Chapter 6

Elastic Stability Analysis of Euler Columns Using Analytical Approximate Techniques

Safa Bozkurt Coşkun and Baki Öztürk

Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/45940

1. Introduction

In most of the real world engineering applications, stability analysis of compressed members is very crucial. There have been many researches dedicated to the buckling behavior of axially compressed members. On the other hand, obtaining analytical solutions for the buckling behavior of columns with variable cross-section subjected to complicated load configurations are almost impossible in most of the cases. Some of the works related to obtaining analytical or analytical approximate solutions for the column buckling problem are provided below.

The problems of buckling of columns under variable distributed axial loads were solved in detail by Vaziri and Xie [1] and others. Some analytical closed-form solutions are given by Dinnik [2], Karman and Biot [3], Morley[4], Timoshenko and Gere [5] and others. One of the detailed references related to the structural stability topic is written by Simitses and Hodges [6] with detailed discussions. Iyengar [7] made some analysis on buckling of uniform with several elastic supports. Wang et al. [8] have given exact mathematical solutions for buckling of structural members for various cases of columns, beams, arches, rings, plates and shells. Ermopoulos [9] found the solution for buckling of tapered bars axially compressed by concentrated loads applied at various locations along their axes. Li [10] gave the exact solution for buckling of non-uniform columns under axially concentrated and distributed loading. Lee and Kuo [11] established an analytical procedure to investigate the elastic stability of a column with elastic supports at the ends under uniformly distributed follower forces. Furthermore, Gere and Carter [12] investigated and established the exact analytical solutions for buckling of several special types of tapered columns with simple boundary conditions. Solution of the problem of buckling of elastic columns with step varying thickness is established by Arbabei and Li [13]. Stability problems of a uniform bar with several elastic supports using the moment-

distribution method were analyzed by Kerekes [14]. The research of Siginer [15] was about the stability of a column whose flexural stiffness has a continuous linear variation along the column. Moreover, the analytical solutions of a multi-step bar with varying cross section were obtained by Li et al. [16-18]. The energy method was used by Sampaio et al. [19] to find the solution for the problem of buckling behavior of inclined beam-column. Some of the important researchers who studied the mechanical behavior of beam-columns are Keller [20], Tadjbakhsh and Keller [21] and Taylor [22]. Later on, analytical approximate techniques were used for the stability analysis of elastic columns. Coşkun and Atay [23] and Atay and Coşkun [24] studied column buckling problems for the columns with variable flexural stiffness and for the columns with continuous elastic restraints by using the variational iteration method which produces analytical approximations. Coşkun [25, 26] used the homotopy perturbation method for buckling of Euler columns on elastic foundations and tilt-buckling of variable stiffness columns. Pınarbaşı [27] also analyzed the stability of nonuniform rectangular beams using homotopy perturbation method. These techniques were also used successfully in the vibration analysis of Euler-Bernoulli beams and in the vibration of beams on elastic foundations. [28-29]

Recently, by the emergence of new and innovative semi analytical approximation methods, research on this subject has gained momentum. Analytical approximate solution techniques are used widely to solve nonlinear ordinary or partial differential equations, integrodifferential equations, delay equations, etc. The main advantage of employing such techniques is that the problems are considered in a more realistic manner, and the solution obtained is a continuous function which is not the case for the solutions obtained by discretized solution techniques.

The methods that will be used throughout this study are, Adomian Decomposition Method (ADM), Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM). Each technique will be explained first, and then all will be applied to a selected case study related to the topic of the article.

2. Problem formulation

Derivation of governing equations related to stability analysis is given in detail in Timoshenko and Gere [5], Simitses and Hodges [6], and Wang et al. [8]. The reader can also refer to any textbook related to the subject. In this section, only the governing equation will be given for the related cases.

Consider the elastic columns given in Fig.1. The governing equation for the buckling of such columns is

$$\frac{d^2y}{dx^2}\left[EI(x)\frac{d^2y}{dx^2}\right]+P\frac{d^2y}{dx^2}=0 \tag{1}$$

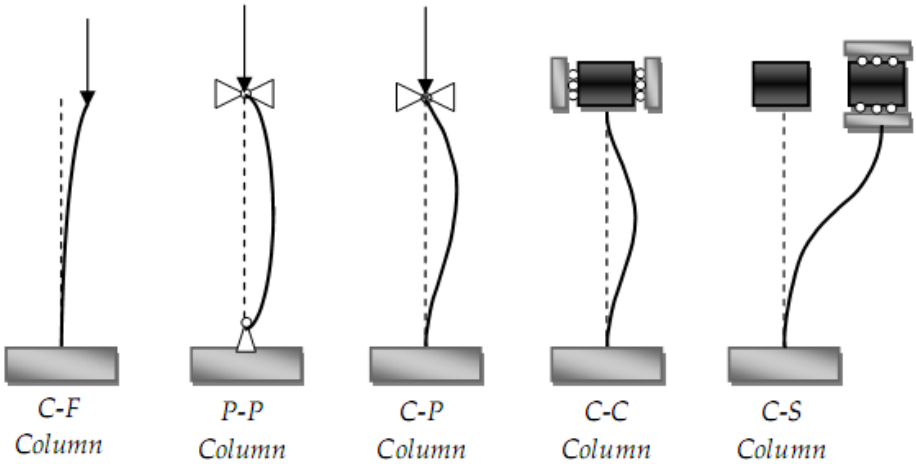

Figure 1. Elastic columns with various end conditions

In the case of constant flexural rigidity (*i.e. EI* is constant), Eq.(1) becomes

$$\frac{d^4y}{dx^4}+\frac{P}{EI}\frac{d^2y}{dx^2}=0 \tag{2}$$

where *EI* is the flexural rigidity of the column, and *P* is the applied load. Both Eqs. (2) and (3) are solved due to end conditions of the column. Some of these conditions are shown in Fig.1. In this figure, letters are used for a simplification to describe the support conditions of the column. The first letter stands for the support at the bottom and the second letter for the top. Hence, CF is *Clamped-Fixed*, PP is *Pinned-Pinned*, C-P is *Clamped-Pinned* and C-S is *Clamped-Sliding Restraint*.

