INFORMATION LENGTH AND LOCALIZATION IN ONE DIMENSION

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Abstract - The scaling properties of the wave functions in finite samples of the one dimensional Anderson model are analyzed. The states have been characterized using a new form of the information or entropic length, and compared with analytical results obtained by assuming an exponential envelope function. A perfect agreement is obtained already for systems of 10^3–10^4 sites over a very wide range of disorder parameter $10^{-4} < W < 10^4$. Implications for higher dimensions are also presented.

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The numerical detection of exponential localization in finite, random systems is not a trivial task, especially in the weak disorder limit when the localization length of the eigenstates is expected to be larger than the system size. This may be a problem even in one dimension (1D), where rigorous results [1] affirm complete exponential localization for any strength of disorder.

The model under consideration can be described by a tight–binding Schrödinger equation in the nearest neighbor approximation as

$$u_{n+1} + u_{n-1} + V_n u_n = E u_n, \tag{1}$$

where V_n are independent random variables with uniform distribution over the $[-W/2 \dots W/2]$ interval, E is the eigenenergy and u_n is the amplitude of the eigenfunction on site n. The wave functions are expected to behave asymptotically up to oscillations as

$$u_n \sim \exp(-\gamma n), \tag{2}$$

where $\gamma = \xi^{-1}$ is the inverse localization length or Lyapunov exponent that may be numerically obtained as [1]

$$\gamma \approx \gamma_N(E) = \frac{1}{N} \sum_{i=1}^{N} \ln \left| \frac{u_n}{u_{n-1}} \right| . \tag{3}$$

The exponential localization has been corroborated by the one–parameter scaling theory [2]. This was built essentially on the basis of the Thouless number [3] that is related to the dimensionless conductance [4]. Inspired by that scaling law in a recent paper Casati *et al.* [5] have introduced the concept of *information localization length* [6] and investigated numerically the possible scaling properties of the eigenstates themselves. Their study has also been motivated by previous results obtaining a scaling law for an analogous model in quantum chaos, the kicked rotator [7], as well as for band random matrices [8] using the same concept.

The main idea of Ref. [5] is to take a suitable ensemble of eigenstates and to calculate the information length as

$$\beta_C(E, N, W) = \exp(\overline{S} - S_{ref}), \tag{4}$$

where N stands for the system size, W describes the strength of disorder. Index C is used to label β in order to refer to the definition of Casati *et al.* $\overline{S}$ is the averaged information entropy,

$$S = -\sum_{n=1}^{N} u_n^2 \ln u_n^2, \tag{5}$$

of normalized eigenstates in a window around energy E for different realizations of the random potential. The S_{ref} in Eq. (1) stands for the entropy of a reference state

$$u_n \sim \sin(\varphi n) \qquad \text{where} \qquad \cos\varphi = E/2. \tag{6}$$

This wave function is the exact solution of Eq. (1) in the absence of disorder ($W = 0$) with $u_0 = 0$ and $u_1 = 1$. A straightforward calculation yields the asymptotic form $S_{ref}(N) \to \ln(2N) - 1$ as $N \to \infty$. We have to indicate that the particular choice of S_{ref} in Eq. (4) involves a delicate problem, that we wish to discuss below.

The principal aim of definition (4) is that β_C should give the portion of the sites significantly populated by the eigenstates compared to that of the reference state (6). It is clear that with the increase of the system size N we expect $\exp(S) \propto N$ in the case of extended states and $\exp(S) \rightarrow$ const. for localized states. Note, that since for large N, $\exp(-S_{ref}) \sim c/N$, its role is normalization.

Casati *et al.* have numerically established the scaling law [5]

$$\ln \frac{\beta_C}{1-\beta_C} = \ln\left(\frac{\xi_\infty}{N}\right) + C \tag{7}$$

with $C \approx 1$. ξ_∞ has been calculated as $\xi_\infty = 1/\gamma_N$ (cf. Eq. (3)) for strong disorder ($1/\gamma_N \leq N$). For weak disorder ($1/\gamma_N > N$) instead of numerically calculating the Lyapunov exponent the authors of Ref. [5] used ξ_∞ given by the perturbative calculation [9]. They have found, however, no theoretical explanation for (7).

