, 426: 810-814 (2007).
- [49] A. Pinkus, Spectral properties of totally positive kernels and Matrices, In *total Positivity and its application*, 1-35 (1995).
- [50] O. Rojo, et al., An always non-trivial upper bound for Laplacian graph eigenvalues *Linear Algebra and its Applications*, 312: 155-159 (2000).
- [51] R. Sharma, Some more inequalities for arithmetic Mean, Harmonic Mean and Variance, *Journal of Mathematical Inequalities*, 2: 109-114 (2008).
- [52] R. Sharma, et al., Some Bounds on sample parameters with refinements of Samuelson and Brunk inequalities, *Journal of Mathematical Inequalities*, 13:99-106 (2009).
- [53] R. Sharma, et al., Some better bounds on the Variance with application, *Journal of Mathematical Inequalities*, 4: 355-363 (2010) .
- [54] J.L. Shu, et al., A Sharp upper bound on the largest eigenvalue of the Laplacian matrix of a graph, *Linear Algebra and its Applications*, 347: 123-129 (2002).
- [55] J. L. Shu and Y. Wu, Sharp upper bounds on the spectral radius of graphs, *Linear Algebra and its Applications*, 377: 241-248 (2004).

- [56] G. W. Stewart and Ji –guang Sun, *Matrix Perturbation Theory*, Acad. Press (1990).
- [57] L. Wang, et al., Some lower bounds for the spectral radius of matrices using traces, *Linear Algebra and its Applications*, 432: 1007-1016 (2010).
- [58] D.G.West, *Graph Theory*, Prentice Hall (2003).
- [59] H. Wolkowicz and G.P.H. Styan, Bounds for eigenvalues using traces, *Linear Algebra and its Applications*, 29: 471-506 (1980).
- [60] H. Wolkowicz and G.P.H. Styan, More bounds for eigenvalues using traces *Linear Algebra and its Applications*, 31: 1-17 (1980).
- [61] Y. Z. Zong, Schur Complements and determinant Inequalities, *Journal of Mathematical Inequalities*, 3: 161-167 (2009).