A New Approach to the Solution of the ℓ_1 Control Problem: The Scaled-Q Method

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Abstract

In this paper we explore a new approach for solving MIMO ℓ_1 optimal control problems. This approach, which we refer to as the Scaled-Q approach, is introduced in order to alleviate many of the difficulties facing the numerical solution of optimal ℓ_1 control problems. In particular, the computations of multivariable zeros and their directions are no longer required. The Scaled-Q method also avoids the pole-zero cancellation difficulties that existing methods based on zero-interpolation face when attempting to recover the optimal controller from an optimal closed-loop map. Since the Scaled-Q approach is based on solving a regularized auxiliary problem for which the solution is always guaranteed to exist, it can be used no matter where the system zeros are(including the stability boundary). Upper and lower bounds which converge to the optimal cost are provided, and all solutions are based on finite dimensional linear programming for which efficient software exists.

1 Introduction

The ℓ_1 optimal control problem addresses the design of optimal controllers which minimize the peak errors due to unknown bounded disturbances. The mathematical formulation of this objective requires the design of a of a linear time-invariant controller which minimizes the induced ℓ_∞ norm of the transfer function relating the disturbances to the error signals of interest (or any other output to be regulated). Since the induced ℓ_∞ norm of a transfer function corresponds to the ℓ_1 norm of the impulse response, the resulting optimization problem is a ℓ_1 norm minimization problem.

The ℓ_1 norm minimization problem was posed by M. Vidyasagar [1] along similar lines to the H_∞ problem. This problem was also addressed in [2]. The first general solution of the ℓ_1 problem based on linear programs was provided by Dahleh and Pearson [3]. This work utilized duality theory to reduce the infinite dimensional ℓ_1 problem to a finite dimensional linear programming problem, in the one block case. In [4], Mendlovitz showed that once the dual problem indicated the solution is finite

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dimensional, the primal problem can be used directly to obtain the solution. In [5], and later in [6] and [7] approximate solutions to the multi-block ℓ_1 problem were proposed. These methods are based on characterizing the closed loop maps using zero interpolation conditions obtained from the Smith-McMillan form. The work reported in [6] and [7] also addressed the issue of redundancy of constraints arising from the zero interpolation constraints. Up to this point, the methods available provided only converging upper bounds to test for optimality. Duality notions were first used to obtain converging lower bounds which can be used in combination with upper bounds to guarantee nearness to optimality in [8] and [9]. The approach giving rise to converging upper bounds to the optimum was termed the Finitely-Many Variables (FMV) approach, while that providing lower bounds was termed the Finitely-Many Equations Approach (FME). In [10, 11] a geometric approach based on dynamic programming was proposed. The Delay Augmentation method for the ℓ_1 problem was proposed by Diaz-Bobillo and Dahleh in [12] to obtain further information about the structure of the optimal controller, hence avoiding unnecessarily high order controllers.

In the approaches mentioned above based on linear programming, computation of zeros and zeros directions is used to place interpolation conditions on the closed loop transfer function. Aside from the issue of constraint redundancy, computation of the zero directions can cause numerical difficulties. This is especially the case when certain eigenvalues are close to the unit circle [12]. Perhaps the most serious drawback of utilizing zero interpolation conditions is in recovering the optimal controller once the optimal closed-loop function has been constructed. This closed loop function must satisfy the interpolation conditions *exactly* to ensure that the appropriate cancellations take place when solving for the controller. Given that equality constraints in linear programs are only satisfied up to certain tolerances, the task of recovering the optimal controller is complicated by the need to determine which nearby poles and zeros should cancel each other, and which ones should not.

Recently, two new methods which avoid zero interpolation altogether have been introduced [14, 15]. The method in [14] is based on solving a standard H_2 problem and a sequence of finite dimensional semi-definite quadratic programming problems. In this paper, the Scaled-Q approach is presented and its features explored. This method is motivated by our numerical experience which indicates that solving the ℓ_1 problem by directly approximating the optimal Q parameter has nice numerical properties, even in the presence of zeros on the stability boundary. In principle, this is not unlike the Q design approach in [16]. One can easily show that by truncating the Q parameter in the ℓ_1 problem, finite linear programs whose optimal solution converges to the optimal ℓ_1 norm are obtained. By far, the biggest shortcoming of Q truncation approaches has been the lack of converging lower bounds which indicate how close the converging upper bounds are to the optimal solution. The Scaled-Q approach overcomes this difficulty by providing converging lower bounds for the MIMO ℓ_1 optimal solution which, similar to the upper bounds, are obtained without the need for zero interpolations. This is achieved by formulating a related auxiliary (regularized) problem to the ℓ_1 problem for which a solution always exists. Like the ℓ_1 problem itself, the auxiliary problem is infinite dimensional. Utilizing duality theory, finite linear programs obtained from appropriate truncations provide converging upper and lower bounds for the solution of the auxiliary problem. Finally, it is shown how to obtain the solution of the ℓ_1 problem itself from that of the auxiliary problem.

2 Preliminaries

In this section we define some notation. We also set up the spaces needed to pose our problem, and state a duality theorem which will be used in the problem solution.

