On Fuzzy Almost $r$-minimal Continuous Functions between Fuzzy Minimal Spaces and Fuzzy Topological Spaces

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Abstract

The purpose of this paper is to introduce and investigate the concept of fuzzy almost $r$-minimal continuous function between fuzzy minimal spaces and fuzzy topological spaces. Particularly, we investigate characterizations for the fuzzy almost $r$-minimal continuity by using generalized fuzzy $r$-open sets.

Key words: $r$-minimal structure, fuzzy $r$-minimal continuous, fuzzy weakly $r$-minimal continuous, fuzzy almost $r$-minimal continuous

1. Introduction

The concept of fuzzy set was introduced by Zadeh [13]. Chang [2] defined fuzzy topological spaces using fuzzy sets. The concept of smooth topological space was introduced in [3, 10] by Chattopadhyay, Hazra, Samanta, and Ramadhan, which is a generalization of fuzzy topological space. Yoo et al. [11] introduced the concept of fuzzy $r$-minimal space which is an extension of the smooth topological space. The author introduced the concepts of fuzzy $r$-minimal continuous function [8] and fuzzy weakly $r$-minimal continuous function [9] between fuzzy $r$-minimal spaces and fuzzy topological spaces. The purpose of this paper is to generalize the concept of fuzzy $r$-minimal continuous function. So, in this paper, we introduce the concept of fuzzy almost $r$-minimal continuous function between a fuzzy $r$-minimal space and a fuzzy topological space. In particular, we investigate characterizations for the fuzzy almost $r$-minimal continuity by using generalized fuzzy $r$-open sets - fuzzy $r$-semiopen sets, fuzzy $r$-preopen sets, fuzzy $r$-$\beta$-open sets, fuzzy $r$-regular open sets.

2. Preliminaries

Let $I$ be the unit interval $[0, 1]$ of the real line. A member $A$ of $I^X$ is called a fuzzy set of $X$. By $\emptyset$ and $\bar{I}$ we denote constant maps on $X$ with value $0$ and $1$, respectively. For any $A \subseteq I^X$, $A^c$ denotes the complement $\bar{I} - A$. All other notations are standard notations of fuzzy set theory.

A fuzzy point $x_\alpha$ in $X$ is a fuzzy set $x_\alpha$ defined as follows

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

A fuzzy point $x_\alpha$ is said to belong to a fuzzy set $A$ in $X$, denoted by $x_\alpha \in A$, if $\alpha \leq A(x)$ for $x \in X$. A fuzzy set $A$ in $X$ is the union of all fuzzy points which belong to $A$.

Let $f : X \to Y$ be a function and $A \subseteq I^X$ and $B \subseteq I^Y$. Then $f(A)$ is a fuzzy set in $Y$, defined by

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

for $y \in Y$, and $f^{-1}(B)$ is a fuzzy set in $X$, defined by

$$f^{-1}(B)(x) = B(f(x)), x \in X.$$

A fuzzy topology (or smooth topology) [3, 10] on $X$ is a map $T : I^X \to I$ which satisfies the following properties:

1. $T(\emptyset) = T(\bar{I}) = 1$.
2. $T(A_1 \cap A_2) \geq T(A_1) \wedge T(A_2)$ for $A_1, A_2 \subseteq I^X$.
3. $T(A_1 \cup A_2) \geq T(A_1) \vee T(A_2)$ for $A_1 \subseteq I^X$.

The pair $(X, T)$ is called a fuzzy topological space [11].

A fuzzy set $A$ is said to be fuzzy $r$-open (resp., fuzzy $r$-closed) [6] if $T(A) \geq r$ (resp., $T(A^c) \geq r$).

The $r$-closure of $A$, denoted by $cl(A, r)$, is defined as $cl(A, r) = \cap\{B \subseteq I^X : A \subseteq B \text{ is fuzzy } r\text{-closed}\}$.

