Abstract: The Signal-to-Interference-plus-Noise Ratio (SINR) is an important metric of wireless communication link quality. SINR estimates have several important applications. These include optimizing the transmit power level for a target quality of service, assisting with handoff decisions and dynamically adapting the data rate for wireless Internet applications. Accurate SINR estimation provides for both a more efficient system and a higher user-perceived quality of service. In this paper, we develop new SINR estimators and compare their mean squared error (MSE) performance. We show that our new estimators dominate estimators that have previously appeared in the literature with respect to MSE. The sequence of transmitted bits in wireless communication systems consists of both pilot bits (which are known both to the transmitter and receiver) and user bits (which are known only by the transmitter). The SINR estimators we consider alternatively depend exclusively on pilot bits, exclusively on user bits, or simultaneously use both pilot and user bits. In addition, we consider estimators that utilize smoothing and feedback mechanisms. Smoothed estimators are motivated by the fact that the interference component of the SINR changes relatively slowly with time, typically with the addition or departure of a user to the system. Feedback estimators are motivated by the fact that receivers typically decode bits correctly with a very high probability, and therefore user bits can be thought of as quasipilot bits. For each estimator discussed, we derive an exact or approximate formula for its MSE. Satterthwaite approximations, noncentral F distributions (singly and doubly) and distribution theory of quadratic forms are the key statistical tools used in developing the MSE formulas. In the case of approximate MSE formulas, we validate their accuracy using simulation techniques. The approximate MSE formulas, of interest in their own right for comparing the quality of the estimators, are also used for optimally combining estimators. In particular, we derive optimal weights for linearly combining an estimator based on pilot bits with an estimator based on user bits. The optimal weights depend on the MSE of the two estimators being combined, and thus the accurate approximate MSE formulas can conveniently be used. The optimal weights also depend on the unknown SINR, and therefore need to be estimated in order to construct a useable combined estimator. The impact on the MSE of the combined estimator due to estimating the weights is examined. © 2004 Wiley Periodicals, Inc. Naval Research Logistics 51: 720–740, 2004.
1. INTRODUCTION

In all communication systems, noise generated by circuit components in the receiver is a source of signal corruption. The ratio of the signal power to noise power, called the signal-to-noise ratio (SNR), is an important indicator of communication link quality. In wireless communications systems (particularly in mobile cellular communication systems) interference from other users in the system is a more significant source of signal corruption than the noise from circuitry. An adjusted indicator of link quality is the signal-to-interference-plus-noise ratio (SINR). Traditionally, a target link quality is characterized by a tolerable bit error rate that, in turn, maps to a required SINR. The SINR is then used to determine the transmitter power, which will deliver the target link quality. The SINR is also utilized by various control actions in wireless communication systems including handoff decisions and data rate adaptation algorithms (see, for example, Furuskar, Mazur, Muller, and Olofsson [4] and Viterbi [19]). We set the context for SINR estimation by first outlining the essential elements of a generic wireless communication system depicted in Figure 1.

An information source outputs an analog (e.g., voice) or digital signal (e.g., data) to a source encoder. The source encoder digitizes the signal, if needed, and typically performs bit compression. The resultant bits are passed to a channel encoder that introduces controlled redundancy (e.g., parity bits) to protect against channel errors. The modulator then maps fixed-size subsets of the bits into one of a finite number of amplitude and/or phase combinations that get impressed on a carrier waveform. The resultant carrier waveform is transmitted through the atmosphere (i.e., through a channel) that randomly attenuates the strength of the signal and shifts its phase. At the receiver, the demodulator is a matched filter that tries to recover the amplitude and/or the phase of the transmitted carrier waveform by “matching” the received signal to the known combinations of amplitude and/or phase that are being used by the modulator at the transmitter. The goal of the demodulator is to undo the random effects of the channel and predict the transmitted bits. It is at the output of the demodulator that SINR estimation is undertaken.

Generally, wireless transmission is organized by time slots wherein some pilot bits (i.e., overhead bits that are known a priori to both the transmitter and the receiver) are sent followed by user bits (i.e., information bits that are known to the transmitter only). Pilot bits allow the receiver to learn the channel attenuation/phase shift and then undo their effect from the user bits, and are also used in maintaining synchronization between the transmitter and the receiver. As the pilot bits are pure overhead, their number relative to the number of user bits in a time slot is small. With an additive white Gaussian noise (AWGN) model for the noise and interference (Proakis [10]), the output of the demodulator (measured in volts) corresponding to the jth bit of the ith time slot is

\[ Y_{ij} = a_{ij} \mu_i + \epsilon_{ij}, \quad i \geq 1, j = 1, \ldots, N, \]

where \( a_{ij} \) is \(-1\) or \(+1\) depending on whether the bit sent was 0 or 1, respectively (assuming Binary Phase Shift Keying). \( \mu_i \) is an unknown constant during the ith time slot that is a function of the channel attenuation, \( \epsilon_{ij} \) are independent and identically distributed Gaussian random variables with zero mean and variance \( \sigma^2 \), and \( N \) is the number of bits in a time slot. In cases
of very fast fading, \(N\) may need to be small in order for the assumption that \(\mu_i\) is constant during the time slot to hold. The \(e_{ij}\) in the AWGN model represent the effect of noise introduced by the receiver as well as interference from other radio transmissions in the area. The Gaussian assumption for \(e_{ij}\) will hold in CDMA systems provided the processing gain is sufficient.

The ratio \(\theta_i = \mu_i^2/\sigma_i^2\) is the SINR during the \(i\)th time slot and is the parameter that we wish to estimate. For certain applications, such as handoff decisions and slow rate adaptation (e.g., GSM GPRS), SINR is alternatively defined using an average value of \(\mu_i\) over several adjacent time slots. Estimation methods discussed in Brandao [1], Turkboylari and Stuber [18], and Ramakrishna et al. [11] are more suitable in those contexts than the methods that are discussed here. Applications where our estimation methods apply include power control and fast rate adaptation [e.g., systems such as High Speed Downlink Packet Access (HSDPA), Third Generation Evolution Data Only (3G1x-EVDO), and Third Generation Evolution Data and Voice (3G1x-EVDV)], where it is desired to track the short-term fading as much as possible.