The governing equations (1) and (2) are both solved with respect to the problem's end conditions. The end conditions for the columns shown in Fig.1 are given below:

Pin support:

$$y=0 \text{ and } \frac{d^2y}{dx^2}=0 \tag{3}$$

Clamped support:

$$y=0 \text{ and } \frac{dy}{dx}=0 \tag{4}$$

Free end:

$$\text{and } \frac{d^3y}{dx^3}+\frac{P}{EI}\frac{dy}{dx}=0 \tag{5}$$

Sliding restraint:

$$\frac{dy}{dx} = 0 \text{ and} \tag{6}$$

The governing equation given in Eq.(2) is a fourth order differential equation with constant coefficients which makes it possible to obtain analytical solutions easily. However, Eq.(1) includes variable coefficients due to variable flexural rigidity. For this type of differential equations, analytical solutions are limited for the special cases of *EI(x)* only. It is not possible to obtain a solution for any form of the function *EI(x)*.

In some problems, obtaining analytical solutions is very difficult even for a constant coefficient governing equation. Consider the buckling of a column on an elastic foundation shown in Fig.2.

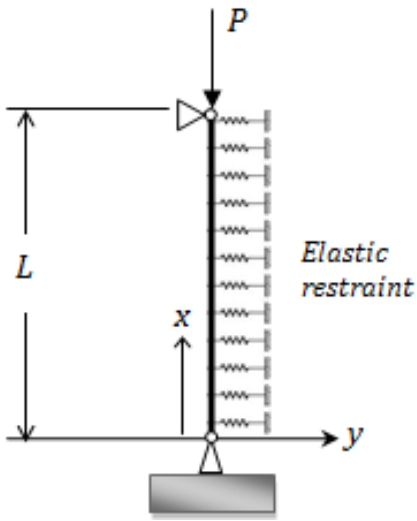

Figure 2. Column with continuous elastic restraints.

The governing equation for the column in Fig.2 is

$$\frac{d^2y}{dx^2}\left[EI(x)\frac{d^2y}{dx^2}\right] + P\frac{d^2y}{dx^2} + ky = 0 \tag{7}$$

which, for the constant *EI* becomes

$$\frac{d^4y}{dx^4} + \frac{P}{EI}\frac{d^2y}{dx^2} + \frac{k}{EI}y = 0 \tag{8}$$

In Eqs.(7) and (8), *k* is the stiffness parameter for the elastic restraint. The solution of Eq.(8) for the *CF* column is given in [8] as

$$\left[\alpha(S^2+T^2)-2S^2T^2\right]\cos T\cos S-\alpha(S^2+T^2)+(S^4+T^4)+ \\ ST\left[2\alpha-(S^2+T^2)\right]\sin T\sin S=0 \tag{9}$$

Although Eq.(8) is a linear equation with constant coefficients, obtaining a solution from Eq.(9) is not that easy. It is very interesting that, even with a software, one can not easily produce the buckling loads in a sequential order from Eq.(9). In view of this experience, an analytical solution for Eq.(7) is almost impossible to obtain except very limited *EI(x)* choices.

Hence, analytical approximate techniques are efficient alternatives for solving these problems. By the use of these techniques, a solution which is continuous in the problem domain is possible for any variation in flexural rigidity. These techniques produce the buckling loads in a sequential order, and it is also very easy to obtain the buckling mode shapes from the solution provided by the method used. These are great advantages in the solution of such problems.

3. The methods used in the elastic stability analysis of Euler columns

3.1. Adomian Decomposition Method (ADM)

In the ADM a differential equation of the following form is considered

$$Lu+Ru+Nu=g(x) \tag{10}$$

where, L is the linear operator which is highest order derivative, R is the remainder of linear operator including derivatives of less order than L, Nu represents the nonlinear terms, and g is the source term. Eq.(10) can be rearranged as

$$Lu=g(x)-Ru-Nu \tag{11}$$

Applying the inverse operator L^{-1} to both sides of Eq.(11) and employing given conditions; we obtain

$$u=L^{-1}\{g(x)\}-L^{-1}(Ru)-L^{-1}(Nu) \tag{12}$$

After integrating source term and combining it with the terms arising from given conditions of the problem, a function *f(x)* is defined in the equation as

$$u=f(x)-L^{-1}(Ru)-L^{-1}(Nu) \tag{13}$$

The nonlinear operator $Nu=F(u)$ is represented by an infinite series of specially generated (Adomian) polynomials for the specific nonlinearity. Assuming Nu is analytical, we write

$$F(u)=\sum_{k=0}^{\infty}A_k \tag{14}$$

The polynomials A_k's are generated for all kinds of nonlinearity, so that they depend only on u_o to u_k components and can be produced by the following algorithm.