The $r$-interior of $A$, denoted by $int(A, r)$, is defined as $int(A, r) = \cup\{B \subseteq I^X : A \subseteq B \text{ is fuzzy } r\text{-open}\}$.

Definition 2.1 ([11]). Let $X$ be a nonempty set and $r \in (0, 1] = I_0$. A fuzzy family $\mathcal{M} : I^X \to I$ on $X$ is said to have a fuzzy $r$-minimal structure if the family

$$\mathcal{M}_r = \{A \subseteq I^X \mid \mathcal{M}(A) \geq r\}$$

contains $\emptyset$ and $\bar{I}$.

Then the $(X, \mathcal{M})$ is called a fuzzy $r$-minimal space (simply $r$-FMS) if $\mathcal{M}$ has a fuzzy $r$-minimal structure. Every member of $\mathcal{M}_r$ is called a fuzzy $r$-minimal open set. A fuzzy set $A$ is called a fuzzy $r$-minimal closed set if the...
complement of $A$ (simply, $A^c$) is a fuzzy r-minimal open set.

Let $(X, M)$ be an r-FMS and $r \in I_0$. The fuzzy $r$-minimal closure and the fuzzy $r$-minimal interior of $A$ [11], denoted by $mC(A, r)$ and $mI(A, r)$, respectively, are defined as

$$mC(A, r) = \cap \{ B \subseteq I^X : B^c \in M_r \text{ and } A \subseteq B \},$$

$$mI(A, r) = \cup \{ B \subseteq I^X : B \in M_r \text{ and } B \subseteq A \}.$$

**Theorem 2.2 ([11]).** Let $(X, M)$ be an r-FMS and $A, B$ in $I^X$.

1. $mI(A, r) \subseteq A$ and if $A$ is a fuzzy $r$-minimal open set, then $mI(A, r) = A$.
2. If $A \subseteq B$, then $mI(A, r) \subseteq mI(B, r)$ and $mC(A, r) \subseteq mC(B, r)$.
3. If $A \subseteq B$, then $mI(A, r) \subseteq mI(B, r)$ and $mC(A, r) \subseteq mC(B, r)$.
4. $mI(mI(A, r), r) = mI(A, r)$ and $mC(mC(A, r), r) = mC(A, r)$.
5. $\tilde{1} - mC(A, r) = mI(\tilde{1} - A, r)$ and $\tilde{1} - mI(A, r) = mC(\tilde{1} - A, r)$.

3. Fuzzy Almost $r$-minimal Continuous Functions

**Definition 3.1.** Let $(X, M_X)$ be an r-FMS and $(Y, \sigma)$ a fuzzy topological space. Then $f : X \to Y$ is said to be **fuzzy almost $r$-minimal continuous** if for a fuzzy point $x_\alpha$ and for each fuzzy $r$-open set $V$ with $f(x_\alpha) \in V$, there exists a fuzzy $r$-minimal open set $U$ such that $x_\alpha \in U$ and $f(U) \subseteq \text{int}(cl(V, r), r)$.

We recall that: Let $(X, M_X)$ be an r-FMS and $(Y, \sigma)$ a fuzzy topological space. Then $f : X \to Y$ is said to be

1. **fuzzy $r$-minimal continuous** [8] if for every fuzzy $r$-open set $A$ in $Y$, $f^{-1}(A)$ is fuzzy $r$-minimal open in $X$;
2. **fuzzy weakly $r$-minimal continuous** [9] if for a fuzzy point $x_\alpha$ and for each fuzzy $r$-open set $V$ with $f(x_\alpha) \in V$, there exists a fuzzy $r$-minimal open set $U$ such that $x_\alpha \in U$ and $f(U) \subseteq cl(V, r)$.