Let P_N be the truncation operator on sequences. Thus for a given sequence $x = \{x(k)\}_{k=0}^{\infty}$

$$(P_N x)(k) = \begin{cases} x(k) & k \leq N \\ 0 & k > N \end{cases}$$

Let c_0 denote the space of sequences of real numbers which converge to zero. For $x \in c_0$, $\|x\|_{c_0}$ is taken to be $\max_k |x(k)|$. The space $c_0^{m\times n}$ denotes $m \times n$ matrices whose elements are in c_0. The norm of $x \in c_0^{m\times n}$ is taken to be

$$\|x\|_{c_0} := \sum_i \max_j \|x_{ij}\|_{c_0}.$$

Let ℓ_1 denote the space of sequences of real numbers which are absolutely summable. For $x \in \ell_1$, $\|x\|_{\ell_1}$ (usually expressed as $\|x\|_1$) is taken to be $\sum_{k=0}^{\infty} |x(k)|$. The set $\ell_1^{m\times n}$ is the vector space of $m \times n$ matrices whose elements are in ℓ_1. The norm of $x \in \ell_1^{m\times n}$ is

$$\|x\|_1 := \max_i \sum_j \|x_{ij}\|_1.$$

Given a normed vector space X and a real number $\alpha > 0$, the space $\mathcal{S}^{\alpha}X$ consists of elements in X with the norm scaled by α. So for $x \in \mathcal{S}^{\alpha}X$,

$$\|x\|_{\mathcal{S}^{\alpha}X} := \alpha \|x\|_X.$$

It is simple to verify that $\mathcal{S}^{\alpha}X$ is a normed vector space.

Given a normed vector space X, the dual space of X, denoted by X^*, is the space of bounded linear functionals on X. Following [17], we use the notation $< x, x^* >$ to denote $x^*(x)$, the value of the linear functional x^* at x. For $x^* \in X^*$,

$$\|x^*\|_{X^*} = \sup_{\substack{x \in X \\ x \neq 0}} \frac{| < x, x^* > |}{\|x\|}.$$

It may be verified that the dual of $c_0^{m\times n}$ is $\ell_1^{m\times n}$, where for $x \in c_0^{m\times n}$ and $x^* \in \ell_1^{m\times n}$,

$$< x, x^* > = \sum_i \sum_j \sum_{k=0}^{\infty} x_{ij}(k) x_{ij}^*(k).$$

It can also be verified that for $\alpha > 0$, the dual of $\mathcal{S}^{\alpha}X$ is $\mathcal{S}^{1/\alpha}X^*$.

If $M \subset X$, $M^{\perp} \subset X^*$ is defined to be the subspace of all $m^* \in X^*$ which annihilate all the elements of M, i.e. those elements in X^* for which $< m, m^* > = 0$ for all $m \in M$.

We now state a duality theorem for minimum distance problems.

Theorem 1 ([17]) *Let M be a subspace in a real normed space X. Let $x^* \in X^*$ be a distance d from $M^{\perp}$. Then*

$$d = \min_{m^* \in M^{\perp}} \|x^* - m^*\| = \sup_{\substack{x \in M \\ \|x\| \leq 1}} < x, x^* >$$

where the minimum on the left is achieved for some $m_0^* \in M^{\perp}$.

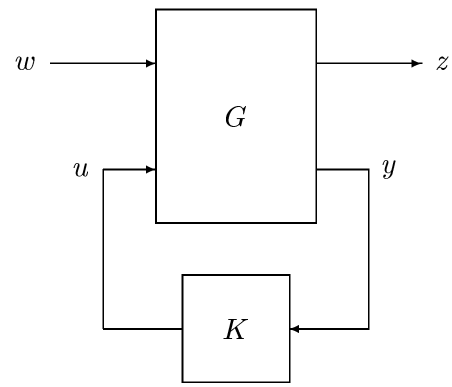

Figure 1: The Configuration Used in the ℓ_1 Problem Formulation

3 Problem Setup

Consider the system given in Fig. 1. G represents the linear time-invariant (LTI) discrete-time generalized plant, K the LTI discrete-time controller, u the control input, y the measured output, w the external inputs, and z the contolled output. We assume the dimensions of w, z, u, and y are n_w, n_z, n_u, and n_y respectively. The objective is to find an LTI discrete-time controller which minimizes the ℓ_1 norm of the impulse response of the function mapping w to z. It can be shown (see [13] for example), that this problem can be formulated as that of finding:

$$\gamma^{opt} = \inf_{Q \in \ell_1^{n_u \times n_y}} \|H - U * Q * V\|_1 \tag{1}$$

where $*$ denotes convolution, $H \in \ell_1^{n_z \times n_w}$, $U \in \ell_1^{n_z \times n_u}$, and $V \in \ell_1^{n_y \times n_w}$ are fixed and depend on the problem data: G, n_w, n_z, n_u, and n_y. It can be assumed without loss of generality that $\hat{U}$ the z-transform matrix of U has full column rank, and that $\hat{V}$ the z-transform matrix of V has full row rank. Otherwise, the formulation has redundant control and/or measurement signals which can be eliminated from consideration. Also, it can be assumed that U and V have finite support, i.e. that $\hat{U}$ and $\hat{V}$ are polynomial in z^{-1}. Otherwise if $\hat{U}$ and/or $\hat{V}$ were rational in z^{-1}, the denominators can be absorbed in $\hat{Q}$.

4 An Auxiliary Problem

In this section we use the ℓ_1 problem at hand to set up an auxiliary problem in a different space. The solution of the auxiliary problem requires no zero interpolation conditions, and therein lies its advantage. Upon solving the auxiliary problem, we show in the next section how to use this solution to get the solution of the original ℓ_1 problem.

Consider the following auxiliary problem which depends on the parameter $\alpha > 0$:

$$\inf_{Q \in \ell_1} \max\{\|H - U * Q * V\|_1 \;, \quad \alpha\|Q\|_1\} =: \gamma^{opt}(\alpha)$$

This problem differs from the ℓ_1 optimization problem previously formulated in that the Q parameter is included in the objective function.

We shall now formulate this problem as a minimum distance problem in a dual space X^*. The formulation in a dual space has two advantages. First, it will show the existence of optimal solutions, and second it will allow us to obtain computable upper and lower bounds which converge to $\gamma^{opt}(\alpha)$.