We would like to point out that the approach of Casati *et al.* [5] resides on the supposition that in Eq. (7) $\beta_C < 1$ i.e. $\overline{S} < S_{ref}$. There is, however, no rigorous proof for these relations. This is particularly crucial in the weak localization limit $W \rightarrow 0$, $\beta_C \approx 1$. Indeed, in actual numerical calculations for small ensembles we have found occasionally $\beta_C > 1$, as well. We could prove, however, that $\overline{S} \leq S_{ref}(N)$ for sufficiently large ensembles. The proof is based on the idea that any small perturbation $\{Q'_n = Q_n(1+\delta_n)\}$, $\overline{\delta}_n = 0$ of an arbitrary probability distribution $\{Q_n\}$ leads to $\overline{S}' \leq S$, where $\overline{S}'$ is an average over an infinite number of realizations of δ_n. As a byproduct, it is also evident that for the calculation of β_C, $S_{ref} = S_{ref}(N)$ has to be applied instead of using the asymptotic form $\ln(2N) - 1$.

Furthermore, as we will show it later, for finite systems the functional form of the scaling law (7) does not seem to be valid in the weak localization limit ($W \rightarrow 0$). This is mainly due to the improper application of the perturbative treatment.

In this Letter we propose a new form of β for resolving the above mentioned difficulties and appropriately handling the $W \to 0$ limit. We will demonstrate that our β function is, apart from showing special scaling properties, capable to prove the exponential localization in one dimension.

On the basis of our recently introduced classification scheme [10], the information entropy (5) of any general, normalized, nonnegative lattice distribution can be split as a sum of two terms

$$S = S_{str} + \ln D, \tag{8}$$

where D is the delocalization measure [11] or participation number [12]

$$D = \left(\sum_{n=1}^{N} u_n^4 \right)^{-1}, \tag{9}$$

and S_{str} is the *structural entropy* of the distribution. Parameter D is widely used in the literature giving the number of sites the eigenstate extends to. Therefore it is bounded as $1 \leq D \leq N$. Using D we may introduce a normalized quantity q, the spatial filling factor or participation ratio as

$$q = D/N, \qquad \text{for which} \qquad 0 < q \leq 1. \tag{10}$$

The structural entropy in equation (8) has been shown [10] to be nonnegative with bounds

$$0 \leq S_{str} \leq -\ln q. \tag{11}$$

Using the quantities discussed above we now propose an alternative form of the normalized information length of an eigenstate as

$$\beta = \frac{1}{N} \exp S, \tag{12}$$

which using expressions (8) (9) and (10) becomes

$$\beta = q \exp(S_{str}). \tag{13}$$

Due to the well-known properties $0 \leq S \leq \ln N$ the following restrictions are imposed on β

$$0 < \beta \leq 1. \tag{14}$$

These bounds are valid for each state separately, whereas $\beta_C \leq 1$ can only be guaranteed after an appropriate averaging process. Obviously, an eigenstate expanding uniformly over the whole system will have $u_n^2 = 1/N$ so that $\beta = 1$. In the other extreme, for localized states $D \sim 1$ and $S_{str} \approx 0$, in a finite system one obtains $\beta \approx 1/N$.

Returning now to the Anderson model (1) with non-zero disorder $W \neq 0$, one expects according to (2) and (6) the charge distribution of the solution of the form

$$|u_n|^2 \sim f(\gamma n) \sin^2(\alpha n), \tag{15}$$

where f is a slowly varying envelope function due to the presence of the perturbing random potential. Obviously, in our case function f is expected to take an exponential form $f(\rho) = \exp(-\rho)$. The value of α is roughly φ defined in (6). Further specification of α is needless for our purposes, the only restriction we impose is $\alpha \gg 1/N$ that is always fulfilled except very close to the band-edges. As we have shown [10,13] for multiplicative superstructures of the form (15) we get

$$\ln q = \ln q^f + \ln q^0 \tag{16}$$

and

$$S_{str} = S_{str}^f + S_{str}^0, \tag{17}$$

where the upper index f stands for the values obtained for the charge distribution $f(\gamma n)$ alone and upper index 0 indicates the ones for $\sin^2(\alpha n)$. It is possible to show that (independently of α)

$$q^0 = 2/3 \qquad \text{and} \qquad S_{str}^0 = \ln 3 - 1. \tag{18}$$

Using (16) and (17) as well as definition (13) we get

$$\beta = \beta_f \, \beta_0, \tag{19}$$

where $\beta_0 = q^0 \exp(S_{str}^0) \approx 0.7357$. In the limit of vanishing disorder $W \to 0$ one expects $\gamma \to 0$, $f(\rho) \to 1$. For such distribution using (5), (8) and (9) $q^f \to 1$ and $S_{str}^f \to 0$ therefore $\beta_f \to 1$. For strong disorder on the other hand

$\gamma \gg 1$ which yields $D \approx 1$, therefore in finite systems $q^f \approx 1/N$ and $S_{str}^f \approx 0$ resulting $\beta_f \approx 1/N$.

