UNIQUELY PARTITIONABLE PLANAR GRAPHS
WITH RESPECT TO PROPERTIES HAVING
A FORBIDDEN TREE

JOZEF BUCKO
Department of Mathematics, Technical University
Hlavná 6, 040 01 Košice, Slovak Republic
e-mail: bucko@ccsun.tuke.sk

AND

JAROSLAV IVANČO
Department of Geometry and Algebra
P.J. Šafárik University, Jesenná 5
041 54 Košice, Slovak Republic
e-mail: ivanco@duro.upjs.sk

Abstract

Let \( P_1, P_2 \) be graph properties. A vertex \((P_1, P_2)\)-partition of a graph \( G \) is a partition \( \{V_1, V_2\} \) of \( V(G) \) such that for \( i = 1, 2 \) the induced subgraph \( G[V_i] \) has the property \( P_i \). A property \( R = P_1 \lor P_2 \) is defined to be the set of all graphs having a vertex \((P_1, P_2)\)-partition. A graph \( G \in P_1 \lor P_2 \) is said to be uniquely \((P_1, P_2)\)-partitionable if \( G \) has exactly one vertex \((P_1, P_2)\)-partition. In this note, we show the existence of uniquely partitionable planar graphs with respect to hereditary additive properties having a forbidden tree.

Keywords: uniquely partitionable planar graphs, forbidden graphs.

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1. Introduction

Let us denote by \( I \) the class of all finite undirected graphs without loops and multiple edges. If \( P \) is a proper isomorphism closed subclass of \( I \), then \( P \) will also denote the property that a graph is a member of the set \( P \). We shall use the terms set of graphs and property of graphs interchangeably.
A property $\mathcal{P}$ is said to be hereditary if, whenever $G \in \mathcal{P}$ and $H$ is a subgraph of $G$, then also $H \in \mathcal{P}$. A property $\mathcal{P}$ is called additive if for each graph $G$ all of whose components have the property $\mathcal{P}$ it follows that $G \in \mathcal{P}$, too.

For every hereditary property $\mathcal{P}$ there is a nonnegative integer $c(\mathcal{P})$ such that $K_{c(\mathcal{P})+1} \not\in \mathcal{P}$ but $K_{c(\mathcal{P})+2} \in \mathcal{P}$ called the completeness of $\mathcal{P}$. For example $c(\mathcal{O}) = 0$, $c(\mathcal{D}_1) = 1$, $c(\mathcal{T}_2) = 2$, $c(\mathcal{T}_3) = 3$, where $\mathcal{O}$ is the class of all totally disconnected graphs, $\mathcal{D}_1$ is the class of acyclic graphs, $\mathcal{T}_2$ is the class of outerplanar graphs and $\mathcal{T}_3$ is the class of planar graphs.

Any hereditary property $\mathcal{P}$ is uniquely determined by the set

$$\mathcal{F}(\mathcal{P}) = \{ G \in \mathcal{I} | G \notin \mathcal{P}, \text{but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P} \}$$

of its minimal forbidden subgraphs.

Let $\mathcal{P}_1, \mathcal{P}_2$ be arbitrary hereditary properties of graphs. A vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition of a graph $G$ is a partition $\{ V_1, V_2 \}$ of $V(G)$ such that for $i = 1, 2$ the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$.

A property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is defined to be the set of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition. It is easy to see that if $\mathcal{P}_1, \mathcal{P}_2$ are additive and hereditary, then $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ is additive and hereditary, too.

A graph $G \in \mathcal{P}_1 \circ \mathcal{P}_2$ is said to be uniquely $(\mathcal{P}_1, \mathcal{P}_2)$-partitionable if $G$ has exactly one (unordered) vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition. For the concept of uniquely partitionable graphs we refer the reader to [1]. Basic properties of uniquely partitionable graphs are discussed in [1] and [4].

**Proposition 1** [1]. Let $\mathcal{P}$ be an additive hereditary property. Then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable graph $G$ if and only if $\mathcal{P} \neq \mathcal{O} \circ \mathcal{Q}$.