Define the vector space X as follows:

$$X := c_0^{n_z \times n_w} \times \mathcal{S}^{1/\alpha} c_0^{n_u \times n_y},$$

where the norm of $x = (x_1, x_2) \in X$ is defined to be $\|x\| = \|x_1\|_{c_0} + \|x_2\|_{\mathcal{S}^{1/\alpha} c_0}$. It can be easily verified that X^*, the dual space of X, is given by

$$X^* := \ell_1^{n_z \times n_w} \times \mathcal{S}^{\alpha} \ell_1^{n_u \times n_y},$$

where the norm of $x^* = (x_1^*, x_2^*) \in X^*$ is taken to be $\|x^*\| = \max\{\|x_1^*\|_1\ ,\ \|x_2\|_{\mathcal{S}^\alpha \ell_1}\}$. By definition, the elements in X^* are the bounded linear functionals on X. In particular, an element $x^* = (x_1^*, x_2^*) \in X^*$ acts on $x = (x_1, x_2) \in X$ as follows

$$< x, x^* > = < x_1, x_1^* > + < x_2, x_2^* > .$$

We shall pose the auxiliary problem as a minimum-distance-to-a-subspace problem. We define the desired subspace as:

$$S := \{(R, Q) \in X^* : R = U * Q * V\}.$$

Clearly S is a nonempty subspace of X^*. Before specifying the minimum distance problem, we will characterize the set S.

For $i = 1, \ldots, n_z$ and $j = 1, \ldots, n_w$, let $Z^{ij} \in \ell_1^{n_u \times n_y}$ be defined by $Z^{ij} := U_{i,\cdot}^T * V_{\cdot,j}^T$ where $U_{i,\cdot}$ is the ith row of U and $V_{\cdot,j}$ is the jth column of V. Now using Z define $W^{ijk} \in c_0^{n_u \times n_y}$ as follows:

$$W^{ijk} := \{Z^{ij}(k), \ldots, Z^{ij}(0), 0, \ldots\}.$$

We also define $E^{ijk} \in c_0^{n_z \times n_w}$ as follows:

$$E^{ijk}(l) = \begin{cases} A & l = k \\ 0 & \text{otherwise} \end{cases} .$$

where A is a matrix having 1 in its ijth entry, and 0 in the remaing entries.

Using the above notation, we characterize in the next lemma pairs (R, Q) in ℓ_1 which satisfy $R = U * Q * V$.

Lemma 1 *Let $Q \in \ell_1^{n_u \times n_y}$ and $R \in \ell_1^{n_z \times n_w}$. Then $(U * Q * V)_{ij}(k) = R_{ij}(k)$ if and only if*

$$< W^{ijk}, Q > \quad = \quad < E^{ijk}, R > .$$

Proof. For $i \in \{1, \ldots, n_z\}$, and $j \in \{1, \ldots, n_w\}$ the following holds:

$$(U * Q * V)_{ij} \quad = \quad \sum_{m=1}^{n_u} U_{im} * (Q * V)_{mj}$$

$$
\begin{aligned}
&= \sum_{m=1}^{n_u} U_{im} * \sum_{l=1}^{n_y} Q_{ml} * V_{lj} \\
&= \sum_{m=1}^{n_u} \sum_{l=1}^{n_y} (U_{im} * V_{lj}) * Q_{ml} \\
&= \sum_{m=1}^{n_u} \sum_{l=1}^{n_y} Z_{ml}^{ij} * Q_{ml}
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
(U * Q * V)_{ij}(k) &= \sum_{m=1}^{n_u} \sum_{l=1}^{n_y} (Z_{ml}^{ij} * Q_{ml})(k) \\
&= \sum_{m=1}^{n_u} \sum_{l=1}^{n_y} < W_{ml}^{ijk}, Q_{ml} > \\
&= < W^{ijk}, Q >
\end{aligned}
$$

Combining this with the fact that

$$R_{ij}(k) =< E^{ijk}, R >,$$

it follows that $(U * Q * V)_{ij}(k) = R_{ij}(k)$ if and only if $< W^{ijk}, Q >= R_{ij}(k)$. This proves the lemma. ■

We are now ready to state the minimum distance optimization problem:

$$\inf_{(Q,R) \in S} \|(H, 0) - (R, Q)\|_{X^*} =: \gamma^{opt}(\alpha). \tag{2}$$

Note that the dependence on α is reflected in the norm on the space X^* which does depend on α.

In order to set up an equivalent problem in the space X, we will show that $S = M^\perp$ for some subspace $M \subset X$.

Lemma 2 *Let*

$$
\begin{aligned}
M \quad := \quad span\{(E^{ijk}, -W^{ijk}), \quad & i = 1, \dots, n_z, \\
& j = 1, \dots, n_w, \ \ k = 0, 1, \dots \quad \}
\end{aligned}
$$

Then $S = M^\perp$.

Proof. Let $(R, Q) \in X^*$. Then

$$
\begin{aligned}
(R, Q) \in M^\perp \quad &\Leftrightarrow \quad < (E^{ijk}, -W^{ijk}), (R, Q) > \ = 0 \\
& \qquad \forall i, j, k \\
&\Leftrightarrow \quad U * Q * V = R \\
&\Leftrightarrow \quad (R, Q) \in S,
\end{aligned}
$$

where Lemma 1 was used to show that

$$< (E^{ijk}, -W^{ijk}), (R, Q) > \ = 0 \ \ \forall i, j, k$$

if and only if $U * Q * V = R$. ■

Corollary 1 *An optimal solutions to (2) always exists, i.e. for any $\alpha > 0$, there exists $Q \in \ell_1^{n_u \times n_y}$ such that*

$$\max\{\|H - U * Q^{opt} * V\|_1 \;,\quad \alpha\|Q\|_1\} =: \gamma^{opt}(\alpha)$$

Proof. Follows from the above lemma and Theorem 1.