The \(a_{ij}\) values in AWGN model are determined by the bit stream, and as such are known by the transmitter. The receiver uses the demodulator output value for each bit to predict \(a_{ij}\) and then map that value (0 or 1) to a predicted bit value (0 or 1). The receiver knows the position of pilot bits within a time slot and it also knows that for pilot bits \(a_{ij} \equiv 1\). Let \(n\) denote the number of pilot bits in a time slot (implying \(m = N - n\) user bits), and let \(P_{ij}\) and \(U_{ij}\) denote the demodulator output values for pilot and user bits, respectively. It follows that, for \(i = 1, \ldots, n, P_{ij}\) has a Gaussian distribution with mean \(\mu_i\) and variance \(\sigma_i^2\). Moreover, the \(P_{ij}\) are mutually independent. The distribution of \(U_{ij}\) is a mixture of two Gaussian distributions that each have variance \(\sigma_i^2\) but have respective means of \(-\mu_i\) and \(\mu_i\), respectively, since \(a_{ij}\) in this case is a random variable taking on each of the values \(-1\) and \(+1\) with probability 0.5.

It is clear that the pilot bits provide an opportunity to estimate \(\theta_i\). In particular, letting \(\hat{P}_i = \sum_{j=1}^{n} P_{ij}/n\) and \(S_i^2 = \sum_{j=1}^{n} (P_{ij} - \hat{P}_i)^2/(n - 1)\), a “plug-in” (PI) estimator of \(\theta_i\) based on pilot bits only, is \(\hat{\theta}_i^P = \hat{P}_i^2/S_i^2\). On the other hand, it is less obvious how to use the user bits to estimate \(\theta_i\), since, for example, the expected value of \(U_{ij}\) is zero, not \(\mu_i\). While \(\hat{\theta}_i^P\) is an intuitive estimator, its drawback is that \(n\) is typically small with the consequence being that the mean squared error (MSE) of \(\hat{\theta}_i^P\) is too large for most applications. In Section 2 we consider

![Figure 1. Generic wireless communication system.](image)
estimation of $\theta_i$ based exclusively on pilot bits and show that $\hat{\theta}_i^{PI}$ is dominated in terms of MSE by its bias-corrected version. In Section 3 we consider estimation of $\theta_i$ based on user bits. We first review an ad hoc approach for using user bits that exists in the literature and then develop a novel estimator that uses the $\hat{a}_{ij}$ predictions as part of a feedback loop to improve upon $\hat{\theta}_i^{PI}$. In Section 4 we develop further improvements by combining estimators of $\theta_i$ that are based on pilot and user bits, respectively. In particular, we derive weights to use in the context of forming a weighted average of the two estimators. In Section 5 we discuss the use of variance-smoothing techniques as an alternative to effectively increase the number of bits available for estimating the SINR. Utilizing both variance-smoothing and combining, we present a very precise SINR estimator. We use MSE as the criteria for evaluating the alternative estimators and derive approximate expressions for the MSE for nearly all of the estimators proposed in this paper. Simulations are used to validate the assumptions underlying the approximation arguments. An overview of the simulation study is provided in an Appendix.

2. ESTIMATORS USING PILOT BITS

We first consider the MSE properties of $\hat{\theta}_i^{PI}$, some of which have been detailed in Thomas [16]. First, note that $nP_i^2/S_i^2$ has a noncentral F distribution (see, for example, Searle [15], Section 2.4i) with numerator and denominator degrees of freedom equal to 1 and $n-1$ respectively, and a noncentrality parameter equal to $\lambda_i = n\theta_i/2$. Using the formulas in Searle [15] for the mean and variance of the noncentral F distribution, it follows that

$$MSE(\hat{\theta}_i^{PI}) = \frac{2(n-1)^2}{n^2(n-3)} \left[ \frac{(1+n\theta)^2}{(n-3)(n-5)} + \frac{1+2n\theta_i}{n-5} \right] + \left[ \frac{n-1}{n-3} \left( \frac{1}{n} + \theta_i \right) - \theta_i \right]^2.$$  (1)

The estimator $\hat{\theta}_i^{PI}$ is a scaled version of the maximum likelihood estimator of SINR derived by Thomas [16]. A bias-corrected (BC) estimator of $\theta_i$ can be formed by linearly transforming $\hat{\theta}_i^{PI}$. Indeed,

$$\hat{\theta}_i^{BC} = \frac{\hat{P}_i^2}{n-1} - \frac{1}{n} \frac{S_i^2}{n-3}$$

removes a variable bias by adjusting the divisor used when estimating $\sigma_i^2$ and removes a constant bias by subtracting $1/n$. It follows that the MSE of $\hat{\theta}_i^{BC}$ is simply its variance and thus

$$MSE(\hat{\theta}_i^{BC}) = \frac{2(n-3)}{n^2} \left[ \frac{(1+n\theta)^2}{(n-3)(n-5)} + \frac{1+n\theta_i}{n-5} \right].$$  (2)

The Lehman-Scheffe theorem (see, for example, Graybill [6], Theorem 2.7.7) implies that $\hat{\theta}_i^{BC}$ is the uniformly minimum variance unbiased estimator (UMVUE) of $\theta_i$ when inference is restricted to using pilot bits from the $i$th time slot. The UMVUE property of $\hat{\theta}_i^{BC}$ was noted in Rukhin [13] in a different application context.

Figure 2 shows the Root Mean Squared Error (RMSE) of $\hat{\theta}_i^{PI}$ and $\hat{\theta}_i^{BC}$ for $\theta_i$ in the range $-2$ dB to 10 dB [SINR is generally expressed in decibels (dB), defined as $10 \log_{10}(\theta_i)$] labeled as PI (A) and BC (A), respectively, for the case $n = 8$. The range $-2$ dB to 10 dB is a practical
range of SINR values for most wireless communication systems (see, for example, Viterbi [19]). The improvement offered by $\hat{\theta}_i^{\text{BC}}$ compared to $\hat{\theta}_i^{\text{PI}}$ is quite evident. Also shown in Figure 2 are the RMSE values for $\hat{\theta}_i^{\text{PI}}$ and $\hat{\theta}_i^{\text{BC}}$ obtained by simulating 50,000 values of each estimator. These values are labeled as PI (S) and BC (S), respectively. Although the simulated RMSE values in Figure 2 are not necessary, since the formulas in (2) and (3) are exact, the good match between the simulated and exact RMSE values suggests that 50,000 will be a sufficient sample size to validate the approximate RMSE formulas in what follows.