From the above definitions, easily we have the following implications:

fuzzy $r$-minimal continuity $\Rightarrow$ fuzzy almost $r$-minimal continuity $\Rightarrow$ fuzzy weakly $r$-minimal continuity

**Example 3.2.** Let $X = I$ and let $A, B, C$ and $D$ be fuzzy sets as the following:

$$A(x) = \frac{1}{2} x, \quad x \in I;$$

$$B(x) = -\frac{1}{2} (x - 1), \quad x \in I;$$

$$C(x) = \begin{cases} \frac{1}{2} (x + 1), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ -\frac{1}{2} (x - 2), & \text{if } \frac{1}{2} < x \leq 1; \end{cases}$$

and

$$D(x) = \begin{cases} -\frac{1}{2} (2x - 1), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2} (2x - 1), & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Let us consider two fuzzy topologies $M_1$ and $M_2$ defined as the following:

$$M_1(\mu) = \begin{cases} 1, & \text{if } \mu = 0_X, 1_X, \\ \frac{1}{2}, & \text{if } \mu = C, \\ 0, & \text{otherwise}; \end{cases}$$

$$M_2(\mu) = \begin{cases} 1, & \text{if } \mu = 0_X, 1_X, \\ \frac{1}{2}, & \text{if } \mu = D, \\ 0, & \text{otherwise}. \end{cases}$$

And let us consider a fuzzy r-minimal structure $N$ defined as the following:

$$N(\mu) = \begin{cases} 1, & \text{if } \mu = 0_X, 1_X, \\ \frac{1}{2}, & \text{if } \mu = A, B, \\ 0, & \text{otherwise}. \end{cases}$$

Then:

1. The identity function $f : (X, N) \to (X, M_1)$ is fuzzy almost $\frac{1}{2}$-minimal continuous but not fuzzy $\frac{1}{2}$-minimal continuous.

2. The identity function $g : (X, N) \to (X, M_2)$ is fuzzy weakly $\frac{1}{2}$-minimal continuous but not fuzzy almost $\frac{1}{2}$-minimal continuous.

Let $(X, M)$ be a FTS and $A \in I^X$. Then a fuzzy set $A$ is said to be **fuzzy $r$-regular open** (resp., fuzzy $r$-regular closed) [7] if $A = \text{int}(cl(A, r), r)$ (resp., $A = cl(\text{int}(A, r), r)$).

**Theorem 3.3.** Let $f : X \to Y$ be a function between an r-FMS $(X, M_X)$ and a fuzzy topological space $(Y, \sigma)$. Then the following statements are equivalent:

1. $f$ is fuzzy almost $r$-minimal continuous.
2. $f^{-1}(B) \subseteq mI(f^{-1}(\text{int}(cl(B, r), r)), r)$ for each fuzzy $r$-open set $B$ of $Y$.
3. $mC(f^{-1}(\text{int}(cl(F, r), r)), r) \subseteq f^{-1}(F)$ for each fuzzy $r$-closed set $F$ in $Y$.
4. $f^{-1}(F) = mC(f^{-1}(F), r)$ for an fuzzy $r$-regular closed set $F$ in $Y$.
5. $f^{-1}(V) = mI(f^{-1}(V), r)$ for an fuzzy $r$-regular open set $V$ in $Y$. 

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This implies $mC(f^{-1}(cl(int(cl(V,r),r)),r)) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4) For any fuzzy r-regular closed set $F$ of $Y$, since $F = cl(int(F,r))$ and fuzzy r-closed, we have $mC(f^{-1}(F),r) = mC(f^{-1}(cl(int(F,r),r)),r) \subseteq f^{-1}(F).$ So $f^{-1}(F) = mC(f^{-1}(F),r)$.

(4) $\Rightarrow$ (5) Obvious.

(5) $\Rightarrow$ (1) Let $V$ be a fuzzy r-open set containing $f(x_a)$. Since $int(cl(V,r),r)$ is fuzzy r-open, from (5),

$$x_a \in f^{-1}(V) \subseteq f^{-1}(int(cl(V,r),r)) = mI(f^{-1}(int(cl(V,r),r),r)).$$

So there is a fuzzy r-minimal open set $U$ such that $x_a \in U \subseteq f^{-1}(int(cl(V,r),r))$. This implies $f(U) \subseteq int(cl(V,r),r)$, and so $f$ is fuzzy almost r-minimal continuous.