Corollary 2

$$\gamma^{opt}(\alpha) = \sup_{\substack{(m_1,m_2)\in M \\ \|(m_1,m_2)\|_X \leq 1}} < (m_1, m_2), (H, 0) > \tag{3}$$

Proof. The proof follows from the above lemma and Theorem 1.

The optimization problem in Corollary 2 is formulated in the space X. Like its dual problem (2), it is infinite dimensional. Next we show how to obtain a sequence of simple finite dimensional problems whose solutions give monotonically decreasing upper bounds and monotonically decreasing lower bounds for $\gamma^{opt}(\alpha)$. Furthermore, we show show that the sequence of upper bounds and the sequence of lower bounds converge to $\gamma^{opt}(\alpha)$.

We begin by defining

$$M_N := \left\{ m = \sum_{k=0}^{N} \sum_{i,j} a_{ijk}(E^{ijk}, -W^{ijk}) : a_{ijk} \in \mathbb{R} \right\}.$$

Consider the following (finite-dimensional) optimization problem:

$$\underline{\gamma}_N(\alpha) := \sup_{\substack{(m_1,m_2)\in M_N \\ \|(m_1,m_2)\|_X \leq 1}} < (m_1, m_2), (H, 0) > \tag{4}$$

Proposition 1 $\left\{\underline{\gamma}_N(\alpha)\right\}$ *forms a monotonically increasing sequence of lower bounds to $\gamma^{opt}(\alpha)$. Furthermore, $\underline{\gamma}_N(\alpha) \nearrow \gamma^{opt}(\alpha)$ as $N \to \infty$.*

Proof. The first claim follows from the fact that

$$M_0 \subset M_1 \subset M_2 \subset \cdots.$$

That $\underline{\gamma}_N(\alpha) \nearrow \gamma^{opt}(\alpha)$ as $N \to \infty$ follows from the fact that

$$\bigcup_{N=0}^{\infty} M_N = M.$$

■

It may be already apparent from (4) that computing $\underline{\gamma}_N(\alpha)$ can be accomplished by solving a linear program in the variables a_{ijk}. The key is to observe that the constraint $\|(m_1, m_2)\|_X \leq 1$ in (4) can be expressed equivalently as a finite set of linear constraints in the variables a_{iji}. While one can certainly solve this linear program to arrive at $\underline{\gamma}_N(\alpha)$, it is far more convenient and and revealing to solve its dual program. As will be seen later, the dual minimization problem is closely related to the original infinite dimensional program. In fact, it is obtained by merely truncating the set of infinite constraints.

To carry out this develpment, we define

$$S_N := \{(R, Q) : \quad P_N R = P_N(U * Q * V)\}.$$

Then the following holds true:

Proposition 2

$$\underline{\gamma}_N(\alpha) = \min_{(Q,R)\in S_N} \|(H,0)-(R,Q)\|_{X^*}$$

Proof.

We will show that $S_N = M_N^{\perp}$, and then the proof of the proposition will follow by invoking Theorem 1.

$$\begin{aligned}(R,Q)\in M_N^{\perp} \;&\Leftrightarrow\; <(E^{ijk},-W^{ijk}),(R,Q)> \;=0 \\ &\qquad \forall i,\forall j, k=0,\ldots,N \\ &\Leftrightarrow\; R_{ij}(k) = (U*Q*V)_{ij}(k) \\ &\qquad \forall i,\forall j, k=0,\ldots,N \\ &\Leftrightarrow\; P_N R = P_N(U*Q*V) \\ &\Leftrightarrow\; (R,Q)\in S_N,\end{aligned}$$

where Lemma 1 was used to show that $<(E^{ijk},-W^{ijk}),(R,Q)> \;=0$, $\forall i$, $\forall j$ if and only if $(U*Q*V)(k) = R(k)$. ■

From the definition of X^*, the optimization problem

$$\min_{(Q,R)\in S_N} \|(H,0)-(R,Q)\|_{X^*}$$

can be rewritten as

$$\begin{aligned}&\min\max\{\|H-R\|_1 \;\; \alpha\|Q\|_1\} \\ \text{subject to}\quad & \\ &P_N R = P_N(U*Q*V).\end{aligned}$$

which is in fact a finite linear programming problem involving only $Q(0),\ldots,Q(N)$ and $R(0),\ldots,R(N)$. It can be viewed as the finite linear program arising by appropriately truncating the *constraints* of the infinite linear program

$$\begin{aligned}&\min\max\{\|H-R\|_1,\;\alpha\|Q\|_1\} \\ \text{subject to}\quad & \\ &R = U*Q*V\end{aligned}$$

whose optimal objective has been defined to be $\gamma^{opt}(\alpha)$.

We now show how to obtain converging upper bounds. This is a simpler matter. Indeed, consider the following optimization problem:

$$\overline{\gamma}_N(\alpha) := \inf_{(Q,R)\in S^N} \|(H,0)-(R,Q)\|_{X^*} \tag{5}$$

where

$$S^N \;:=\; \{(R,Q)\in X^* : R = U*P_N Q*V\}. \tag{6}$$

Proposition 3 *$\{\overline{\gamma}_N(\alpha)\}$ forms a monotonically decreasing sequence of upper bounds. Furthermore, $\overline{\gamma}_N(\alpha) \searrow \gamma^{opt}(\alpha)$ as $N\to\infty$.*

Proof. The proof follows immediately since finitely supported sequences in ℓ_1 are dense. ■

Here again, the problem

$$\min_{(Q,R)\in S^N} \|(H,0)-(R,Q)\|_{X^*}$$

can be rewritten as

$$\begin{array}{ll} & \min\max\{\|H-R\|_1,\ \alpha\|Q\|_1\} \\ \text{subject to} & \\ & R=(U*P_NQ*V). \end{array}$$