The proof used non-planar graphs. The constructions of uniquely $(\mathcal{O}, \mathcal{P})$-partitionable outerplanar and planar graphs were presented in [2]. The following results have been proved:

**Proposition 2** [2]. Let $\mathcal{P}$ be an additive hereditary property of completeness 1. Then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable outerplanar graph $G$ if and only if there is a tree $T$ which is forbidden for $\mathcal{P}$.

**Proposition 3** [2]. Let $\mathcal{P}$ be an additive hereditary property of completeness 1. Then there exists a uniquely $(\mathcal{O}, \mathcal{P})$-partitionable planar graph $G$ if and only if either some odd cycle $C_{2q+1}$ has property $\mathcal{P}$ or there is a bipartite planar graph $H$ which is forbidden for $\mathcal{P}$. 
Our first result shows that the restriction on the completeness is not necessary for the existence of uniquely \((\mathcal{O}, \mathcal{P})\)-partitionable planar graphs.

**Theorem 1.** Let \(\mathcal{P}\) be an additive hereditary property. If there is a tree \(T \in F(\mathcal{P})\), then there exists a uniquely \((\mathcal{O}, \mathcal{P})\)-partitionable planar graph.

Furthermore, let us consider \((\mathcal{D}_1, \mathcal{D}_1)\)-partitions of planar graphs. The following result is presented in [3]:

**Proposition 4** [3]. There are no uniquely \((\mathcal{D}_1, \mathcal{D}_1)\)-partitionable planar graph.

In this note, we shall show that the property \(\mathcal{D}_1 \circ \mathcal{D}_1\) is in some sense “a minimal property” having no uniquely partitionable planar graphs. More precisely, we will prove the following result:

**Theorem 2.** Let \(\mathcal{P}, \mathcal{Q}\) be the additive hereditary properties of graphs with completeness 1. If there is a tree \(T \in F(\mathcal{P})\), then there exists a uniquely \((\mathcal{P}, \mathcal{Q})\)-partitionable planar graph.

2. Proofs of the Main Results

**Proof** of **Theorem 1.** Let \(T\) be a forbidden tree for a property \(\mathcal{P}\). As every connected bipartite planar graph is uniquely \((\mathcal{O}, \mathcal{O})\)-partitionable, we can assume that \(T\) has at least 3 vertices. Then \(T\) contains a path \(wuv_1\), where \(v_1\) is an end vertex of \(T\). Denote by \(T'\) the graph which we obtain from \(T\) by adding the edge \(wv_1\). \(T'\) is outerplanar and so the join \(K_1 + T'\) is a planar graph. Let \(G(T, 1)\) be the graph which we obtain from \(K_1 + T'\) by deleting the edge \(av_1\), where \(a\) denotes the vertex of \(K_1\). Evidently, \(G(T, 1)\) may be embedded on the plane such that the vertices \(a\) and \(v_1\) lie in the exterior face (see Figure 1).

![Figure 1](image-url)
$G(T, k)$, for $k > 1$, is a planar graph which we obtain from $G(T, 1)$ by adding the vertices $v_2, v_3, \ldots, v_k$ and edges $uv_2, uv_3, \ldots, uv_k, wv_2, wv_3, \ldots, wv_k$. The vertex $a$ is called the root of $G(T, k)$ and vertices $v_1, v_2, \ldots, v_k$, are called leaves of $G(T, k)$. Moreover, for every leaf $v_i$ we define its successor $s(v_i)$ by $s(v_1) = a$ and $s(v_i) = v_{i-1}$, if $i = 2, 3, \ldots, k$. Obviously, $G(T, k)$ may be embedded on the plane such that both vertices $v_i$ and $s(v_i)$ lie in a common face (see Figure 1).