The role of β_0 in (19) is similar to the factor $\exp(S_{ref})$ in (4) introduced by Casati *et al.* [5], however, β_0 is a constant independent of the system size N, and its derivation is based on the separation of the wave function to an envelope and a strongly oscillating part (15).

Before performing the numerical simulation we still have to give the explicit q^f and S_{str}^f values as a function of γ. In Ref. [10] the general form of $q^f(z)$ and $S_{str}^f(z)$ functions with $z = \gamma N = N/\xi$ is given for arbitrary dimensionality, that have been calculated applying a continuous lattice approximation, i.e. the relevant scale (γ^{-1}) was assumed to extend over many lattice spacings. It has been shown in [10] that in one dimension

$$q^f(z) = \frac{[F(z)]^2}{z\, G(z)}, \tag{20a}$$

and

$$S_{str}^f(z) = \frac{H(z)}{F(z)} + \ln\left(\frac{G(z)}{F(z)}\right), \tag{20b}$$

where functions $F(z)$, $G(z)$, and $H(z)$ are defined as

$$F(z) = \int_0^z f(\rho)\, d\rho, \tag{21a}$$

$$G(z) = \int_0^z f^2(\rho)\, d\rho, \tag{21b}$$

$$H(z) = -\int_0^z f(\rho) \ln[f(\rho)]\, d\rho. \tag{21c}$$

Inserting the general functions given in expressions (21) into (20) one obtains for β_f in the continuous limit

$$\beta_f(z) = q^f(z) \exp(S_{str}^f(z)) = z^{-1} F(z) \exp\left(\frac{H(z)}{F(z)}\right). \tag{22}$$

It is straightforward to calculate the $\beta_f(z)$ function for any envelope shape $f(\rho)$. For exponential decay, $f(\rho) = \exp(-\rho)$, we get

$$\beta_{\exp}(z) = \frac{\exp z - 1}{z \exp z} \exp\left(1 - \frac{z}{\exp z - 1}\right). \tag{23}$$

Expressions (22) and (23) are the principal results of this Letter showing the scaling property of β_f provided that a reasonable definition for the $f(\rho)$ decay function (15) exists.

Let us turn now to the asymptotic properties of $\beta_f(z)$. In the case of strong localization $z \to \infty$ ($N \gg \xi$), since both $F(\infty)$ and $H(\infty)$ are finite for most of the practical cases, from the general expression (22) one gets

$$\beta_f(z) \sim z^{-1} \sim \frac{\xi}{N}, \tag{24}$$

as expected. In the other limit of delocalization as $z \to 0$ (e.g. $\xi \to \infty$ keeping N fixed) we found that the asymptotic form of $\beta_f(z)$ is governed by the short range properties of the form function $f(\rho)$, i.e. it depends on the derivative of $f(\rho)$ at the origin $\rho = 0$. Namely, if $f'(0) \neq 0$ then

$$\beta(z) \approx 1 - \frac{1}{24} z^2, \tag{25}$$

while for a Gaussian form function e.g. where $f'(0) = 0$ the first nonvanishing term is of the order of z^4.

Instead of the $\beta_f(z)$, after Ref. [5], we define

$$y(z) = \frac{\beta_f(z)}{1 - \beta_f(z)}, \tag{26}$$

in order to emphasize both the localized and delocalized limits. In our numerical simulation we have compared Eq. (26) for exponential form function (22) with the calculated

$$y = \frac{\beta_f}{1 - \beta_f} \tag{27}$$

values, where β_f is defined here as

$$\beta_f = \frac{1}{\beta_0} \overline{q} \exp \overline{S}_{str}. \tag{28}$$

As $0 < \beta_f \leq 1$ is true already for individual wave functions it is not necessary to perform averaging over states taken from an energy window, however, at a certain energy we calculate the statistical means $\overline{q}$ and $\overline{S}_{str}$ over many realizations of the random potential.

The eigenvectors in our simulation were obtained by the iteration of the recurrence relation of Eq. (1) with initial conditions $u_0 = 0$ and $u_1 = 1$. In all of the presented results the length of the system was $N = 10^4$ and the number of samples used for averaging was $M = 10^3$. The energy was fixed to $E = 0.1$. The localization length was obtained as

$$\xi(E) = \overline{\gamma}(E)^{-1}, \tag{29}$$

were $\gamma(E)$ was calculated according to Eq. (3). In a finite lattice two relevant length scales characterize the system the chain length N and the lattice spacing ($a = 1$). The relation of ξ with respect to these length scales is essential in such type of calculations.