$$A_0 = F(u_0) \tag{15}$$

$$A_1 = u_1 F'(u_0) \tag{16}$$

$$A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0) \tag{17}$$

$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0) \tag{18}$$

$$\vdots$$

The reader can refer to [30, 31] for the algorithms used in formulating Adomian polynomials. The solution *u(x)* is defined by the following series

$$u = \sum_{k=0}^{\infty} u_k \tag{19}$$

where, the components of the series are determined recursively as follows:

$$u_0 = f(x) \tag{20}$$

$$u_{k+1} = -L^{-1}\left(Ru_k\right) - L^{-1}\left(A_k\right), \quad k \geq 0 \tag{21}$$

3.2. Variational Iteration Method (VIM)

According to VIM, the following differential equation may be considered:

$$Lu + Nu = g(x) \tag{22}$$

where L is a linear operator, N is a nonlinear operator, and *g(x)* is an inhomogeneous source term. Based on VIM, a correct functional can be constructed as follows:

$$u_{n+1} = u_n + \int_0^x \lambda(\xi)\left\{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right\} d\xi \tag{23}$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript n denotes the nth-order approximation, $\tilde{u}$ is considered as a restricted variation *i.e.* $\delta\tilde{u} = 0$. By solving the differential equation for λ obtained from Eq.(23) in view of $\delta\tilde{u} = 0$ with respect to its boundary conditions, Lagrangian multiplier $\lambda(\xi)$ can be obtained. For further details of the method the reader can refer to [32].

3.3. Homotopy Perturbation Method (HPM)

HPM provides an analytical approximate solution for problems at hand as the other previously explained techniques. Brief theoretical steps for the equation of following type can be given as

$$L(u) + N(u) = f(r) \ , \ r \in \Omega \tag{24}$$

with boundary conditions $B(u, \partial u / \partial n) = 0$. In Eq.(24) L is a linear operator, N is a nonlinear operator, B is a boundary operator, and $f(r)$ is a known analytic function. HPM defines homotopy as

$$v(r,p) = \Omega \times [0,1] \to R \tag{25}$$

which satisfies the following inequalities:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \tag{26}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{27}$$

where $r \in \Omega$ and $p \in [0,1]$ is an imbedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. Obviously, from Eq.(26) and Eq.(27) , we have :

$$H(v,0) = L(v) - L(u_0) = 0 \tag{28}$$

$$H(v,1) = L(v) + N(v) - f(r) = 0 \tag{29}$$

As p is changing from zero to unity, so is that of $v(r,p)$ from u_0 to $u(r)$. In topology, this deformation $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are called homotopic. The basic assumption is that the solutions of Eq.(34) and Eq.(35) can be expressed as a power series in p such that:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + ... \tag{30}$$

The approximate solution of $L(u) + N(u) = f(r) \ , \ r \in \Omega$ can be obtained as :

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + ... \tag{31}$$

The convergence of the series in Eq.(31) has been proved in [33]. The method is described in detail in references [33-36].

4. Case studies

4.1. Buckling of a clamped-pinned column

The governing equation for this case was previously given in Eq.(1). ADM, VIM and HPM will be applied to this equation in order to compute the buckling loads for the clamped-

pinned column with constant flexural stiffness, *i.e.* constant *EI*, and variable flexural stiffness, *i.e.* variable *EI* with its corresponding mode shapes. To achieve this aim, a circular rod is defined with an exponentially varying radius. The case is given in Fig.3, and rod and its associated boundary conditions are also provided in Eqs.(3-4). As a case study, first the formulations for constant stiffness column by using ADM, VIM and HPM are given, and then applied to the governing equation of the problem. Afterwards, a variable flexural rigidity will be defined for the same column, and the same techniques will be used for the analysis.

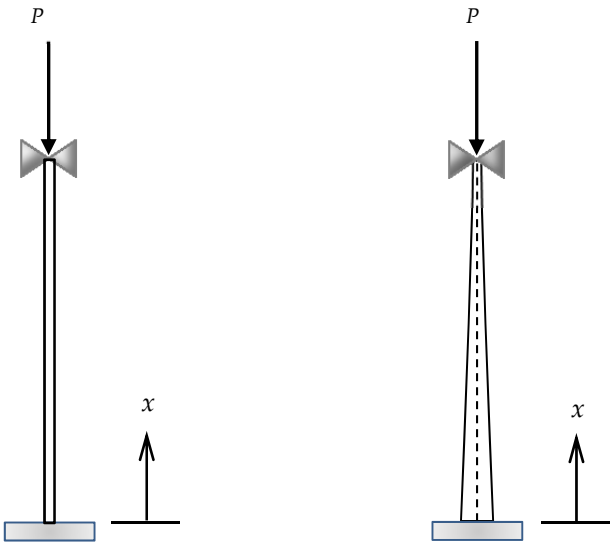

Figure 3. *CP* column with constant and variable flexural rigidity

4.2. Formulation of the algorithms for uniform column

4.2.1. ADM

The linear operator and its inverse operator for Eq.(2) is

$$L(\cdot) = \frac{d^4}{dx^4}(\cdot) \tag{32}$$

$$L^{-1}(\cdot) = \int_0^x \int_0^x \int_0^x \int_0^x (\cdot)\, dx\, dx\, dx\, dx \tag{33}$$