Now we construct a planar graph $H(T, k, d)$ using the induction on $d$. $H(T, k, 1)$ is a graph which we obtain from $k$ copies of $G(T, k)$ by identifying their roots. The vertex arisen by the identification is called the root of $H(T, k, 1)$. The leaves of copies of $G(T, k)$ are leaves of $H(T, k, 1)$. Similary, the successor of a leaf in $H(T, k, 1)$ is equal to the successor of this leaf in the corresponding copy of $G(T, k)$. For $d > 1$, $H(T, k, d)$ is a planar graph which we obtain from $H(T, k, 1)$ and $k^2$ copies of $H(T, k, d - 1)$ by identifying each leaf of $H(T, k, 1)$ with the root of a copy of $H(T, k, d - 1)$. Evidently, a copy of $H(T, k, d - 1)$ can be inserted into a face of $H(T, k, 1)$ which contains a corresponding leaf $x$ of $H(T, k, 1)$ and its successor $s_1(x)$ in $H(T, k, 1)$ (see Figure 2).
The root of $H(T, k, d)$ is the root of $H(T, k, 1)$ and the leaves of $H(T, k, d)$ are leaves of copies of $H(T, k, d - 1)$. Denote by $s_d(y)$ and $s_{d-1}(y)$ the successor of a leaf $y$ in $H(T, k, d)$ and in a corresponding copy of $H(T, k, d - 1)$. Then

$$s_d(y) = \begin{cases} s_1(x), & \text{if } s_{d-1}(y) \text{ was identified with } x, \\ s_{d-1}(y), & \text{otherwise.} \end{cases}$$

Finally, $H^*(T, k, d)$ is a planar graph which we obtain from $H(T, k, d)$ such that we connect each leaf of $H(T, k, d)$ with its successor by a copy of $G(T, 1)$ identifying the leaf with the root of $G(T, 1)$ and the successor with the leaf of $G(T, 1)$ (see Figure 2).

Put $V_1 = \{ x \in V(H^*(T, k, d)) \mid d(r, x) \equiv 0 \pmod{2} \}$, where $r$ denotes the root of $H(T, k, d)$ and $d(y, z)$ is the length of the shortest path between $y$ and $z$ in $H(T, k, d)$. The vertices belonging to $V_1$ are depicted by white in Figure 2. It is easy to see that $V_1$ is an independent set of $H^*(T, k, d)$. Moreover, the set $V_2 = V(H^*(T, k, d)) - V_1$ induces a subgraph of $H^*(T, k, d)$ each of whose components is isomorphic to $T - v_1$. So, $\{V_1, V_2\}$ is a vertex $(O, P)$-partition of $H^*(T, k, d)$.

Suppose that $\{U_1, U_2\}$ is a vertex $(O, P)$-partition of $H^*(T, k, d)$. Consider two cases:

Case 1. $U_1 \cap V_1 \neq \emptyset$. Let $x \in U_1 \cap V_1$ and let $y$ be any vertex of $V_1 - \{x\}$. From the construction of $H^*(T, k, d)$ it can easily be seen that there exists
a sequence \( x = x_1, x_2, \ldots, x_t = y \) satisfying: For every \( i = 1, \ldots, t - 1 \), there is a subgraph \( G_i \) of \( H^*(T, k, d) \) isomorphic to \( G(T, k) \) (or \( G(T, 1) \)), where \( x_i \) is its root and \( x_{i+1} \) is its leaf. As \( x_1 \) belongs to \( U_1 \), all vertices of \( G_1 \) adjacent to \( x_1 \) belong to \( U_2 \). However, these neighbours of \( x_1 \) together with \( x_2 \) induce a subgraph of \( G_1 \) containing \( T \). Therefore, \( x_2 \in U_1 \), and by induction, \( y \in U_1 \). Since \( y \) is any vertex of \( V_1 \setminus \{ x \} \), \( V_1 \subseteq U_1 \). The set \( V_1 \) is a domination set of \( H(T, k, d) \), and so, \( V_1 = U_1 \), i.e., \( \{ U_1, U_2 \} = \{ V_1, V_2 \} \).