3. ESTIMATORS USING USER BITS

3.1. Absolute Value Estimators

Typically there are far fewer pilot bits than there are user bits. As we have previously mentioned, the difficulty with using user bits is that the expected value of $U_{ij}$ is zero instead of $\mu_i$. Hence, it is not possible to estimate $\theta_i$ using the $U_{ij}$ values directly. An ad-hoc approach for using user bits, first proposed by Gilchriest [5], is to estimate $\theta_i$ by computing either $\hat{\theta}_i^{\text{PI}}$ or $\hat{\theta}_i^{\text{BC}}$ using $Z_{ij}$ in place of $P_{ij}$.

The ad hoc approach for using the user bits can be heuristically motivated as follows. Suppose $\theta_i$ is large, implying $\mu_i$ is large relative to $\sigma_i$. Given that $a_{ij} = -1$, $U_{ij} < 0$ with a high probability. It follows that $Z_{ij} \approx -U_{ij}$. Since the distribution of $U_{ij}$, given $a_{ij} = -1$, is Gaussian with mean $-\mu_i$ and variance $\sigma_i^2$, it follows that the conditional distribution of $Z_{ij}$, given $a_{ij} = -1$, is approximately Gaussian with mean $\mu_i$ and variance $\sigma_i^2$. Similarly, given that $a_{ij} = 1$, $U_{ij} > 0$ with a high probability and thus $Z_{ij} \approx U_{ij}$. Since the distribution of $U_{ij}$, given $a_{ij} = 1$, is Gaussian with mean $\mu_i$ and variance $\sigma_i^2$, the conditional distribution of $Z_{ij}$, given $a_{ij} = 1$, is approximately Gaussian with mean $\mu_i$ and variance $\sigma_i^2$. Since the conditional distributions of $Z_{ij}$, given $a_{ij} = 1$ and given $a_{ij} = -1$, are the same and there are no other possible values for $a_{ij}$, it follows the unconditional distribution for $Z_{ij}$ is approximately Gaussian with mean $\mu_i$ and variance $\sigma_i^2$. The ad hoc approach for estimating $\theta_i$ relies on the fact that the $Z_{ij}$ values are roughly stochastically equivalent to $P_{ij}$ values. Clearly, for small or even intermediate values of $\theta_i$, the $Z_{ij}$ values will have a substantially different distribution than $P_{ij}$ values, and the ad-hoc approach estimators will not have good MSE properties.

Define $\hat{\theta}_i^{\text{PI-}Z} = \bar{Z}_i^2 / T_i^2$, where $\bar{Z}_i = \sum_{j=1}^{m} Z_{ij} / m$, $T_i^2 = \sum_{j=1}^{m} (Z_{ij} - \bar{Z}_i)^2 / (m - 1)$. Layland [9] gives large $m$ approximations for the mean and variance of $\hat{\theta}_i^{\text{PI-}Z}$. Let $\Phi(\cdot)$ denote the

![Figure 2. RMSE of pilot-based estimators.](image-url)
cumulative distribution function of the Gaussian distribution that has mean 0 and variance 1. The mean and variance of \( Z_{ij} \) are \( \mu_{Z,ij} = \sigma \sqrt{2/\pi} e^{-\mu/(2\sigma^2)} + 2\mu_\phi \mu_{ij} - \mu_i \) and \( \sigma^2_{Z,ij} = \mu_i^2 + \sigma_i^2 - \mu_{Z,ij}^2 \) (see, for example, Johnson, Kotz, and Balakrishnan [8], pp. 453–454). We use the central limit theorem to approximate the distribution of \( \sqrt{mZ_i}/\sigma_{Z,ij} \) by a Gaussian distribution with mean \( \sqrt{m\theta_{Z,ij}} \) and variance 1, where \( \theta_{Z,ij} = \mu_{Z,ij}^2/\sigma_{Z,ij}^2 \). It follows that \( V_i^2 = m\tilde{Z}_{ij}^2/\sigma_{Z,ij}^2 \) has an approximate noncentral chi-square distribution with 1 degree of freedom and noncentrality parameter \( \lambda_i^2 = m\theta_{Z,ij}/2 \). We use a Satterthwaite [14] chi-square approximation for the distribution of \( W_i^2 = (m - 1)T_i^2/\sigma_{Z,ij}^2 \), writing \( W_i^2 \sim g_i\chi^2_{\eta_i} \) where \( g_i = g(\theta_i) \) and \( \eta_i = \eta(\theta_i) \) are the scale and degree-of-freedom constants developed in the Appendix A.

If we assume that \( \tilde{Z}_i \) and \( T^2 \) are independently distributed, then it would follow that \( m\tilde{Z}_{ij}^2/T_i^2 \) has an approximate noncentral F distribution with numerator and denominator degrees of freedom equal to 1 and \( \eta_i \), respectively, and noncentrality parameter equal to \( \lambda_i^2 \). In general, the asymmetry of the distribution of \( Z_{ij} \) prevents \( \tilde{Z}_i \) and \( T_i^2 \) from being uncorrelated (see, for example, Randles and Wolfe [12], p. 24, Corollary 1.3.33), and therefore they will not be independent. We do note that for large \( \theta_i \), \( \tilde{Z}_i \) and \( T_i^2 \) will be approximately independent since the distribution of \( Z_{ij} \) is then approximately Gaussian. We absorb the error in assuming \( \tilde{Z}_i \) and \( T_i^2 \) are independent as part of the approximate noncentral F distribution we use for \( m\tilde{Z}_{ij}^2/T_i^2 \). It follows that

\[
MSE(\hat{\theta}_i^{PL-Z}) = \frac{2\eta_i^2}{m^2(\eta_i - 2)} \left[ \frac{(1 + m\theta_{Z,ij})^2}{(\eta_i - 2)(\eta_i - 4)} + \frac{1 + 2m\theta_{Z,ij}}{\eta_i - 4} \right] + \left[ \frac{\eta_i}{\eta_i - 2} \left( \frac{1}{m} + \theta_{Z,ij} \right) - \theta_i \right]^2.
\]

(3)

It follows from results in Appendix A that as \( \theta_i \) gets large, \( \eta_i \to m - 1 \) and \( \theta_{Z,ij} \to \theta_i \). Thus, the squared bias term in (3) suggests a bias corrected estimator of the form \( \hat{\theta}_i^{BC-Z} = (m - 3)\hat{\theta}_i^{PL-Z}(m - 1) - 1/m \), which is

\[
MSE(\hat{\theta}_i^{BC-Z}) = \frac{2(m - 3)^2\eta_i^2}{m^2(m - 1)^2(\eta_i - 2)} \left[ \frac{(1 + m\theta_{Z,ij})^2}{(\eta_i - 2)(\eta_i - 4)} + \frac{1 + 2m\theta_{Z,ij}}{\eta_i - 4} \right] + \left[ \frac{m - 3}{m - 1} \frac{\eta_i}{\eta_i - 2} \left( \frac{1}{m} + \theta_{Z,ij} \right) - \frac{1}{m - \theta_i} \right]^2.
\]