In Figure 1 we have plotted the localization length as a function of the strength of disorder ranging from $W = 10^{-4}$ up to $W = 10^4$. Our numerical calculations confirm the theoretically expected behavior $\xi^{-1} \sim \ln W$ for large disorder. In the case of vanishing disorder $W \to 0$ perturbation theory predicts $\xi^{-1} \sim W^2$ in the thermodynamic limit $N \to \infty$ [9]. This consideration, however, fails for finite N as one can see in the low W part of Figure 1, where we have numerically found $\xi^{-1} \sim W$. The figure clearly shows that this finite size behavior becomes relevant for such disorder values where the localization length is comparable to or greater than the system size. It seems that the behavior of the wave function on intermediate length scales is governed by a different characteristic length ξ_i. As is shown in Figure 2 on this latter scale exponential localization can be detected, as well.

In Figure 2 we have compared β_C of Casati *et al.* and our numerical β_f and theoretical $\beta_{\exp}$ as a function of ξ/N. In each of these cases we have plotted $y = \beta/(1-\beta)$ vs ξ/N in a log-log plot. The results of the simulation follow the curve for exponential localization. The deviation for $\ln(\xi/N) \leq -9$ is present because the value of the localization length becomes comparable to the lattice constant as $W \to \infty$ i.e. we obtain $\ln y \to -\ln N$. Both our analytical and numerical results confirm the expected high disorder behavior of $\ln y \approx \ln \beta \sim \ln \xi$ (see e.g. (24)), as well as the low disorder behavior of $\ln y \approx \ln(1-\beta) \sim 2\ln \xi$ (see e.g. (25)). For strong disorder we observe perfect agreement with the scaling law set up by Casati *et al.* [5] in Eq. (7). Our analytical results (23) confirm this scaling law

$$\ln \beta_{\rm exp}(z) \rightarrow -\ln z + 1, \tag{30}$$

in the limit of strong localization $z = N/\xi \rightarrow \infty$.

For weak disorder, however, Figure 2 shows a considerable disagreement between our results and that of Ref. [5]. This is easy to understand considering that for this regime Casati *et al.* have used the predictions of the perturbative calculation valid in the thermodynamic limit. As we pointed out earlier this approach needs a careful analysis. They compare quantities y and ξ_∞ where y is calculated from wave functions characterized by the intermediate length scale ξ_i. On the other hand calculating the ξ according to Eqs. (3) and (29) we obtain an almost perfect agreement between the numerical simulation and our analytical expressions for exponential form function. This shows that, apart from the exponential long range behavior for $\xi_\infty < N$, in the intermediate range ($\xi_\infty > N$) the same kind of decay was found with a different scale constant ξ_i, as well.

Just to have a feeling how well the charge distribution of the form of Eq. (15) describes the average properties of the wave functions in the Anderson model, in Figure 3 we have plotted $\overline{S}_{str}$ as a function of $\overline{q}$. We have compared the results of the simulation with analytical results obtained assuming the form of Eq. (15). A satisfactory agreement can be established between the numerical and analytical results especially for low and high disorder. The charge distribution is clearly not of pure exponential form, but a plane wave modulated by an exponential envelope.

We would like to note that especially for weak disorder, the exponential localization, apart form fluctuations and oscillations, is still strictly true and visible using our construction of β–function at least up to localization lengths several times larger than the size of the system. We believe that after a proper definition of the localization length, a similar procedure could clearly show the expected exponential localization in two dimensions, as well. Results along these lines are to be published in a subsequent paper.

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Figure Captions

Figure 1. The log-log plot of the localization length versus strength of disorder using the numerical simulation. The characteristic relations are also denoted.

Figure 2. Scaling of the entropic length versus the localization length using $\ln y = \ln(\beta/(1-\beta))$ versus $\ln(\xi/N)$. Solid symbols represent the results of our numerical simulation. Dashed line stands for the scaling law found numerically by Casati *et al.* [5]. Note the change in the slope of the continuous curve at around $\xi \approx N$. The continuous curve is our analytical result (see Eqs. (23) and (26)).

Figure 3. The structural entropy versus filling factor in a semi-log plot. The filled symbols represent our simulation, while the solid line stands for the relation assuming a charge distribution of the form of Eq. (15).