**Case 2.** \( V_1 \subseteq U_2 \). It is easy to see that every block of \( H(T, k, d) \) is a copy of \( G(T, k) \), where the root and leaves of the copy belong to \( V_1 \). As the vertices of a block corresponding to \( u \) and \( w \) are adjacent, at least one of them belongs to \( U_2 \). Thus, vertices of a block belonging to \( U_2 \) induce a graph containing a star \( K_{1,k+1} \). From the construction of \( H(T, k, d) \) one can see that vertices of \( H(T, k, d) \) belonging to \( U_2 \) induce a graph containing a complete \( k \)-ary tree with \( 2d+1 \) levels. Therefore, for \( k \geq \Delta(T) \) and \( d \geq \frac{1}{2}\text{rad}(T) \), \( H^*(T, k, d)[U_2] \) contains a subgraph isomorphic to \( T \), a contradiction. Thus, for \( k \geq \Delta(T) \) and \( d \geq \frac{1}{2}\text{rad}(T) \), the graph \( H^*(T, k, d) \) is uniquely \((O, P)\)-partitionable.

**Proof of Theorem 2.** To construct the planar graph \( H_r(s) \), for \( r \geq 1, s \geq 2 \) we will use the induction on \( r \). The first step is the construction of planar graph \( H_1(s) \):

\[
H_1(s) = K_2 + \bigcup_{i=1}^s K_2, \quad \text{where} \quad V(K_2) = \{x_1, x_2\} \quad \text{and} \quad V(\bigcup_{i=1}^s K_2) = \{y_{1i}, y_{2i} \mid i = 1, 2, \ldots, s\}.
\]

The edge \( x_1x_2 \) of \( H_1(s) \) we will call the "major" edge of \( H_1(s) \) and edges \( y_{1i}y_{2i}, i = 1, 2, \ldots, s \) we will call "minor" edges of \( H_1(s) \). For the construction of \( H_1(3) \) see Figure 3.

![Figure 3. The graph \( H_1(3) \)](image-url)
Let us construct the graph $H_{k+1}(s)$ in the following way:

We insert $s$ copies of graph $H_k(s)$ to graph $H_1(s)$ such that we identify the "major" edges of copies of graphs $H_k(s)$ with "minor" edges of $H_1(s)$. For the construction of $H_2(3)$ see Figure 4.

It is easy to see from the construction, that $H_r(s)$ is a planar graph. Now we shall show, that if the maximum degree $(\Delta)$ of the tree $T \in F(P)$ is $\Delta(T) \leq s$ and radius $rad(T)$ of the tree $T$ is $rad(T) \leq r$, then the planar graph $H_r(s)$ is uniquely $(P, Q)$-partitionable.

Let us distinguish two "possible" vertex partitions of the graph $H_r(s)$:

1. The end vertices $x_1, x_2$ of "major" edge of $H_1(s)$ belong to different classes of the vertex partition. From the fact that $K_3$ is forbidden for both properties $P, Q$, it follows that vertices of "minor" edges of $H_1(s)$ belong to different classes of the vertex partition, too. By induction on $r$ in both classes of the partition, it grove the complete $s$-ary tree with $1 + r$ levels, which is, for $r \geq rad(T)$ and $s \geq \Delta(T)$, a supergraph of the forbidden tree $T$. It means, that it is not a $(P, Q)$-partition of $H_r(s)$.

2. Hence the end vertices $x_1, x_2$ of "major" edge of the graph $H_1(s)$ have to belong to the same class of a vertex partition. From the fact that $K_3$ is forbidden for both properties $P, Q$, it follows, that vertices of "minor" edges of $H_1(s)$ have both to belong to the second class of the vertex partition. From
the construction of $H_r(s)$ and from the fact that $K_3$ is forbidden it is easy to see that the partition of $H_r(s)$ is a $(\mathcal{P}, \mathcal{Q})$-partition of $H_r(s)$. Thus $H_r(s)$, for $r \geq \text{rad}(T)$ and $s \geq \Delta(T)$ is a uniquely $(\mathcal{P}, \mathcal{Q})$-partitionable graph. ■

References


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