To keep the formulation a general one for all configurations to be considered, the boundary conditions are chosen as $Y(0) = A$, $Y'(0) = B$, $Y''(0) = C$ and $Y'''(0) = D$. Suitable values should be replaced in the formulation with these constants. In this case, $A = 0$ and $B = 0$ should be inserted for the *CP* column. Hence, the equation to be solved and the recursive algorithm can be given as

$$LY = \zeta Y'' \tag{34}$$

$$Y = A + Bx + C\frac{x^2}{2!} + D\frac{x^3}{3!} + L^{-1}(\zeta Y'') \tag{35}$$

$$Y_{n+1} = L^{-1}(\zeta Y_n''), \quad n \geq 0 \tag{36}$$

Finally, the solution is defined by

$$Y = Y_0 + Y_1 + Y_2 + Y_3 + \ldots \tag{37}$$

4.2.2. VIM

Based on the formulation given previously, Lagrange multiplier, λ would be obtained for the governing equation, *i.e.* Eq.(2), as

$$\lambda(\xi) = \frac{(\xi - x)^3}{3!} \tag{38}$$

An iterative algorithm can be constructed inserting Lagrange multiplier and governing equation into the formulation given in Eq.(31) as

$$Y_{n+1} = Y_n + \int_0^x \lambda(\xi)\left\{Y_n^{iv}(\xi) + \zeta \tilde{Y}_n''(\xi)\right\} d\xi \tag{39}$$

where ζ is normalized buckling load for the column considered. Initial approximation for the algorithm is chosen as the solution of $LY = 0$ which is a cubic polynomial with four unknowns to be determined by the end conditions of the column.

4.2.3. HPM

Based on the formulation, Eq.(2) can be divided into two parts as

$$LY = Y^{iv} \tag{40}$$

$$NY = \zeta Y'' \tag{41}$$

The solution can be expressed as a power series in p such that

$$Y = Y_0 + pY_1 + p^2 Y_2 + p^3 Y_3 + \ldots \tag{42}$$

Inserting Eq.(50) into Eq.(35) provides a solution algorithm as

$$Y_0^{iv} - y_0^{iv} = 0 \tag{43}$$

$$Y_1^{iv} + y_0^{iv} + \zeta^4 Y_0'' = 0 \tag{44}$$

$$Y_n^{iv} + \zeta Y_{n-1}'' = 0, \quad n \geq 2 \tag{45}$$

Hence, an approximate solution would be obtained as

$$Y = Y_0 + Y_1 + Y_2 + Y_3 + \ldots \tag{46}$$

Initial guess is very important for the convergence of solution in HPM. A cubic polynomial with four unknown coefficients can be chosen as an initial guess which was shown previously to be an effective one in problems related to Euler beams and columns [23-29].

4.3. Computation of buckling loads

By the use of described techniques, an iterative procedure is constructed and a polynomial including the unknown coefficients resulting from the initial guess is produced as the solution to the governing equation. Besides two unknowns from the initial guess, an additional unknown ζ also exists in the solution. Applying far end boundary conditions to the solution produces a linear algebraic system of equations which can be defined in a matrix form as

$$\left[M(\zeta)\right]\{\alpha\} = \{0\} \tag{47}$$

where $\{\alpha\} = \langle A, B \rangle^T$. For a nontrivial solution, determinant of coefficient matrix must be zero. Determinant of matrix $\left[M(\zeta)\right]$ yields a characteristic equation in terms of ζ. Positive real roots of this equation are the normalized buckling loads for the Clamped-Pinned column.

4.4. Determination of buckling mode shapes

Buckling mode shapes for the column can also be obtained from the polynomial approximations by the methods considered in this study. Introducing, the buckling loads into the solution, normalized polynomial eigen functions for the mode shapes are obtained from

$$\bar{Y}_j = \frac{Y_N\left(x, \zeta_j\right)}{\left[\int_0^1 \left|Y_N\left(x, \zeta_j\right)\right|^2 dx\right]^{1/2}}, \quad j = 1, 2, 3, \ldots \tag{48}$$

The same approach can also be employed to predict mode shapes for the cases including variable flexural stiffness.

4.5. Analysis of a uniform column

After applying the procedures explained in the text, the following results are obtained for the buckling loads. Comparison with the exact solutions is also provided in order that one can observe an excellent agreement between the exact results and the computed results.

Twenty iterations are conducted for each method, and the computed values are compared with the corresponding exact values for the first four modes of buckling in the following table.

Mode	*Exact*	*ADM*	*VIM*	*HPM*
1	20.19072856	20.19072856	20.19072856	20.19072856
2	59.67951594	59.67951594	59.67951594	59.67951594
3	118.89986916	118.89986857	118.89986857	118.89986868
4	197.85781119	197.88525697	197.88522951	197.88520511

Table 1. Comparison of normalized buckling loads (PL^2 / EI) for the *CP* column

From the table it can be seen that the computed values are highly accurate which show that the techniques used in the analysis are very effective. Only a few iterations are enough to obtain the critical buckling load which is Mode 1. Additional modes require additional iterations. The table also shows that additional two or three iterations will produce an excellent agreement for Mode 4. Even with twenty iterations, the error is less than 0.014% for all the methods used in the analyses.

The buckling mode shapes of uniform column for the first four modes are depicted in Fig.4. To prevent a possible confusion to the reader, the exact mode shapes and the computed ones are not shown separately in the figure since the obtained mode shapes coincide with the exact ones.