(4)

Figure 3 shows the analytic approximations for the RMSE of \( \hat{\theta}_i^{PL-Z} \) and \( \hat{\theta}_i^{BC-Z} \), labeled as PI-Z (A) and BC-Z (A), respectively, for the case \( m = 20 \). The curves, labeled PI-Z (S) and BC-Z (S), are the respective RMSE curves obtained via simulation. The nonmonotone nature of the curves in Figure 3 over the interval -2 dB to 6 dB is explained by the fact that initially the decreasing squared bias component of MSE offsets the increasing variance component, but eventually the bias becomes near zero and the MSE becomes essentially equal to the variance. From Figure 3 we conclude that the analytic RMSE approximations (3) and (4) for \( \hat{\theta}_i^{PL-Z} \) and \( \hat{\theta}_i^{BC-Z} \), respectively, are quite good when the SINR is greater than or equal to 5 dB. Moreover, by comparing the ordinate scales of Figures 2 and 3, we see that the RMSE values of both \( \hat{\theta}_i^{PL-Z} \) and \( \hat{\theta}_i^{BC-Z} \) are considerably smaller than the RMSE values of \( \hat{\theta}_i^{BC} \) for the companion case \( n = 8 \).

3.2. Feedback Estimators

In this approach, we take advantage of the fact that the receivers (in typical operation) make correct predictions of the \( a_{ij} \) values with a high probability. Let \( \hat{a}_{ij} \) denote the receiver’s
predicted value of $a_{ij}$. Using a likelihood ratio test, the value of $\hat{a}_{ij}$ is +1 and -1 when $U_{ij}$ is greater and less than zero, respectively. Define $D_{ij} = \hat{a}_{ij}U_{ij} = a_{ij}^*\mu_i + \epsilon_{ij}^*$, where $a_{ij}^* = \hat{a}_{ij}a_{ij}$ and $\epsilon_{ij}^* = \hat{a}_{ij}\epsilon_{ij}$. Since both $\hat{a}_{ij}$ and $a_{ij}$ are $\{-1, +1\}$ random variables, $a_{ij}^*$ is also a $\{-1, +1\}$ random variable and is equal to +1 if and only if $\hat{a}_{ij} = a_{ij}$. It is straightforward to show that $\Pr(a_{ij}^* = 1) = p(\theta_i) = \Phi(\sqrt{\theta_i})$.

We now derive the conditional distribution of $\epsilon_{ij}^*$, given $a_{ij}^*$. First note that $D_{ij}$ is nonnegative. Thus, conditional on $a_{ij}^*$ we must have $x_{ij}^* = \epsilon_{ij}^*$. Consider first the case where $a_{ij}^* = 1$. We have

$$
\Pr(\epsilon_{ij}^* < x | a_{ij}^* = 1) = \frac{\Pr(\epsilon_{ij}^* < x, \hat{a}_{ij} = -1, a_{ij} = -1) + \Pr(\epsilon_{ij}^* < x, \hat{a}_{ij} = 1, a_{ij} = 1)}{\Pr(\hat{a}_{ij} = -1, a_{ij} = -1) + \Pr(\hat{a}_{ij} = 1, a_{ij} = 1)}
$$

Substituting $U_{ij} < 0$ ($U_{ij} > 0$) for $\hat{a}_{ij} = -1$ ($\hat{a}_{ij} = 1$), and using $U_{ij} = a_{ij}\mu_i + \epsilon_{ij}$ leads to

$$
\Pr(\epsilon_{ij}^* < x | a_{ij}^* = 1) = \frac{\Pr(-x < \epsilon_{ij} < \mu_i, a_{ij} = -1) + \Pr(-\mu_i < \epsilon_{ij} < x, a_{ij} = 1)}{\Pr(\epsilon_{ij} < \mu_i, a_{ij} = -1) + \Pr(\epsilon_{ij} > -\mu_i, a_{ij} = 1)}
$$

$$
= \frac{\Phi\left(\frac{x}{\sigma_i}\right) - [1 - \Phi(\sqrt{\theta_i})]}{\Phi(\sqrt{\theta_i})}, \quad x \geq -\mu_i.
$$

It follows that the conditional distribution of $\epsilon_{ij}^*$, given $a_{ij}^* = 1$, is a zero mean, $\sigma_i^2$ variance Gaussian distribution truncated at the point $-\mu_i$. In a similar way, it can be shown that

$$
\Pr(\epsilon_{ij}^* < x | a_{ij}^* = -1) = \frac{\Phi\left(\frac{x}{\sigma_i}\right) - \Phi(\sqrt{\theta_i})}{1 - \Phi(\sqrt{\theta_i})}, \quad x \geq \mu_i.
$$
and thus the conditional distribution of \( e_{ij} \), given \( a_{ij}^* = -1 \) is a zero mean, \( \sigma_i^2 \) variance Gaussian distribution truncated at the point \( \mu_i \). For \( |\mu_i| > 2 \) the conditional distributions of \( e_{ij} \), given \( a_{ij}^* \), will each be close to a zero mean \( \sigma_i^2 \) variance Gaussian distribution. Let \( D_i \) be the \( m \times 1 \) vector of \( \{ D_{ij} \}_{j=1}^m \) values and \( a_i^* \) be the \( m \times 1 \) vector of \( \{ a_{ij}^* \}_{j=1}^m \) values. Conditional on \( a_i^* \), we approximate the distribution of \( D_i \) as multivariate Gaussian with mean vector \( a_i^* \mu_i \) and variance–covariance matrix \( \sigma_i^2 I \), where \( I \) is the \( m \times m \) identity matrix. We utilize this approximation in what follows.