The numerical solution of this problem amounts to solving a finite linear program-min problem involving only $Q(0),\ldots,Q(N)$. Only a finite number of the variables $R(0),R(1),\ldots$ will enter the optimization problem due to the fact that U and V are finitely supported. Hence, a finite number of constraints need be considered. In contrast with the linear program arising from the lower bound calculations, this finite linear program can be viewed as that arising by appropriately truncating the *variables* of the infinite linear program

$$\begin{array}{ll} & \min\max\{\|H-R\|_1,\ \alpha\|Q\|_1\} \\ \text{subject to} & \\ & R=U*Q*V. \end{array}$$

To summarize, we have shown that upper and lower bounds for $\gamma_{opt}(\alpha)$ are given by finite linear programs as follows:

A Lower Bound for $\gamma_{opt}(\alpha)$:

$$\begin{array}{ll} & \underline{\gamma}_N(\alpha)=\min\max\{\|H-R\|_1\ \alpha\|Q\|_1\} \\ \text{subject to} & \\ & P_NR=P_N(U*Q*V). \end{array}$$

An Upper Bound for $\gamma_{opt}(\alpha)$:

$$\begin{array}{ll} & \overline{\gamma}_N(\alpha)=\min\max\{\|H-R\|_1,\ \alpha\|Q\|_1\} \\ \text{subject to} & \\ & R=(U*P_NQ*V). \end{array}$$

As $N\to\infty$, $\underline{\gamma}_N(\alpha)\nearrow\gamma^{opt}(\alpha)$ and $\overline{\gamma}_N(\alpha)\searrow\gamma^{opt}(\alpha)$.

5 Relating the Auxiliary Problem to the ℓ_1 Problem

In this section we make the connection between the ℓ_1 problem in (1) to the auxiliary problem in (2). We do this for the case when $\hat{U}$ and $\hat{V}$ have no zeros on the unit circle. For this case, it has been shown (see [13]) that the ℓ_1 problem (1) has an optimal solution $Q_{opt}\in\ell_1^{n_u\times n_y}$. We now define $\alpha_{max}:=\frac{\gamma^{opt}}{\|Q_{opt}\|_1}$, where γ^{opt} is the optimal solution of the ℓ_1 problem (1). We now relate $\gamma^{opt}(\alpha)$ to γ^{opt}.

Proposition 4 *Suppose* $\gamma^{opt} > 0$. *Then*

$$\gamma^{opt}(\alpha) = \gamma^{opt} \qquad 0 < \alpha \le \alpha_{max}$$

Proof. It is clear that $\gamma^{opt}(\alpha) \ge \gamma^{opt}$. Defining $R_{opt} := U * Q_{opt} * V$, the pair $(R_{opt}, Q_{opt}) \in S$. Furthermore,

$$\begin{aligned} \|(H,0) - (R_{opt}, Q_{opt})\|_{X^*} &= \max\{\|H - R_{opt}\|_1, \\ &\quad \alpha\|Q_{opt}\|_1\} \\ &= \max\{\gamma^{opt}, \alpha\|Q_{opt}\|_1\} \end{aligned}$$

But for $\alpha \le \alpha_{max}$, $\max\{\gamma^{opt}, \alpha\|Q_{opt}\|_1\} = \gamma^{opt}$. Thus for these values of α, $\gamma^{opt}(\alpha) = \gamma^{opt}$, and the result follows. ■

Based on the above result, if α is taken to be the ratio of any positive lower bound for γ^{opt} and any upper bound for $\|Q_{opt}\|$ then the solution of the auxiliary problem for such α gives the solution of the ℓ_1 problem.

We now show how to obtain an upper bound for $\|Q_{opt}\|_1$. Let Q_{opt} be any optimal solution. Let $\hat{U}^{-L}$ be a left inverse for $\hat{U}$ which has full column rank. Since $\hat{U}$ has no zeros on the circle, $\hat{U}^{-L}$ having no poles on the unit circle can be obtained. Let U^{-L} be the two sided inverse z-transform of $\hat{U}^{-L}$. It follows that $U^{-L} \in \ell_1^{n_z \times n_u}(Z)$ where $\ell_1(Z)$ is the space of two-sided absolutely summable sequences. V^{-R} can be defined similarly.

$$\begin{aligned} \|Q_{opt}\|_1 &= \|U^{-L} * U * Q_{opt} * V * V^{-R}\|_1 \\ &\le \|U^{-L}\|_{\ell_1(Z)} \|U * Q_{opt} * V\|_1 \|V^{-R}\|_{\ell_1(Z)} \\ &\le 2\|U^{-L}\|_{\ell_1(Z)} \|V^{-R}\|_{\ell_1(Z)} \|H\|_1 \end{aligned}$$

This gives the desired upper bound for $\|Q^{opt}\|_1$. Finally, it should be mentioned that U^{-L} and V^{-R} both can be computed directly from the state-space representations for $\hat{U}$ and $\hat{V}$.

