LETTER

Basic Bifurcation of Artificial Spiking Neurons with Triangular Base Signal

Toshimitsu OHTANI†, Student Member and Toshimichi SAITO‡, Member

SUMMARY This paper studies a spiking neuron circuit with triangular base signal. The circuit can output rich spike-trains and the dynamics can be analyzed using a one-dimensional piecewise linear map. This system exhibits period doubling bifurcation, tangent bifurcation, super-stable periodic orbit bifurcation and so on. These phenomena can be characterized based on the inter-spike intervals. Using the maps, we can analyze the phenomena precisely. By presenting a simple test circuit, typical phenomena are confirmed experimentally.

key words: spiking neurons, bifurcation, chaos

1. Introduction

The spiking neurons (SKN) are known as artificial neuron models having rich dynamics [1],[2]. Repeating integrate-and-fire behavior between threshold and base signals the SKN can output a variety of periodic/chaotic spike-trains. The SKNs can be a building block of Pulse-coupled neural networks (PCNN) having rich synchronization phenomena [3]–[5]. The SKN and PCNN have many potential applications including spike-based A/D converters [6], spike-based communications [2] [7], [8], associative memories [9] and image processing [10]. The SKN is a kind of switched dynamical systems [11] and analysis of bifurcation is important from both fundamental and application viewpoints.

This paper studies the dynamics of a SKN with a triangular base signal [12]. Since the base signal is piecewise linear (PWL), the dynamics can be analyzed using a 1D PWL map of spike-phase. The SKN has rich chaotic, periodic and superstable spike-trains. The superstable behavior corresponds to superstable periodic orbit (SSPO) [13] of the map that can not appear in smooth 1D maps. We consider the dynamics based on inter-spike interval (ISI). This system exhibits period doubling bifurcation, tangent bifurcation, SSPO bifurcation and so on. We consider bifurcation sets of these phenomena and their intersection (co-dimension two bifurcation set). In the intersection of tangent and SSPO bifurcation sets, we have confirmed an interesting change of the ISI density (IID). It may provide basic information for applications based on spike-trains. By presenting a simple test circuit, typical phenomena are confirmed experimentally.

Motivations for studying the SKN include three points.

First, precise analysis is possible for the PWL map. SKNs with sinusoidal base signal [1] have been studied, however, the maps are smooth and precise analysis is hard. Second, bifurcation of superstable phenomena relates deeply to various models, e.g., flat-top tent maps with relation to optimal limiter control [13]. Also, transient to SSPOs is very fast and analysis of SSPOs is fundamental to construct robust and fast system. Third, bifurcation of IID is basic to understanding information processing in spiking neurons and to consider applications. Preliminary results of the bifurcation can be found in [14].

In our previous paper [12], SKN with triangular base is presented and its basic dynamics is analyzed; however, the ISI is not discussed, co-dimension two bifurcation is not discussed and bifurcation analysis of single SKN is not sufficient.

2. Artificial Spiking Neuron and Phase Map

Figure 1 shows dynamics of the SKN where \( b(\tau) \) is the triangular base signal with period 1:

\[
\begin{align*}
\tau &< 1 & \Rightarrow & b(\tau + 1) = b(\tau) \\
1 &\leq \tau & \Rightarrow & b(\tau + 1) = b(\tau + 1)
\end{align*}
\]

The state variable \( x \) rises and when it reaches the threshold, it jumps to the base \( b(\tau) \) and SKN outputs a spike \( y = 1 \).

\[
\begin{align*}
\dot{x} &= s, \quad y(\tau) = 0 \quad \text{for } x < 0 \\
x(\tau^+) &= b(\tau^+), \quad y(\tau^+) = 1 \quad \text{if } x(\tau) \geq 0
\end{align*}
\]

where \( \dot{x} \equiv \frac{dx}{dt} \). This system has two parameters \( s \) and \( k \) that

![Fig. 1 Artificial spiking neuron with a triangular base signal.](image-url)

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The authors are with the Hosei University, Koganei-shi, 184-002 Japan.

a) E-mail: tsaito@hosei.ac.jp

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control $x$-slope and $b(\tau)$, respectively. Let $\tau_n$ denote $n$-th spike-position where $n$ is a positive integer.