Let $X$ be a nonempty set and $M : I^X \rightarrow I$ a fuzzy family on $X$. The fuzzy family $M$ is said to have the property $(U)$ [11] if for $A_i \subseteq M (i \in J)$,

$$M(\bigcup A_i) \geq \wedge M(A_i).$$

**Theorem 3.4** ([11]). Let $(X, M)$ be an r-FMS with the property $(U)$. Then

1. For $A \subseteq I^X, mI(A,r) = A$ if and only if $A$ is fuzzy r-minimal open.
2. For $F \subseteq I^X, mC(F,r) = F$ if and only if $F$ is fuzzy r-minimal closed.

**Corollary 3.5.** Let $f : X \rightarrow Y$ be a function between an r-FMS $(X, M_X)$ and a fuzzy topological space $(Y, \sigma)$. If $M_X$ has the property $(U)$, then the following statements are equivalent:

1. $f$ is fuzzy almost r-minimal continuous.
2. $f^{-1}(B) \subseteq mI(f^{-1}(cl(cl(B,r),r)),r)$ for each fuzzy r-open set $B$ of $Y$.
3. $mC(f^{-1}(cl(cl(F,r),r)),r) \subseteq f^{-1}(F)$ for each fuzzy r-closed set $F$ in $Y$.
4. $f^{-1}(B)$ is fuzzy r-minimal open for each fuzzy r-regular open set $B$ of $Y$.
5. $f^{-1}(B)$ is fuzzy r-minimal closed for each fuzzy r-regular closed set $B$ of $Y$.

**Definition 3.6.** Let $(X, \tau)$ be a FTS and $A \subseteq I^X$. Then a fuzzy set $A$ is said to be

1. fuzzy r-semiopen [6] if $A \subseteq cl(int(cl(A,r),r))$;
2. fuzzy r-preopen [5] if $A \subseteq int(cl(A,r),r)$;
3. fuzzy r-β-open [1] if $A \subseteq cl(int(cl(A,r),r),r)$.

A fuzzy set $A$ is called a fuzzy r-semiopen (resp., fuzzy r-preopen, fuzzy r-β-open) set if the complement of $A$ is a fuzzy r-semiopen (resp., fuzzy r-preopen, fuzzy r-β-open) set.

**Theorem 3.7.** Let $f : X \rightarrow Y$ be a function between an r-FMS $(X, M_X)$ and a fuzzy topological space $(Y, \sigma)$. Then the following statements are equivalent:

1. $f$ is fuzzy almost r-minimal continuous.
2. $mC(f^{-1}(G),r) \subseteq f^{-1}(cl(G,r))$ for each fuzzy r-β-open set $G$ in $Y$.
3. $mC(f^{-1}(G),r) \subseteq f^{-1}(cl(G,r))$ for each fuzzy r-semiopen set $G$ in $Y$.

*Proof.* (1) $\Rightarrow$ (2) Let $G$ be a fuzzy r-β-open set. Then since $cl(G,r)$ is fuzzy r-regular closed, from Theorem 3.3 (4), it follows

$$mC(f^{-1}(G),r) \subseteq mC(f^{-1}(cl(G,r)),r) = f^{-1}(cl(G,r)).$$

(2) $\Rightarrow$ (3) Since every fuzzy r-semiopen set is fuzzy r-β-open, it is obvious.

(3) $\Rightarrow$ (1) Let $F$ be a fuzzy r-regular closed set. Then $F$ is fuzzy r-semiopen, and so from (3), we have

$$mC(f^{-1}(F),r) \subseteq f^{-1}(cl(F,r)) = f^{-1}(F).$$

Hence, from Theorem 3.3, $f$ is fuzzy almost r-minimal continuous.