Nonzero lower bounds for γ^{opt} can be computed using H_∞ optimal solutions [15]. If good and computationally cheap lower bounds for γ^{opt} are available they should be used. However, such bounds are not essential, and the upper bound for Q_{opt} alone can be used. We demonstrate this for the sequence of converging lower bounds $\underline{\gamma}_N(\alpha)$. Let β be any upper bound for the norm of any optimal Q_{opt}. One such bound has been given above. It is easy to see that there exists $0 < \alpha' < \alpha_{max}$ and an integer N' such that

$$\frac{\underline{\gamma}_N(\alpha')}{\alpha'} > \beta \qquad \forall N \ge N'.$$

Now $\underline{\gamma}_N(\alpha')$ is the solution of the optimization problem

$$\begin{aligned} &\min \max\{\|H - R\|_1,\ \alpha'\|Q\|_1\} \\ \text{subject to} \quad & \\ &P_N R = P_N(U * Q * V). \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &\min \|H - R\|_1 \\ \text{subject to} \quad & \\ &\alpha'\|Q\|_1 \le \underline{\gamma}_N(\alpha') \\ &P_N R = P_N(U * Q * V). \end{aligned}$$

If we now define $\underline{\nu}_N(\beta)$ to be the solution of the finite linear program

$$\begin{array}{ll} & \min \|H - R\|_1 \\ \text{subject to} & \\ & \|Q\|_1 \leq \beta, \\ & P_N R = P_N(U * Q * V) \end{array}$$

it may be seen that $\underline{\gamma}_N(\alpha') \leq \underline{\nu}_N(\beta)$ for all $N \geq N'$. As $\underline{\nu}_N(\beta) \leq \gamma^{opt}$ for all N, it follows that $\underline{\nu}_N(\beta) \nearrow \gamma^{opt}$. We have thus shown that only an upper bound for an optimal Q_{opt} is really needed to get converging lower bound. Finally, we can define $\overline{\nu}_N(\beta)$ using truncations of Q in the obvious fashion to get a converging upper bound to γ^{opt}.

We now address the case when either U or V has zeros on the unit circle. Clearly, the existence of solutions for the auxiliary problem is still guaranteed as is the convergence of the upper and lower bounds. In this case, however, the connection between the auxiliary problem and the ℓ_1 problem may not be as given in Proposition 4. The main difficulty arises because in such problems an optimal Q may not exist for the ℓ_1 problem. If such a solution exists, then everything goes through as before. But it may well happen that an optimal solution to the ℓ_1 problem does not exist, and that the optimal value of the objective function can be approached only with a sequence of Q parameters whose norms increase without bound. In these cases, the optimal value of the objective funtion for the auxiliary problem does not coincide with that of the ℓ_1 problem. No matter what bound on the norm of Q is used in solving the auxiliary problem, the objective will have a higher value that the optimal ℓ_1 objective, with the difference between the two numbers approaching zero as the bound on Q increases to infinity. In such cases solving the auxiliary problem with a reasonable bound on the norm of Q and using the solution to construct the controller can be considered to be more desirable than obtaining a sequence of solutions which correspond to Q parameters whose norms grow arbitrarily large in order to achieve objective values arbitrarily close to the optimal. In other words, in the cases we are addressing suboptimal soultions are the only desirable solutions, and the auxiliary problem gives just such solutions.

6 Convergence Properties

We now shed some light on some connections between the optimal solution of the original problem, and solutions obtained in the process of computing the upper bounds. We limit our discussion to the case when the optimal solution to the ℓ_1 problem is unique.

Proposition 5 *Suppose the standard ℓ_1 problem*

$$\gamma^{opt} := \inf \left\{ \|\Phi\|_1 \ : \ \Phi = H - U * Q * V, \quad Q \in \ell_1^{n_u \times n_y} \right\}$$

has a unique solution $\Phi_o \in \ell_1^{n_z \times n_w}$. Then at least n_u rows of Φ_o are active, i.e.

$$\|(\Phi_o)_i\|_1 = \gamma_{opt}$$

for all i in some subset of $\{1, \ldots, n_z\}$ having at least n_u elements.

Remark 1 *Without loss of generality, we may assume that the regulated outputs z have been numbered so that the active rows of Φ_o are listed first. Accordingly, H and U can be partitioned such that,*

$$\Phi_o := \begin{bmatrix} \Phi_{o,1} \\ \Phi_{o,2} \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} - \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} * Q * V$$

and $\|(\Phi_{o,1})_i\|_1 = \gamma_{opt}$, while $\|(\Phi_{o,2})_i\|_1 < \gamma_{opt}$.

Proof. To prove the above proposition, we will show that $\hat{U}_1(\lambda)$ must have at least n_u rows. Since $\hat{U}_1(\lambda)$ has n_u columns, it will be enough to show that $\hat{U}_1(\lambda)$ has full column rank (over the field of real rational functions in λ). This is accomplished by proving that for any polynomial $\hat{Q}_1(\lambda)$, $\hat{U}_1(\lambda)\hat{Q}_1(\lambda) = 0$ implies $\hat{Q}_1(\lambda) = 0$.

So let $\hat{Q}_1(\lambda)$ be a polynomial such that $\hat{U}_1(\lambda)\hat{Q}_1(\lambda) = 0$. We claim that $\hat{U}_2(\lambda)\hat{Q}_1(\lambda)$ must also be equal to zero. Since

$$\hat{U}(\lambda) = \begin{bmatrix} \hat{U}_1(\lambda) \\ \hat{U}_2(\lambda) \end{bmatrix}$$

has full column rank, we can then conclude that $\hat{Q}_1(\lambda) = 0$ and the proof would be complete. To prove this claim suppose that $\hat{U}_2(\lambda)\hat{Q}_1(\lambda) \neq 0$. Since

$$\hat{V}(\lambda) = \begin{bmatrix} \hat{V}_1(\lambda) & \hat{V}_2(\lambda) \end{bmatrix}$$

has full row rank, we have $\hat{U}_2(\lambda)\hat{Q}_1(\lambda)\hat{V}(\lambda) \neq 0$. There exists a sufficintly small $\epsilon > 0$ such that

$$\|H_2 - U_2 * (Q_o + \epsilon Q_1) * V\|_1 < \gamma^{opt}.$$

Since $H_1 - U_1 * (Q_o + \epsilon Q_1) * V = H_1 - U_1 * Q_o * V$, it follows that $Q_o + \epsilon Q_1$ is an optimal solution. Since $Q_1 \neq 0$ this implies that the optimal solution is not unique. ■