Define \( V_i^F = mD_i^2/\sigma_i^2 \). Noting that \( V_i^F = D_i^T A D_i \), where \( A = J/(m\sigma_i^2) \) and \( J \) is an \( m \times m \) matrix of ones, it follows from Searle (see [15], p. 57, Theorem 2) that, conditional on \( a_i^* \), \( V_i^F \) has an approximate noncentral chi-square distribution with \( 1 \) degree of freedom and noncentrality parameter \( \lambda_1 = \theta_i (\sum_{j=1}^m a_{ij}^* - \bar{a}_i^*)^2/(2m) \). Next let \( U_i^2 = \sum_{j=1}^m (D_{ij} - \bar{D}_i)^2/(m - 1) \) and define \( W_i^F = (m - 1)U_i^2/\sigma_i^2 \). Noting that \( W_i^F = D_i^T B D_i \), where \( B = (I - J/m)/\sigma_i^2 \), it similarly follows that, conditional on \( a_i^* \), \( W_i^F \) has an approximate noncentral chi-square distribution with \( m - 1 \) degrees of freedom and noncentrality parameter \( \lambda_2 = \theta_i \sum_{j=1}^m (a_{ij}^* - \bar{a}_i^*)^2/2 \). Moreover, since \( AB = 0 \), \( V_i^F \) and \( W_i^F \) are independently distributed (see, for example, Searle [15], p. 59, Theorem 3). Define \( \hat{\theta}_{i}^{pF-F} = \hat{D}_i^2/\hat{U}_i^2 \). It follows that, conditional on \( a_i^* \),

\[
F_i^F = \frac{V_i^F}{W_i^F} = m\hat{\theta}_{i}^{pF-F}
\]

has an approximate doubly noncentral F distribution (see, for example, Tiku [17]) with numerator and denominator degrees of freedom equal to 1 and \( m - 1 \), respectively, and numerator and denominator noncentrality parameters equal to \( \lambda_1 \) and \( \lambda_2 \), respectively. Using results from Tiku [17], we have

\[
E(\hat{\theta}_{i}^{pF-F}|a_i^*) = \frac{1}{m - 3} \left( 1 + \frac{2\lambda_1}{m - 1} \right)
\]

\[
Var(\hat{\theta}_{i}^{pF-F}|a_i^*) = \frac{2}{m^2} \left( 1 - \frac{1}{m - 1} \right)^2 \left[ \frac{(1 + 2\lambda_1)^2}{(m - 3)(m - 5)} + \frac{4\lambda_1}{m - 5} \right] \left( 1 + \frac{2\lambda_2}{m - 1} \right)^{-2}.
\]

Since \( a_{ij}^* \) is a \{-1, +1\} random variable, it is easy to verify that \( \lambda_1 \) and \( \lambda_2 \) can equivalently be expressed as \( \lambda_1 = \theta_i (m - 2N_i - m)^2/(2m) \) and \( \lambda_2 = 2\theta_i N_i (m - N_i)/m \), where \( N_i \) is the number of \( \{ a_{ij}^* \}_{j=1}^m \) values that are equal to \(+1\). Consequently, the unconditional mean of \( \hat{\theta}_{i}^{pF-F} \) can be approximated by

\[
E(\hat{\theta}_{i}^{pF-F}) = \frac{1}{m - 3} E \left[ \frac{1 + 2\lambda_1}{1 + 2\lambda_2/(m - 1)} \right],
\]

where the expectation on the right-hand side is with respect to the distribution of \( N_i \) which is binomial with parameters \( m \) and \( p(\theta_i) \). Similarly, the unconditional variance of \( \hat{\theta}_{i}^{pF-F} \) can be approximated by
where both the mean and variance operators on the right-hand side are with respect to the distribution of $N_i$. Equations (5) and (6) together provide the components needed to evaluate the MSE of $\hat{\theta}_i^{PI-F}$.

An approximate bias correction for $\hat{\theta}_i^{PI-F}$ can be motivated from the special case where $p(\theta_i)$ is close to unity and therefore $N_i$ is stochastically close to $m$. In the special case where $N_i = m$, $\lambda_{1i}$ and $\lambda_{2i}$ reduce to $m\theta_i/2$ and zero, respectively. It follows from (5) that

$$E(\hat{\theta}_i^{PI-F}) = \frac{1}{m} \frac{m-1}{m-3} (1 + m\theta), \quad \text{for } p(\theta_i) \approx 1,$$

which suggests an approximate bias-corrected version of $\hat{\theta}_i^{PI-F}$ is $\hat{\theta}_i^{BC-F} = (m-3)\hat{\theta}_i^{PI-F} / (m-1) - 1/m$. It follows from (5) and (6) that

$$E(\hat{\theta}_i^{BC-F}) = \frac{1}{m} \left[ E\left\{ \frac{1 + 2\lambda_{1i}}{1 + \left( \frac{2\lambda_{2i}}{m-1} \right)} \right\} - 1 \right],$$

$$Var(\hat{\theta}_i^{BC-F}) = \frac{2(m-3)}{m^2} E\left\{ \left[ \frac{1 + 2\lambda_{1i}}{(m-3)(m-5)} + \frac{1 + 4\lambda_{1i}}{m-5} \right] \left( 1 + \frac{2\lambda_{2i}}{m-1} \right)^{-2} \right\}$$

$$+ \frac{1}{m^2} Var\left\{ \frac{1 + 2\lambda_{1i}}{1 + \left( \frac{2\lambda_{2i}}{m-1} \right)} \right\}.$$
The following lemma shows how to linearly combine two estimators, based on pilot bits and user bits respectively, to yield a new estimator that has smaller MSE.

**LEMMA 1**: Suppose \( T_1 \) and \( T_2 \) are independent estimators of a parameter \( \delta \) with biases \( B_1 \) and \( B_2 \), respectively, and variances \( \sigma_1^2 \) and \( \sigma_2^2 \), respectively. Consider the class \( C \) of estimators of \( \delta \) which have the form \( T(\alpha_1, \alpha_2) = \alpha_1 T_1 + \alpha_2 T_2 \), where the weights \( \alpha_1 \) and \( \alpha_2 \) are arbitrary real numbers. The weights of the estimator in \( C \) that has the smallest MSE are

\[
\alpha_i^{opt} = \frac{\left( \frac{\delta + B_i}{\sigma_i} \right)^2}{1 + \frac{B_i}{\delta}} \left[ 1 + \left( \frac{\delta + B_1}{\sigma_1} \right)^2 + \left( \frac{\delta + B_2}{\sigma_2} \right)^2 \right] \quad (i = 1, 2),
\]

and the resulting MSE of \( T(\alpha_1^{opt}, \alpha_2^{opt}) \) is

\[
MSE[T(\alpha_1^{opt}, \alpha_2^{opt})] = \frac{\delta^2}{1 + \left( \frac{\delta + B_1}{\sigma_1} \right)^2 + \left( \frac{\delta + B_2}{\sigma_2} \right)^2}.
\]