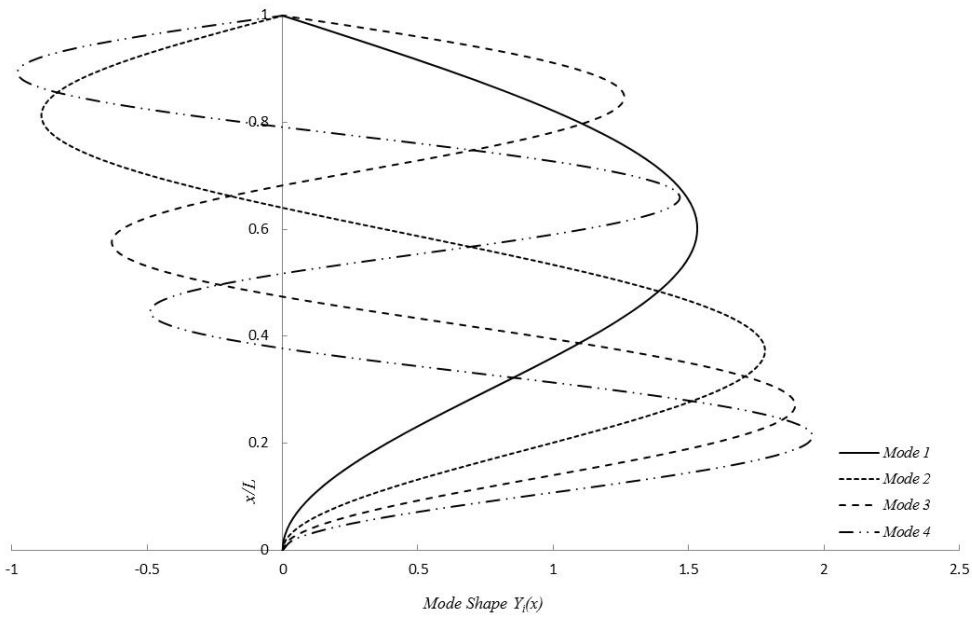

Figure 4. Buckling modes of *CP* column.

4.6. Buckling of a rod with variable cross-section

A circular rod having a radius changing exponentially is considered in this case. Such a rod is shown below in Fig.5. The function representing the radius would be as

$$R(x) = R_0 \, \mathrm{e}^{-ax} \tag{49}$$

where R_o is the radius at the bottom end, L is the length of the rod and $aL \le 1$.

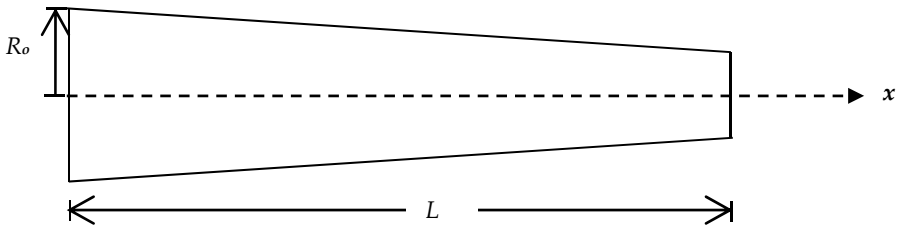

Figure 5. Circular rod with a variable cross-section

Employing Eq.(49), cross-sectional area and moment of inertia for a section at an arbitrary point x becomes:

$$A(x) = A_0 \, \mathrm{e}^{-2ax} \tag{50}$$

$$I(x) = I_0 \, \mathrm{e}^{-4ax} \tag{51}$$

where

$$A_0 = \pi R_0^2 \tag{52}$$

$$I_0 = \frac{\pi R_0^4}{4} \tag{53}$$

Governing equation for the rod was previously given in Eq.(1) as

$$\frac{d^2y}{dx^2}\left[EI(x)\frac{d^2y}{dx^2}\right] + P\frac{d^2y}{dx^2} = 0$$

4.6.1. Formulation of the algorithms

4.6.1.1. ADM

Application of ADM leads to the following

$$Y^{iv} - 8aY''' + \left(16a^2 + \zeta\right)\psi(x)Y'' = 0 \tag{54}$$

where

$$\psi(x) = \mathrm{e}^{4ax} \tag{55}$$

and, where ζ is normalized buckling load PL^2 / EI_0. Once ζ is provided by ADM, buckling mode shapes for the rod can also be easily produced from the solution.

ADM gives the following formulation with the previously defined fourth order linear operator.

$$Y = A\frac{x^2}{2!} + B\frac{x^3}{3!} + L^{-1}\left(8aY''' - \left(16a^2 + \zeta\right)\psi(x)Y''\right) \tag{56}$$

4.6.1.2. VIM

Lagrange multiplier is the same as used in the uniform column case due to the fourth order derivative in Eq.(38). Hence, an algorithm by using VIM can be constructed as

$$Y_{n+1} = Y_n + \int_0^x \lambda(\xi)\left\{Y_n^{iv} - 8a\tilde{Y}_n''' + \left(16a^2 + \zeta\right)\psi(x)\tilde{Y}_n''\right\} d\xi \tag{57}$$

4.6.1.3. HPM

Application of HPM produces the following set of recursive equations as the solution algorithm.

$$Y_0^{iv} - y_0^{iv} = 0 \tag{58}$$

$$Y_1^{iv} + y_0^{iv} - 8aY_0''' + \left(16a^2 + \zeta\right)\psi(x)Y_0'' = 0 \tag{59}$$

$$Y_n - 8aY_{n-1}''' + \left(16a^2 + \zeta\right)\psi(x)Y_{n-1}'' = 0, \quad n \geq 2 \tag{60}$$

4.6.2. Results of the analyses

The proposed formulations are applied for two different variations, *i.e.* $aL = 0.1$ and $aL = 0.2$. Twenty iterations are conducted for each method, and the computed normalized buckling load PL^2 / EI_0 values are given for the first four modes of buckling in Tables 2 and 3.