Proposition 6 *Suppose the standard ℓ_1 problem*

$$\gamma^{opt} := \inf\left\{\|\Phi\|_1 \;:\; \Phi = H - U * Q * V, \quad Q \in \ell_1^{n_u \times n_y}\right\}$$

has a unique solution $\Phi_o \in \ell_1^{n_z \times n_w}$, and let Q_o be the unique element in $\ell_1^{n_u \times n_y}$ such that $\Phi_o = H - U * Q_o * V$. Let $\{Q_N\}$ be any bounded sequence of elements in $\ell_1^{n_u \times n_y}$ such that $\lim_{N\to\infty} \|H - U * Q_N * V\|_1 = \gamma_{opt}$, and define

$$\Phi_N := \begin{bmatrix} \Phi_{N,1} \\ \Phi_{N,2} \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} - \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} * Q_N * V.$$

Then

1. $\lim_{N\to\infty} \|\Phi_{N,1} - \Phi_{o,1}\|_1 = 0$.
2. *If $\hat{U}_1$ has no zeros on the unit circle,* $\lim_{N\to\infty} \|\Phi_N - \Phi_o\|_1 = 0$.
3. *If $\hat{U}_1$ has no zeros on the unit circle and a square submatrix formed from columns of $\hat{V}$, say $\hat{V}_1 = \begin{bmatrix}\hat{V}_{i_1} \dots \hat{V}_{i_{n_y}}\end{bmatrix}$, has no zeros on the unit circle, then*

$$\lim_{N\to\infty} \|Q_N - Q_o\|_1 = 0.$$

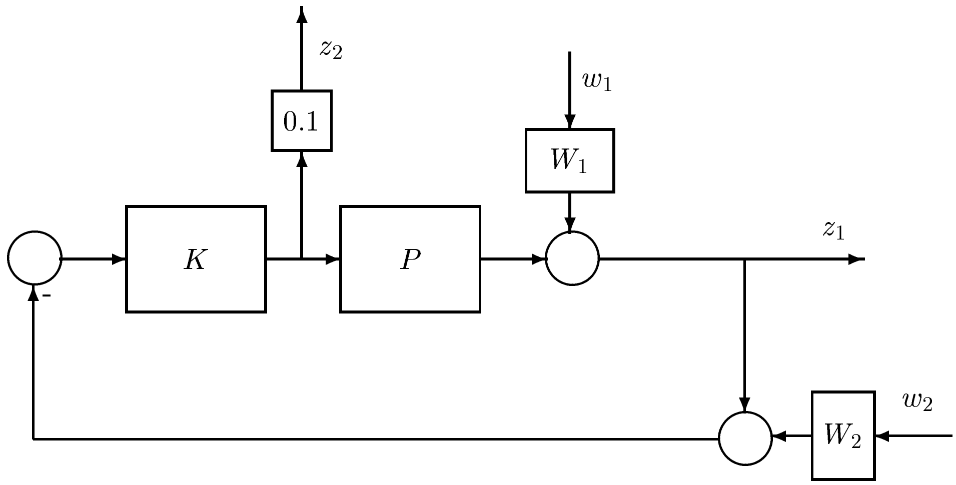

Figure 2: Example Problem

Proof. We prove item 1. As $\{Q_N\}$ is a bounded sequence in $\ell_1^{n_u \times n_y}$, it follows from Alaoglu's theorem that there exists a subsequence $\{Q_{N_k}\}$ which converges weak-star (wk^*) to an element $Q' \in \ell_1^{n_u \times n_y}$. From this it follows that $H - U * Q_{N_k} * V$ converges wk^* to $H - U * Q' * V$. Thus,

$$\|H - U * Q' * V\|_1 \leq \liminf_k \|H - U * Q_{N_k} * V\|_1 = \gamma^{opt}.$$

This implies that Q' is an optimal solution. By uniqueness of the optimal solution, it follows that $Q' = Q_o$.

From the above, it follows that for each i, $(\Phi_{N_k,1})_i$ converges wk^* to $(\Phi_{o,1})_i$. In addition, $\|(\Phi_{N_k,1})_i\|_1$ converges to $\|(\Phi_{o,1})_i\|_1$. Together, both of these facts imply that $\lim_{N\to\infty} \|\Phi_{N,1} - \Phi_{o,1}\|_1 = 0$. (See [13],page 286).

We next prove item 2. If $\hat{U}_1$ has no zeros on the unit circle, then $\hat{U}_1^{-L}$ has no poles on the unit circle and the two-sided inverse transform U_1^{-L} has its elements in $\ell_1(Z)$. Now, $U_1 * Q_N * V$ converges in the ℓ_1 norm to $U_1 * Q_o * V$ and thus $U_1^{-L} * (U_1 * Q_N * V) = Q_N * V$ converges in the ℓ_1 norm to $U_1^{-L} * (U_1 * Q_o * V) = Q_o * V$. It follows that $\lim_{N\to\infty} \|U * Q_N * V - U * Q_o * V\|_1 = 0$, and hence $\lim_{N\to\infty} \|\Phi_N - \Phi_0\|_1 = 0$.