**PROOF OF LEMMA 1**: The MSE of an arbitrary estimator of the form \( T(\alpha_1, \alpha_2) = \alpha_1 T_1 + \alpha_2 T_2 \) is \( MSE(T) = \alpha_1^2 \sigma_1^2 + \alpha_2^2 \sigma_2^2 + [\alpha_1(\delta + B_1) + \alpha_2(\delta + B_2) - \delta]^2 \). It follows that

\[
\frac{\partial}{\partial \alpha_i} MSE(T) = 2\alpha_i \sigma_i^2 + 2[\alpha_1(\delta + B_1) + \alpha_2(\delta + B_2) - \delta] \frac{\partial}{\partial \alpha_i} (\delta + B_i),
\]

for \( i = 1, 2 \). Setting the two partial derivative equations equal to zero and solving for \( \alpha_1 \) and \( \alpha_2 \) gives the optimal weights and substituting them into the expression for \( MSE(T) \) gives the minimum MSE value.

The class of estimators defined in Lemma 1 allow arbitrary weights on \( T_1 \) and \( T_2 \). If instead we impose the constraint that \( \alpha_1 + \alpha_2 = 1 \), the estimator \( T(\alpha_1, \alpha_2) = \alpha_1 T_1 + (1 - \alpha_1) T_2 \)
becomes a weighted average of $T_1$ and $T_2$. If both $T_1$ and $T_2$ are unbiased then $T(\alpha_1, \alpha_2)$ will also be unbiased.

**Lemma 2:** Suppose $T_1$ and $T_2$ are independent estimators of a parameter $\delta$ with biases $B_1$ and $B_2$, respectively, and variances $\sigma_1^2$ and $\sigma_2^2$, respectively. Consider the class $C^*$ of estimators of $\delta$ which has the form $T(\alpha) = \alpha T_1 + (1 - \alpha)T_2$, where $0 \leq \alpha \leq 1$. The weight $\alpha$ of the estimator in $C^*$ that has the smallest MSE is

$$\alpha^{opt} = \frac{MSE_2 - B_1B_2}{MSE_1 + MSE_2 - 2B_1B_2},$$

where $MSE_i = \sigma_i^2 + B_i^2$, and the resulting MSE of $T(\alpha^{opt})$ is

$$MSE[T(\alpha^{opt})] = \frac{MSE_1MSE_2 - B_1^2B_2^2}{MSE_1 + MSE_2 - 2B_1B_2}.$$

The proof of Lemma 2 is similar to the proof of Lemma 1 and is thus omitted.

**Corollary 1:** If one of the estimators, say $T_1$, in Lemma 2 is unbiased, then

$$\alpha^{opt} = \frac{MSE_2}{MSE_1 + MSE_2} = \frac{MSE_1^{-1}}{MSE_1^{-1} + MSE_2^{-1}},$$

and

$$MSE[T(\alpha^{opt})] = \frac{MSE_1MSE_2}{MSE_1 + MSE_2}.$$ 

Note that the optimal weight in Corollary 1 can equivalently be expressed as $\alpha^{opt} = \sigma_1^{-2}/(\sigma_1^{-2} + MSE_2^{-1})$, and when both estimators are unbiased, $\alpha^{opt} = \sigma_1^{-2}/(\sigma_1^{-2} + \sigma_2^{-1})$, which is a well-known result (see, for example, Christensen [2], Theorem 2.7.1).

We now apply Corollary 1 using $\hat{\theta}_i^{BC}$ and $\hat{\theta}_i^{BC-Z}$ as $T_1$ and $T_2$, respectively. The condition $B_1 = 0$ is met. Define

$$\hat{\theta}_i^{C1} = \alpha^{opt}(\theta_i)\hat{\theta}_i^{BC} + [1 - \alpha^{opt}(\theta_i)]\hat{\theta}_i^{BC-Z},$$

where

$$\alpha^{opt}(\theta_i) = \frac{MSE(\hat{\theta}_i^{BC-Z})}{MSE(\hat{\theta}_i^{BC}) + MSE(\hat{\theta}_i^{BC-Z})}$$

with $MSE(\hat{\theta}_i^{BC})$ and $MSE(\hat{\theta}_i^{BC-Z})$ given by (2) and (4), respectively. It follows from Corollary 1 that
A practical approach is to estimate the optimal weights using an unbiased estimator such as 
and the MSE of \(\hat{\theta}_{i}^{C1}\) of the weight is given to 
hand, for intermediate to large values of SINR, where the bias in 
SINR (where the bias in 
the case of 
curves), as a function of SINR, for the combined estimators obtained from 
expressions.

An alternative way to combine \(\hat{\theta}_{i}^{BC}\) and \(\hat{\theta}_{i}^{BC-Z}\) is to use Lemma 1 to form the estimator \(\hat{\theta}_{i}^{C2}\) = \(\alpha_{1}^{opt}(\theta_{i})\hat{\theta}_{i}^{BC}\) + \(\alpha_{2}^{opt}(\theta_{i})\hat{\theta}_{i}^{BC-Z}\). Since \(\hat{\theta}_{i}^{BC}\) is unbiased, \(B_{1} = 0\). The second term in (4) is \(B_{2}\), and thus

\[
\alpha_{1}^{opt}(\theta_{i}) = \frac{\theta_{i}^{2}}{\text{Var}(\hat{\theta}_{i}^{bc})} + \frac{(\theta_{i} + B_{2})^{2}}{\text{Var}(\hat{\theta}_{i}^{bc})}.
\]

\[
\alpha_{2}^{opt}(\theta_{i}) = \frac{(\theta_{i} + B_{2})^{2}}{\text{Var}(\hat{\theta}_{i}^{bc-Z})}.
\]

and the MSE of \(\hat{\theta}_{i}^{C2}\) is

\[
\text{MSE}(\hat{\theta}_{i}^{C2}) = \frac{\theta_{i}^{2}}{\text{Var}(\hat{\theta}_{i}^{bc})} + \frac{(\theta_{i} + B_{2})^{2}}{\text{Var}(\hat{\theta}_{i}^{bc-Z})}.
\]

Since both \(\hat{\theta}_{i}^{C1}\) and \(\hat{\theta}_{i}^{C2}\) depend on \(\theta_{i}\) through the optimal weights, neither is directly useable. A practical approach is to estimate the optimal weights using an unbiased estimator such as \(\hat{\theta}_{i}^{BC}\), for example. We thus define

\[
\hat{\theta}_{i}^{EC1} = \alpha_{1}^{opt}(\hat{\theta}_{i}^{BC})\hat{\theta}_{i}^{BC} + [1 - \alpha_{1}^{opt}(\hat{\theta}_{i}^{BC})]\hat{\theta}_{i}^{BC-Z},
\]

\[
\hat{\theta}_{i}^{EC2} = \alpha_{1}^{opt}(\hat{\theta}_{i}^{BC})\hat{\theta}_{i}^{BC} + \alpha_{2}^{opt}(\hat{\theta}_{i}^{BC})\hat{\theta}_{i}^{BC-Z}.
\]

Analytic approximations for the MSE of \(\hat{\theta}_{i}^{EC1}\) and \(\hat{\theta}_{i}^{EC2}\) are largely intractable, but not particularly necessary since we will not be defining other estimators that depend on these expressions.