Mode	*ADM*	*VIM*	*HPM*
1	16.47361380	16.47361380	16.47361380
2	48.69674135	48.69674135	48.69674135
3	97.02096924	97.02096916	97.02096921
4	161.45155447	161.45151518	161.45150000

Table 2. Comparison of normalized buckling loads (PL^2 / EI_0) for $aL = 0.1$

Mode	*ADM*	*VIM*	*HPM*
1	13.35006457	13.35006457	13.35006457
2	39.47004813	39.47004813	39.47004813
3	78.64155457	78.64155458	78.64155466
4	130.86858532	130.86856343	130.86853842

Table 3. Comparison of normalized buckling loads (PL^2 / EI_0) for $aL = 0.2$

The buckling mode shapes of the rod for the first four modes are depicted in between Figs.5-9. To demonstrate the effect of variable cross-section in the results, a comparison is made with normalized mode shapes for a uniform rod which are given in Fig.4. Constant flexural rigidity is defined as $aL = 0.1$ in these figures.

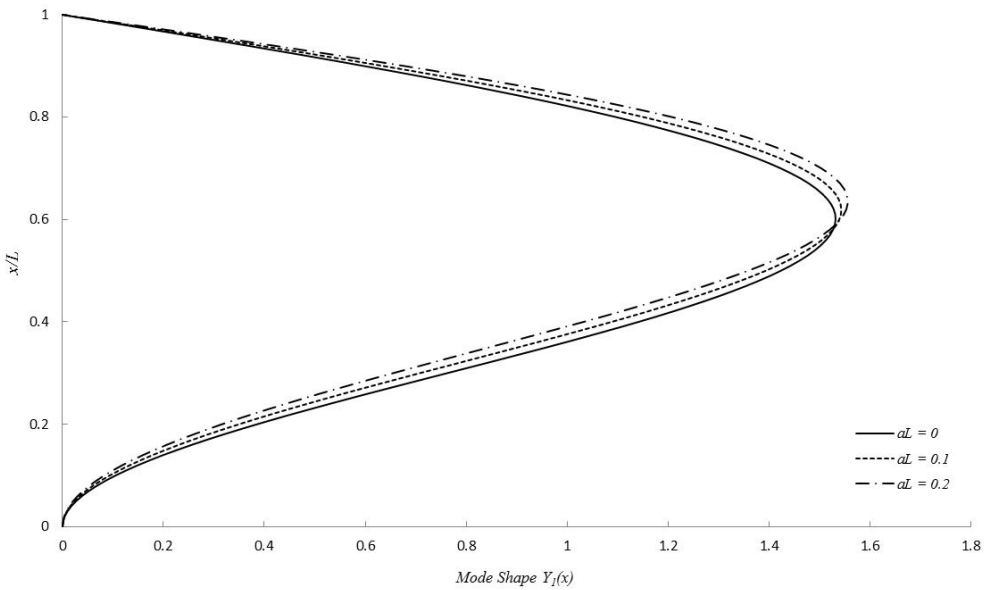

Figure 6. Comparison of buckling modes for *CP* rod (Mode 1)

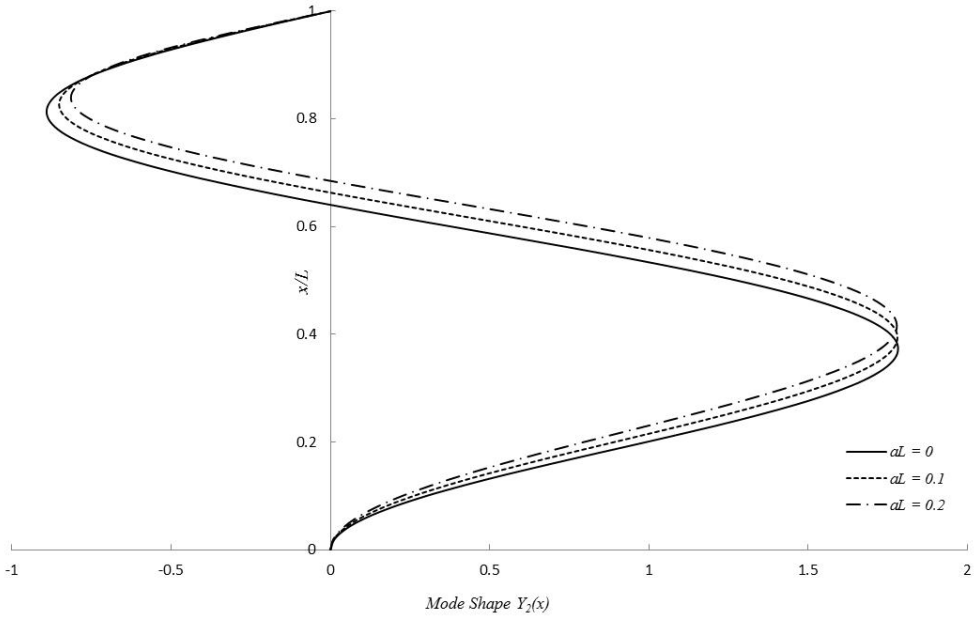

Figure 7. Comparison of buckling modes for *CP* rod (Mode 2)