We now prove item 3. If $\hat{U}_1$ has no zeros on the unit circle, we have shown that $Q_N * V$ converges in the ℓ_1 norm to $Q_o * V$. Since $\hat{V}_1$ has no zeros on the unit circle, $\hat{V}_1^{-1}$ has no poles on the unit circle and its two-sided inverse transform has its elements in $\ell_1(Z)$. This implies that $Q_N * V$ converges in the ℓ_1 norm to Q_o. ■

7 Example

The system in figure 2 has been considered in [13]. The plant P is given by

$$P = \frac{0.5z - 1}{0.1z^2 - 1.05z - 0.5}.$$

Two disturbance signals w_1 and w_2 are affecting the system. They are weigted by

$$W_1(z) = \frac{0.4z}{z - 0.6} \qquad \text{and} \quad W_2 = \frac{z - 0.75}{z - 0.25}$$

respectively. z_1 and z_2 are two outputs of interest, and it is desired to minimize the ℓ_1 norm of the impulse response of the mapping

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The corresponding transfer function is

$$\begin{pmatrix} (I+PK)^{-1}W_1 & PK(I+PK)^{-1}W_2 \\ 0.1K(I+PK)^{-1}W_1 & 0.1K(I+PK)^{-1}W_2 \end{pmatrix}$$

It has been shown in Section 5 that upper bounds and lower bounds for γ_{opt} are given by $\overline{\gamma}_N(\alpha)$ and $\underline{\gamma}_N(\alpha)$. From these, we obtained the related bounds $\overline{\nu}_N(\beta)$ and $\underline{\nu}_N(\beta)$. Figure 3 shows plots of $\overline{\nu}_N(\beta)$ and $\underline{\nu}_N(\beta)$ vs. N for $\beta = 5, 10, 100$, and 1×10^7, all of which can be shown to be upper bounds for some optimal Q_{opt}. It is clearly

Figure 3: Upper and lower bound plots vs. N for various values of β

seen from the figure that the upper and lower bounds converge to γ_{opt} whose value is 71.1146. It can also be seen that is this case the upper bounds $\overline{\nu_N}$ do not change for the selected values of β. It is also interesting to observe that while the lower bounds $\underline{\nu}_N(\beta)$ *do* depend on the size of β (with lower values giving faster convergence), even very bad upper bound estimates for $\|Q_{opt}\|_1$ which are reflected by large β still give reasonably fast convergence for the lower bounds. Indeed, when $\beta = 1 \times 10^7$ is used in this example, errors between $\underline{\nu}_N(\beta)$ and γ_{opt} of less than 1% are observed for $N \geq 14$. The desirable convergence properties observed here, and the extent to which they apply in general, warrant further investigation.

Order of Approximation	Approximation Error Norm	Error Percentage
5	2.2×10^{-5}	0.00058 %
4	6.88×10^{-4}	0.018 %
3	4.86×10^{-3}	0.13 %
2	6.29×10^{-4}	0.16 %

Table 1: Table showing the approximation error for several rational approximations to the 29th order optimal FIR solution Q corresponding to $\underline{\nu}_{30}$

7.1 Controller Order

We now address the issue of controller order. Using an optimal Q from any of the upper bounds one can construct a controller which yields the performance level corresponding to that upper bound. Such Q is FIR and has length equal to N. As N is increased the upper bound approaches the optimal value γ_{opt}, and in the process the length of the resulting Q will be increased as well. However, this does not necessarily lead to large order controllers. In fact in this example, as in several others solved using this approach, the long FIR Q can be very well approximated by a much lower order *rational* stable system. To demonstrate, for $N = 30$ the FIR Q obtained from the upper bound has length 30 and hence a McMillan degree of 29. However, using the Hankel SVD method proposed by S. Kung (implemented in matlab `imp2ss()` command), excellent low order rational approximations have been obtained. These are given in Table 7.1. The order of the controller itself will be equal to the order of the Q used plus 4. Of course the error in the closed-loop norm resulting from the approximation error in Q can be no larger than (and typically much less than) $\|U\|_1\|V\|_1$ times the norm of the approximation error of Q.

8 Conclusions

In this work we have introduced a new approach to solving the ℓ_1 control problem. Like many other solutions to the ℓ_1 problem the underlying solution utilizes finite dimensional linear programming as a method for computation. The solution is obtained through introducing an auxiliary problem with desirable properties and then relating its solution to that of the ℓ_1 problem. The auxiliary problem can be viewed as a regularization of the ℓ_1 problem charactrerized by introducing the Q parameter directly in the optimization by introducing a scaled norm of Q in the objective (or equivalently by adding to the constraints a Q norm constraint). This "regularization" of the problem *in combination* with the fact that the Q parameter variables are explicitly included in the optimization problem leads to a number of desirable features which characterize the auxiliary ℓ_1 problem. First, it is possible to solve the problem without the need for computing system zeros or including zero interpolation conditions at all. Such conditions can, and frequently do, complicate the numerical computations both in the construction of the linear program constraints and in the controller recovery stage after the solution is obtained. Another feature of the proposed approach is that the auxiliary problem *always* has a solution, whether the system has zeros on the unit circle or not.

In fact, in cases when unit circle zeros are present solving the auxiliary problem is the best approach to take since solutions which get arbitrarily close to the optimal index often result in Q parameters with arbitrarily large norms, a situation which is avoided by the auxiliary problem setup. A third feature of the auxiliary problem is one whose absence in FIR Q-approximation methods motivated this whole research direction, namely obtaining lower bounds to the solution which converge to the optimal index. Using ideas from duality theory it has been shown that one can indeed obtain such bounds for the auxiliary problem. Furthermore, it has been shown that such bounds can be obtained by appropriately truncating constraints from the auxiliary problem. Such procedure simply does not yield any results if applied to the original problem formulated for FIR Q approximation.

In the proposed approach, it was suggested that one way to try avoid large order controllers is to approximate a long FIR Q which is close to optimal with a low order rational Q. The reason this approach often yields very good results is that FIR sequences Q which obtain from solving an upper bound problem are approaching a rational optimal solution Q. One problem which remains open is how to arrive at an optimal rational Q directly (when one exists) without attempting to reconstruct it from FIR sequences, while at the same time avoiding zero interpolaton. Another problem of interest is to generalize these ideas for problems where multiple objectives are of interest including possibly other norms in addition to the ℓ_1 norm. Some initial results along these line have been obtained [18], although much remains to be explored.

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