For simplicity we assume

$$-2 < x(0) < 0, \ 0 < k < 4, \ 1 - k/4 < s$$

They guarantee $b(\tau) \leq x(\tau) \leq 0, -2 < b(\tau) < 0$ and $\tau_1 < 1$ for $x(0) = b(0)$. The nonlinear dynamics can be analyzed by a 1D map. Since $\tau_n$ determines $\tau_{n+1}$, we can define the 1D spike-position map $f$ from positive reals to itself:

$$\tau_{n+1} = f(\tau_n) = \tau_n - b(\tau_n)/s$$

where $f(\tau + 1) = f(\tau) + 1$ and

$$f(\tau) = \begin{cases} (1+k/s)\tau + (1-k/4)/s & \text{for } 0 \leq \tau < 1/2 \\ (1-k/s)\tau + (1+3k/4)/s & \text{for } 1/2 \leq \tau < 1 \end{cases}$$

Introducing phase variable $\theta_n \equiv \tau_n \mod 1$, the spike-position map is reduced to a phase map $F$ from $I \equiv [0, 1)$ to itself:

$$\theta_{n+1} = F(\theta_n) \equiv f(\theta_n) \mod 1$$

The phase map is illustrated with the spike-phase map in Fig. 2. That is, dynamics of the SKN is integrated into this 1D-map. Note that the phase map is PWL and has two slopes $1 \pm k/s$.

As preparation to consider the dynamics we introduce some definitions based on the phase map $F$.

**Definition 1:** A point $\theta_f \in I$ is said to be a 1-periodic point or fixed point if $F(\theta_f) = \tau_f$. A point $\theta_p \in I$ is said to be a $k$-periodic point if $F^k(\theta_p) = \theta_p$ and $F^k(\theta_p) \neq \theta_p$ for $1 \leq l < k$ where $F^k$ is the $k$-fold composition of $F$ and $k \geq 2$. A sequence of $k$-periodic points $(F(\theta_p), \ldots, F^k(\theta_p))$

is said to be a $k$-periodic orbit. A $k$-periodic point/orbit is said to be unstable, stable and super-stable for initial state if $|DF^k(\theta_p)| > 1$, $|DF^k(\theta_p)| < 1$ and $|DF^k(\theta_p)| = 0$, respectively where $DF^k$ denotes the derivative of $F^k$. Figure 3(a) shows super-stable fixed point.

**Definition 2:** If $F$ is onto Map and monotone increasing as shown in Fig. 2 then $F$ is equivalent to the rotation of the unit circle. The rotation can exhibit a variety of periodic or quasi-periodic orbit depending on parameter value [15].

**Definition 3:** $F$ is said to generate chaos if there exist some subset $J \subseteq I$ and positive integer $m$ such that $F^m(J) \subseteq J$ and $|DF^m(\theta)| > 1$ (expanding) for almost all $\theta \in J$. In this case $F$ is ergodic on $J$ and has positive Lyapunov exponent [15]. Figure 3(b) shows a chaotic phase map.

It goes without saying that periodic/chaotic orbits of the phase map $F$ corresponds to periodic/chaotic spike-trains of the SKN.

### 3. Bifurcation and ISIs

Here we consider an important bifurcation set that causes super-stability. The super-stable bifurcation set (SS) is defined by

$$SS: \{ (s, k) | s = k \}$$

On the SS, the map has zero-slope $(1 - k/s = 0)$ for $\theta > 1/2$ and exhibits super-stable periodic orbits (SSPO) as shown in Fig. 3(a). As mentioned in Sect. 1, the zero-slope is impossible for smooth maps [5] and transient to the SSPO is very fast. The SS is the border between two phases. For $s > k$, both slopes are positive and $F$ is monotone increasing: $F$ is equivalent to the rotation (Definition 2). For $s < k$, the slope is negative $(1 - k/s < 0)$ for $\theta > 1/2$. The map extrema and can exhibit chaos (Definition 3). If $2s < k$ then $|1 + k/s| > |1 - k/s| > 1$ is satisfied and chaos generation is guaranteed theoretically (Definition 3).