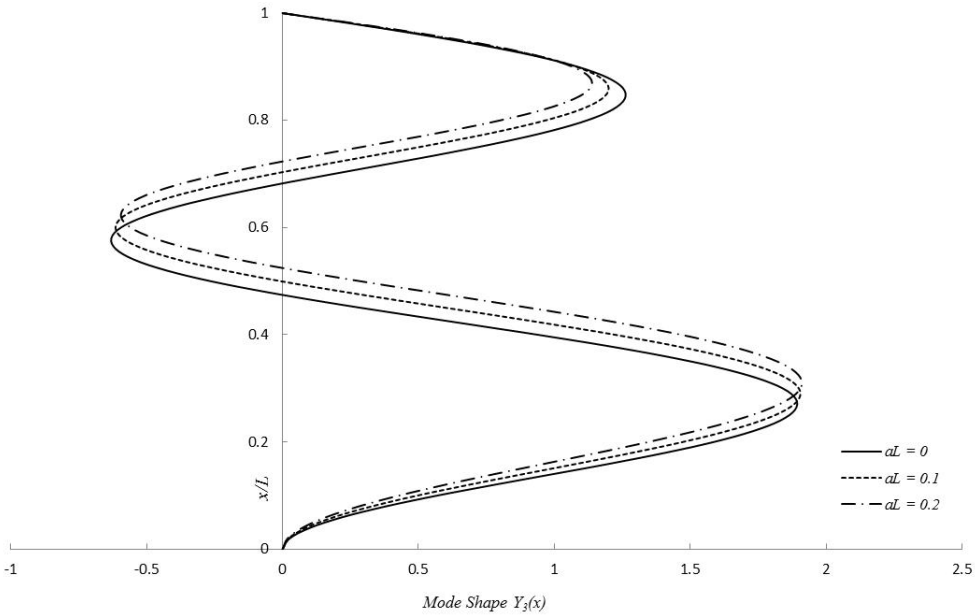

Figure 8. Comparison of buckling modes for *CP* rod (Mode 3)

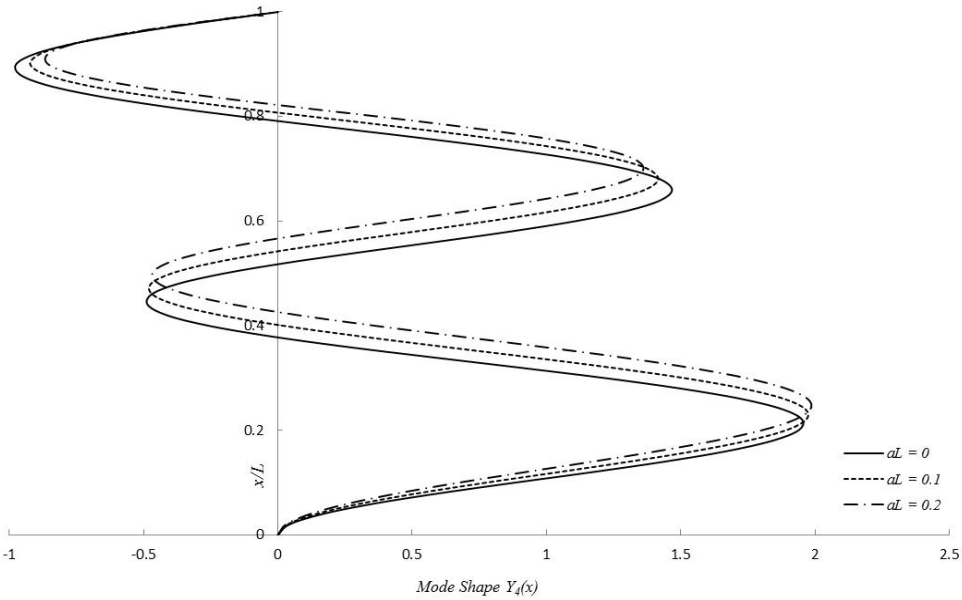

Figure 9. Comparison of buckling modes for *CP* rod (Mode 4)

5. Conclusion

In this article, some analytical approximation techniques were employed in the elastic stability analysis of Euler columns. In a variety of such methods, ADM, VIM and HPM are widely used, and hence chosen for use in the computations. Firstly, a brief theoretical knowledge was given in the text, and then all of the methods were applied to the selected cases. Since the exact values for the buckling of a uniform rod were available, the analyses were initially conducted for that case. Results showed an excellent agreement with the exact ones that all three methods were highly effective in the computation of buckling loads and corresponding mode shapes. Finally, ADM, VIM and HPM were applied to the buckling of a rod having variable cross section. To this aim, a rod with exponentially varying radius was chosen and buckling loads with their corresponding mode shapes were obtained easily.

This study has shown that ADM, VIM and HPM can be used effectively in the analysis of elastic stability problems. It is possible to construct easy-to-use algorithms with these methods which are highly accurate and computationally efficient.

Author details

Safa Bozkurt Coşkun
Kocaeli University, Faculty of Engineering, Department of Civil Engineering Kocaeli, Turkey

Baki Öztürk
Niğde University, Faculty of Engineering, Department of Civil Engineering Niğde, Turkey

6. References

[1] Vaziri H. H., and Xie. J., Buckling of columns under variably distributed axial loads, Comput. Struct., 45(3) (1992), 505-509.

[2] Dinnik, A.N., Design of columns of varying cross-section, Trans. ASME 51, 1929.

[3] Karman, T.R, Biot, M.A., Mathematical Methods in Engineering, McGraw Hill, NewYork, 1940.