Figure 5 shows the optimal constrained (solid curves) and unconstrained weights (dashed curves), as a function of SINR, for the combined estimators obtained from \(\hat{\theta}_{i}^{BC}\) and \(\hat{\theta}_{i}^{BC-Z}\) for the case of \(n = 8\) and \(m = 20\). In the case of constrained weights, Figure 5 shows that for small SINR (where the bias in \(\hat{\theta}_{i}^{BC-Z}\) is significant) most of the weight is given to \(\hat{\theta}_{i}^{BC}\). On the other hand, for intermediate to large values of SINR, where the bias in \(\hat{\theta}_{i}^{BC-Z}\) has become small, most of the weight is given to \(\hat{\theta}_{i}^{BC-Z}\) since it is based on a larger sample size. In the case of unconstrained weights, there is no crossover point. For small SINR values, both unconstrained weights are small which makes the combined estimator relatively small. As the SINR increases, the weight on \(\hat{\theta}_{i}^{BC-Z}\) increases more rapidly than the weight on \(\hat{\theta}_{i}^{BC}\).
Figure 6 shows the simulated RMSE values of $\hat{\theta}_i^{C1}$ and $\hat{\theta}_i^{C2}$ labeled as C1 (S) and C2 (S), and the analytic RMSE approximations labeled as C1 (A) and C2 (A), respectively. The approximate values match the simulated values quite well. By comparing the ordinate scales of Figures 2 and 3 to the same in Figure 6, it is clear that the combined estimator offers significant reductions in RMSE compared to the individual estimators, $\hat{\theta}_i^{BC}$ and $\hat{\theta}_i^{BC-Z}$.

Figure 7 shows the effect of estimating the optimal weights by contrasting the simulated RMSE values of $\hat{\theta}_i^{C1}$ and $\hat{\theta}_i^{C2}$ with the simulated RMSE values of $\hat{\theta}_i^{EC1}$ and $\hat{\theta}_i^{EC2}$, which are labeled as EC1 (S) and EC2 (S), respectively. It is evident from Figure 7 that the cost of estimating the unconstrained weights is appreciably more than the cost of estimating the constrained weights. In fact, although the RMSE of $\hat{\theta}_i^{C2}$ is significantly smaller than the RMSE of $\hat{\theta}_i^{C1}$, the RMSE of $\hat{\theta}_i^{EC2}$ and $\hat{\theta}_i^{EC1}$ are virtually the same. Once again, comparing the ordinate scales of Figures 2 and 3 to the same in Figure 7 demonstrates that the cost of estimating the weights does not diminish the value of using combined estimators.

Figure 5. Ideal constrained and unconstrained weights for combining the BC and BC-Z estimators.

Figure 6. RMSE of combined BC and BC-Z estimators (known weights).
5. SMOOTHED VARIANCE ESTIMATORS

5.1. Pilot-Based Estimator

The AWGN model for the demodulator output discussed in Section 1 shows that the variance of the noise factor, $\sigma_i^2$, varies from time slot to time slot. It has been recognized that $\sigma_i^2$ is a slowly varying function of $i$ (see Viterbi [19]). The average noise plus interference component, for example, would only change with the addition/departure of a call in a CDMA system, or when a discernible “rearrangement” of users has taken place due to mobility. The slowly varying nature of $\sigma_i^2$ suggests using exponential weighting of the within time slot estimates of the noise variance. In particular, we could consider an estimator of $\sigma_i^2$ of the form

$$\hat{\sigma}_i^2 = (1 - r)^{i-1}S_i^2 + \sum_{k=2}^{i} r(1 - r)^{i-k}S_k^2,$$

where $0 < r \leq 1$. An alternative recursive form of $\hat{\sigma}_i^2$ is ($i \geq 2$)

$$\hat{\sigma}_i^2 = rS_i^2 + (1 - r)\hat{\sigma}_{i-1}^2.$$

Note that $r = 1$ delivers $\hat{\sigma}_i^2 = S_i^2$. Higuchi [7] smoothes variance estimates calculated using both pilot and user bits. Analogous to $\hat{\theta}_i^{PI}$, we could consider an estimator of $\theta_i$ of the form $\hat{\theta}_i^{SV} = \hat{P}_i^2/\hat{\sigma}_i^2$, which, of course, reduces to $\hat{\theta}_i^{PI}$ when $r = 1$. Furthermore, we could consider scaled versions of $\hat{\theta}_i^{SV}$ to reduce bias.

The MSE properties of $\hat{\theta}_i^{SV}$ depend on the nature of the $\{\sigma_k^2\}$ sequence. An interesting class of models for the $\{\sigma_k^2\}$ sequence would be an appropriate family of stationary stochastic processes (e.g., ARMA models). The random nature of $\{\sigma_k^2\}$ implies the $\{\theta_k\}$ sequence is also random. The inference problem shifts from estimating the parameter $\theta_i$ to predicting the realized value of the random variable $\theta_i$. The MSE of $\hat{\theta}_i^{SV}$ would then be evaluated as $E(\hat{\theta}_i^{SV} - \theta_i)^2$, where the expectation is with respect to the joint distribution of $\hat{\theta}_i^{SV}$ and $\theta_i$. We defer the evaluation of the MSE of $\hat{\theta}_i^{SV}$ in the context of stochastic models for $\{\sigma_k^2\}$ to a future paper and consider here only the MSE properties of $\hat{\theta}_i^{SV}$ in the special case where $\sigma_k^2 = \sigma^2$. The MSE
results for the special case where \( \sigma_i^2 = \sigma^2 \) are approximations to the case where \( \{\sigma_k^2\} \) is a slowly varying sequence that has a mean equal to \( \sigma^2 \).