Let us assume existence of a fixed point as shown in Fig. 3(a): $\theta_f = F(\theta_f)$ for $1/2 \leq \theta_f < 1$. For this fixed point, the tangent bifurcation set (TB) and period-doubling bifurcation set (PD) are defined by

$$TB \text{ for } \theta_f: \{(s, k) | F(1/2) = -k/s - 1/2 = 1/2 \}$$

$$PD \text{ for } \theta_f: \{(s, k) | DF(\theta_f) = 1 - k/s = -1 \}$$

The TB and PD are illustrated together with the SS in Fig. 4. Based on the SS, TB and PD, we consider the dynamics of the phase map with characteristics for the ISI defined by

$$\Delta \tau_n \equiv \tau_{n+1} - \tau_n$$

The ISI can be calculated by 1D map $f$. Note that the range of ISI is restricted by parameters: if $x$ is reset to the peak of base at $\tau = m$, the ISI is the minimum $\Delta \tau_{\text{min}} = (4 - k)/4s$ and if $x$ is reset to the valley of base at $\tau = m + 0.5$, the ISI is maximum $\Delta \tau_{\text{max}} = (4 + k)/4s$. That is, $\Delta \tau_{\text{min}} < \Delta \tau_n < \Delta \tau_{\text{max}}$. Let us consider the effect of the TB and SS for the ISI distribution. The SS is the border of the following.

1. Let $s > k$. As $s$ increases, the stable periodic orbit
Fig. 4  Basic bifurcation sets. SS: Superstable bifurcation set, TB: Tangent bifurcation set, PD: Period doubling bifurcation set.

Fig. 5  Tangent bifurcation and IID distribution. (a) SPO to torus for $k = 0.79$, (b) SPO to chaos for $k = 3.2$.

Fig. 6  Super-stable orbits. (a) Bifurcation for $s = k$, (b) 2-SSPO for $s = 2.0, k = 2.0$, (c) 3-SSPO for $s = 3.0, k = 3.0$.

(SPO) is changed into torus via the TB as shown in Fig. 5(a). The IID has one peak for the SPO and the IID distributes near the peak for the torus: fundamental IID corresponding to the peak is preserved.

(II) Let $s < k$. As $s$ increases, the SPO is changed into chaos via the TB as shown in Fig. 5(b). The IID has one peak for the SPO and the peak disappears in the IID distribution for chaos: wide band distribution. As $s$ decreases, the SPO is changed as into chaos via the PD as shown in Fig. 3(b).

Note that the intersection of SS and TB causes co-

dimension two bifurcation that is the border of not only between (I) and (II) characterized by disappearance of fundamental ISI but also between SPO and torus/chaos characterized by disappearance of the fixed point. Such intersection cannot exist in 1D maps without a flat segment.

Let us consider bifurcation of SSPOs for $s = k$. For $s = k$ the two slopes are fixed (2 and 0) and the intercept $f(0) = k/2 - 1/4$ varies from 0 to $\infty$. Since phase map $F$ is to be the same shape for all $s$ that gives the same value of $F(0) = f(0) \mod 1$, it is sufficient to consider the case $0 < f(0) < 1$ that is equivalent to $4/5 < s < 4$. In Fig. 6 we can see relatively simple SSPOs and complicated SSPOs which are sensitive to parameter. For $4/5 < s < 4/3$, a 1-SSPO exists ($F(\theta_p) = \theta_p$, Fig. 3(a)). In the borders $s = 4/5$ and $s = 4/3$, the map satisfies $F(0) = 1/s - 1/4 = 0$ and $F(0.5) = 1/s + 3/4 = 0.5$, respectively. For $12/7 < s < 12/5$, a 2-SSPO exists ($F^2(\theta_p) = \theta_p$). In the borders $s = 12/7$ and $s = 12/5$, the map satisfies $F^2(0) = 2(1/s - 1/4) + 1/s - 1/4 = 0$ and $F^2(0.5) = 2(1/s + 3/4) + 1/s - 1/4 = 0.5$, respectively. In general, $k$-SSPO exists for $s < s_k$ where the borders $s_1$ and $s_2$ are roots of equations $F^k(0) = 0$ and $F^k(0.5) = 0.5$, respectively. In parameter region between $k$- and $(k+1)$ SSPOs (e.g., $3/4 < s < 7/12$) the map exhibits complex bifurcation of SSPOs. Note that the intersection of TB and SS relates to 1-SSPO and there exist tangent bifurcation and it intersection to SS relating to $k - SSPOs$ as shown in Fig. 6. Detailed analysis of the bifurcation sets is not easy and remains as an open problem for future work.

4. Experiments

In order to confirm typical phenomena we have fabricated simple test circuit shown in Fig. 7. The capacitor voltage $v$ increases by integrating a constant current $I > 0$. If $v$ reaches
have clarified basic bifurcation phenomena and related IID characteristics. Basic conditions for generation of chaotic, quasi-periodic and periodic spike-trains are also given. By presenting a simple test circuit, typical phenomena are confirmed experimentally. Future problems include detailed analysis of bifurcation, IID characteristics and application to spike-based coding.

References