**Theorem 3.8.** Let $f : X \rightarrow Y$ be a function between an r-FMS $(X, M_X)$ and a fuzzy topological space $(Y, \sigma)$. Then $f$ is fuzzy almost r-minimal continuous if and only if $mC(f^{-1}(cl(cl(F,r),r),r),r) \subseteq f^{-1}(cl(G,r))$ for each fuzzy r-preopen set $G$ in $Y$. 


Proof. Suppose \( f \) is fuzzy almost \( r \)-minimal continuous and let \( G \) be a fuzzy \( r \)-preopen set in \( Y \). Then since \( cl(G, r) = cl(int(cl(G, r), r), r) \) and \( cl(G, r) \) is fuzzy \( r \)-regular closed, from Theorem 3.3,

\[
\begin{align*}
    f^{-1}(cl(G, r)) &= mC(f^{-1}(cl(G, r)), r) \\
    &= mC(f^{-1}(cl(int(cl(G, r), r), r)), r).
\end{align*}
\]

Thus it implies \( mC(f^{-1}(cl(int(cl(G, r), r), r)), r) \subseteq f^{-1}(cl(G, r)). \)

For the converse, let \( A \) be a fuzzy \( r \)-regular closed set in \( Y \). Then since \( int(A, r) \) is fuzzy \( r \)-preopen, from hypothesis and \( A = cl(int(A, r), r) \), it follows

\[
\begin{align*}
    f^{-1}(A) &= f^{-1}(cl(int(A, r), r)) \\
    &= mC(f^{-1}(cl(int(A, r), r), r), r) \\
    &= mC(f^{-1}(cl(A, r), r), r) \\
    &= mC(f^{-1}(A), r).
\end{align*}
\]

This implies \( f^{-1}(A) = mC(f^{-1}(A), r) \), and hence by Theorem 3.3, \( f \) is fuzzy almost \( r \)-minimal continuous. \( \square \)

Theorem 3.9. Let \( f : X \rightarrow Y \) be a function between an \( r \)-FMS \((X, M_X)\) and a fuzzy topological space \((Y, \sigma)\). Then \( f \) is fuzzy almost \( r \)-minimal continuous if and only if \( f^{-1}(G) \subseteq mI(f^{-1}(int(cl(G, r), r)), r) \) for each fuzzy \( r \)-preopen set \( G \) in \( Y \).

Proof. Suppose \( f \) is fuzzy almost \( r \)-minimal continuous and let \( G \) be a fuzzy \( r \)-preopen set in \( Y \). Since \( int(cl(G, r), r) \) is fuzzy \( r \)-regular open, from Theorem 3.3, it follows \( f^{-1}(G) \subseteq f^{-1}(int(cl(G, r), r)) = mI(f^{-1}(int(cl(G, r), r)), r) \).

For the converse, let \( U \) be fuzzy \( r \)-regular open. Then \( U \) is also fuzzy \( r \)-preopen. Since \( U = int(cl(U, r), r) \), by hypothesis, \( f^{-1}(U) \subseteq mI(f^{-1}(int(cl(U, r), r)), r) = mI(f^{-1}(U), r) \). This implies \( f^{-1}(U) = mI(f^{-1}(U), r) \) and so \( f \) is fuzzy almost \( r \)-minimal continuous. \( \square \)

Definition 3.10 ([12]). Let \((X, M_X)\) be an \( r \)-FMS and \( C = \{A_i \in I^X : i \in J\} \). \( C \) is called a fuzzy \( r \)-minimal cover if \( \{A_i : i \in J\} = \overline{1}_X \). It is a fuzzy \( r \)-minimal open cover if each \( A_i \) is a fuzzy \( r \)-minimal open set. A subcover of a fuzzy \( r \)-minimal cover \( A \) is a subfamily of \( A \) which also is a fuzzy \( r \)-minimal cover. \( X \) is said to be fuzzy \( r \)-minimal compact (resp., almost fuzzy \( r \)-minimal compact, nearly fuzzy \( r \)-minimal compact) if for every fuzzy \( r \)-minimal open cover \( C = \{A_i \in I^X : i \in J\} \) of \( X \), there exists \( J_0 = \{j_1, j_2, \ldots, j_n\} \subseteq J \) such that \( \overline{1}_X = \cup_{i \in J_0} A_i \) (resp., \( \overline{1}_X = \cup_{i \in J_0} mC(A_i, r) \), \( \overline{1}_X = \cup_{i \in J_0} mI(mC(A_i, r), r) \)).