[4] Morley A., Critical loads for long tapering struts, Engineering, 104-295, 1917.

[5] Timoshenko, S.P., Gere, J.M., Theory of Elastic Stability, McGraw Hill, NewYork, 1961.

[6] G.J. Simitses , D. H. Hodges, Fundamentals of structural Stability, Elsevier-Butterworth-Heinemann Publishing, 2005.

[7] Iyengar NGR, Structural Stability of Columns and Plates. New York, John Wiley and Sons, 1988

[8] Wang, C.M., Wang, C.Y., Reddy, J.N., Exact Solutions for Buckling of Structural Members, CRC Press LLC, Florida, 2005.

[9] Ermopoulos, J. C., Buckling of tapered bars under stepped axial loading, J. structural Eng. ASCE, 112(6) (1986), 1346-1354.

[10] Li Q. S., exact solutions for buckling of non-uniform columns under axial concentrated and distributed loading, Eur. J. Mech. A / Solids, 20(2001), 485-500.

[11] Lee S.Y. and Kuo Y. H., Elastic stability of non-uniform columns, J. Sound Vibration, 148(1) (1991), 11-24

[12] Gere, J.M., Carter, W.O., Critical buckling loads for tapered columns, J. Struct. Eng.-ASCE, 88(1962), 1-11.

[13] Arbabi. F., Li, F., Buckling of variable cross-section columns – integral equation approach, J. Struct. Eng.-ASCE, 117(8) (1991), 2426-2441.

[14] Kerekes F., Hulsbos CL., Elastic stability of the top chord of a three-span continuous pony truss bridge , Iowa Eng. : Expt. Sta. Bull., (1954), 177.

[15] Siginer, A., Buckling of columns of variable flexural stiffness, J. Eng. Mech.-ASCE, 118(1992), 543-640.

[16] Li, Q.S., Cao, H., Li, G., Stability analysis of bars with multi-segments of varying cross section, Comput. Struct., 53 (1994), 1085-1089.

[17] Li, Q.S., Cao, H., Li, G., Stability analysis of bars with varying cross section, Int. J. Solids Struuct., 32 (1995), 3217-3228.

[18] Li, Q.S., Cao, H., Li, G.,Static and dynamic analysis of straight bars with variable cross-section, Comput. Struct., 59 (1996), 1185-1191.

[19] Sampaio Jr., J.H.B, Hundhausen, J.R.,A mathematical model and analytical solution for buckling of inclined beam columns, Appl. Math. Model., 22 (1998), 405-421.

[20] Keller, J.B.,The shape of the strongest column, Archive for Rational Mechanics and Analysis, 5 (1960), 275-285.

[21] Tadjbakhsh, I., Keller., J.B., Strongest columns and isoperimetric inequalities for eigenvalues, J. Appl. Mech. ASME, 29 (1962), 159-164.

[22] Taylor, J.E.,The strongest column – an energy approach, J. Appl. Mech. ASME, 34 (1967), 486-487.

[23] Coskun, S.B., Atay, M.T., Determination of critical buckling load for elastic columns of constant and variable cross-sections using variational iteration method, , Computers and Mathematics with Applications, 58 (2009), 2260-2266.
[24] Atay, M.T., Coskun, S.B., Elastic stability of Euler columns with a continuous elastic restraint using variational iteration method, Computers and Mathematics with Applications, 58 (2009), 2528-2534.
[25] Coskun, S.B., Determination of critical buckling loads for Euler columns of variable flexural stiffness with a continuous elastic restraint using Homotopy Perturbation Method, Int. Journal Nonlinear Sci. and Numer. Simulation , 10(2) (2009), 191-197.
[26] Coskun, S.B., Analysis of Tilt-Buckling of Euler Columns with Varying Flexural Stiffness Using Homotopy Perturbation Method, Mathematical Modelling and Analysis, 15(3) (2010), 275-286.
[27] Pinarbasi, S, Stability analysis of nonuniform rectangular beams using homotopy perturbation method, Mathematical Problems in Engineering, 2012 (2012), Article ID.197483.
[28] Safa Bozkurt Coşkun, Mehmet Tarik Atay and Baki Öztürk, Transverse Vibration Analysis of Euler-Bernoulli Beams Using Analytical Approximate Techniques, Advances in Vibration Analysis Research, Dr. Farzad Ebrahimi (Ed.), ISBN: 978-953-307-209-8, InTech, 2011.
[29] Ozturk, B., Coskun, S.B., The Homotopy Perturbation Method for free vibration analysis of beam on elastic foundation, Structural Engineering and Mechanics, 37(4) (2011), 415-425.
[30] Adomian, G., Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, MA, 1994.
[31] Adomian, G., A review of the decomposition method and some recent results for nonlinear equation, Math. Comput. Modell., 13(7) (1992), 17-43.
[32] He, J.H., Variational iteration method: a kind of nonlinear analytical technique, Int. J. Nonlin. Mech., 34 (1999), 699-708.
[33] He, J.H., A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Int. J. Nonlin. Mech., 35 (2000), 37-43.
[34] He, J.H., An elemantary introduction to the homotopy perturbation method, Computers and Mathematics with Applications, 57 (2009), 410-412.
[35] He, J.H., New interpretation of homotopy perturbation method, International Journal of Modern Physics B, 20 (2006), 2561-2568.
[36] He, J.H., The homotopy perturbation method for solving boundary problems, Phys. Lett. A, 350 (2006), 87-88.