Definition 3.11 ([4]). Let \((X, \tau)\) be a fuzzy topological space. \( X \) is said to be fuzzy \( r \)-preopen and fuzzy \( r \)-precompact if for every fuzzy \( r \)-open cover \( C = \{A_i \in I^X : \tau(A_i) \geq r, i \in J\} \) of \( A \), there exists \( J_0 = \{j_1, j_2, \ldots, j_n\} \subseteq J \) such that \( \overline{1}_X = \cup_{i \in J_0} A_i \) (resp., \( \overline{1}_X = \cup_{i \in J_0} mC(A_i, r) \), \( \overline{1}_X = \cup_{i \in J_0} mI(mC(A_i, r), r) \)).

Theorem 3.12. Let \( f : X \rightarrow Y \) be a fuzzy almost \( r \)-minimal continuous surjection between an \( r \)-FMS \((X, M_X)\) and a fuzzy topological space \((Y, \sigma)\). If \( X \) is fuzzy \( r \)-minimal compact, then \( Y \) is \( r \)-fuzzy nearly compact.

Proof. Let \( C = \{B_i \in I^Y : i \in J\} \) be a fuzzy \( r \)-open cover of \( Y \). Then for each \( x(i) \in f^{-1}(B_i) \) for \( B_i \in C \), since \( f \) is fuzzy almost \( r \)-minimal continuous, there exists a fuzzy \( r \)-minimal open set \( U(x(i)) \) such that \( x(i) \in U(x(i)) \subseteq f^{-1}(int(cl(B_i, r), r)) \). So the collection \( \{U(x(i)) : x(i) \in X\} \) is a fuzzy \( r \)-minimal open cover in \( X \). Since \( X \) is fuzzy \( r \)-minimal compact, there exists \( J_0 = \{1, 2, \ldots, n\} \subseteq J \) such that \( \overline{1}_X = \cup_{j \in J_0} U(x(j)) \subseteq \cup_{j \in J_0} f^{-1}(int(cl(B_j, r), r)) \). Hence \( \overline{1}_Y = \cup_{j \in J_0} int(cl(B_j, r), r) \).

\( \square \)

Theorem 3.13. Let \( f : X \rightarrow Y \) be a fuzzy almost \( r \)-minimal continuous surjection between an \( r \)-FMS \((X, M_X)\) and a fuzzy topological space \((Y, \sigma)\). If \( X \) is fuzzy \( r \)-minimal compact and if \( M_X \) has the property \( (U) \), then \( Y \) is \( r \)-fuzzy nearly compact.

Proof. Let \( C = \{B_i \in I^Y : i \in J\} \) be a fuzzy \( r \)-open cover of \( Y \). Then by the property \( (U) \), the fuzzy family \( C' = \{mI(f^{-1}(int(cl(B_i, r), r)), r) : B_i \in C \} \) for \( i \in J \) is a fuzzy \( r \)-minimal open cover of \( X \). Since \( X \) is fuzzy \( r \)-minimal compact, there exists a finite subset \( J_0 \) of \( J \) such that \( \overline{1}_X = \cup_{j \in J_0} mI(f^{-1}(int(cl(B_j, r), r)), r) = \cup_{j \in J_0} f^{-1}(int(cl(B_j, r), r)) \).

This implies \( \overline{1}_Y = \cup_{j \in J_0} int(cl(B_j, r), r) \) and so \( Y \) is \( r \)-fuzzy nearly compact. \( \square \)

References


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