Define \( W = (n - 1) \hat{\sigma}^2 / \sigma^2 = \sum_{k=1}^i w_k(n - 1)S_k^2 / \sigma^2 \), where \( w_1 = (1 - r)^{i-1} \) and \( w_k = r(1 - r)^{i-k} \), for \( 2 \leq k \leq i \). Since the terms \( (n - 1)S_k^2 / \sigma^2 \) are independent and identically distributed chi-square random variables with \( n - 1 \) degrees of freedom, it follows that the mean and variance of \( W \) are \( \mu_W = n - 1 \) and \( \sigma_W^2 = 2(n - 1)r/(2 - r) \), respectively, where for \( \sigma_W^2 \) we have used the fact that \( \sum_{k=1}^i w_k^2 \) converges to \( r/(2 - r) \) as \( i \to \infty \).

We shall approximate the distribution of \( W \) using a Satterthwaite approximation and write \( W \sim h \chi^2(\nu) \), where \( h = \sigma_W^2/(2\mu_W) \) and \( \nu = 2\mu_W / \sigma_W^2 \). Making the appropriate substitutions, we find \( \nu = (n - 1)(2 - r)/r \) and \( h = r/(2 - r) \). Next let \( V_i = n\hat{P}_i^2 / \sigma^2 \). It follows that \( V_i \) has a noncentral chi-square distribution with \( 1 \) degree of freedom and noncentrality parameter \( \lambda_i \). It is well known that \( \hat{P}_i \) (and hence \( V_i \)) is independently distributed of \( S_k^2 \), and it is clearly independent of \( S_1^2, \ldots, S_{i-1}^2 \). Thus, \( V_i \) and \( W \) are distributed independently and consequently

\[
F = \frac{V_i}{(W/h)} = \frac{n\hat{P}_i^2}{\nu} \tag{9}
\]

has an approximate noncentral F distribution with numerator and denominator degrees of freedom equal to \( 1 \) and \( \nu \), respectively, and noncentrality parameter equal to \( \lambda_i \). It follows that

\[
MSE(\hat{\theta}_i^S) = \frac{2\nu^2}{n^2(\nu - 2)} \left[ \frac{(\nu - 2)(\nu - 4)}{\nu - 4} + \frac{1 + 2n/\theta_i}{\nu - 4} \right] + \left[ \frac{\nu}{\nu - 2} \left( \frac{1 + \theta_i}{n} - \theta_i \right) \right]^2 \tag{9}
\]

The squared bias term in (9) suggests a bias corrected estimator of the form \( \hat{\theta}_i^{BCSV} = (\nu - 2)\hat{\theta}_i^S / \nu - (1/n) \) from which it follows that

\[
MSE(\hat{\theta}_i^{BCSV}) = \frac{2(\nu - 2)}{n^2} \left[ \frac{(\nu - 2)(\nu - 4)}{\nu - 4} + \frac{1 + 2n/\theta_i}{\nu - 4} \right]. \tag{10}
\]

We note that in the special case where \( r = 1 \), then \( \nu = n - 1 \) and (9) and (10) reduce to (1) and (2), respectively.

Figure 8 shows the simulated RMSE values of \( \hat{\theta}_i^{BCSV} \), labeled as BCSV (S), for the case where \( n = 8 \) and \( r = 0.1 \). Approximate RMSE values for \( \hat{\theta}_i^{BCSV} \) using (10) agree very closely with the simulated values shown in Figure 8. We also note that there is very little difference in the RMSE values of \( \hat{\theta}_i^S \) and \( \hat{\theta}_i^{BCSV} \), suggesting that bias correction after variance-smoothing has a small effect.

### 5.2. Absolute Value Estimator

Let \( U^2 = \sum_{k=1}^i w_k T_k^2 \) and define \( \hat{\theta}_i^{SV-Z} = Z_i / U^2 \). We now derive an approximation for the MSE of \( \hat{\theta}_i^{SV-Z} \). Recalling from Section 3.1 the definition \( W_k = (m - 1)T_k^2 / \sigma_{Z,k}^2 \), let

\[
W^Z = \frac{(m - 1)U^2}{\sigma^2} = \sum_{k=1}^i \frac{w_k \sigma_{Z,k}^2 W_k}{\sigma^2}.
\]
As in Section 3.1, we approximate the distribution of \( W_k \) as \( g_k \chi^2 \), where \( g_k = g(\theta_k) \) and \( \eta_k = \eta(\theta_k) \) are the scale and degree-of-freedom parameters of a Satterthwaite approximation. Explicit expressions for both \( g_k \) and \( \eta_k \) were previously mentioned as appearing in Appendix A. It follows that approximate expressions for the mean and variance of \( W_k \) are thus

\[
E(W_k) \approx (m - 1) \sum_{k=1}^{i} w_k \frac{\sigma^2_{Z,k}}{\sigma^2}
\]

\[
Var(W_k) \approx 2(m - 1) \sum_{k=1}^{i} w_k^2 \left( \frac{\sigma^2_{Z,k}}{\sigma^2} \right)^2 g_k.
\]

As noted in Section 3.1, \( \sigma^2_{Z,k} \to \sigma^2 \) and \( g_k \to m - 1 \) as \( \theta_k \) gets large. Consequently, for large \( \theta_k \), \( E(W_k) \approx m - 1 \) and \( Var(W_k) \approx 2(m - 1)^2 r/(2 - r) \). A Satterthwaite approximation based on the large \( \theta_k \) approximations for the mean and variance of \( W_k \) implies \( W_k \sim h_k \chi^2_{\nu_k} \), where \( h_k = r/(2 - r) \) and \( \nu_k = (m - 1)(2 - r)/r \) from which it follows that

\[
P_k = \frac{V_{i_k}^2}{W_k} = m_0 \frac{\theta_{SV-Z}}{h_k \nu^2_k}
\]

has an approximate noncentral F distribution with numerator and denominator degrees of freedom equal to 1 and \( \nu^2_k \), respectively, and noncentrality parameter equal to \( \lambda^2_k \). An approximate expressions for the MSE of \( \theta_{SV-Z}^i \) is thus

\[
MSE(\hat{\theta}_{SV-Z}^i) = \frac{2(\nu^2_k)^2}{m^*(\nu^2 - 2)} \left[ \frac{(1 + m\theta_{Z_i})^2}{(\nu^2 - 2)(\nu^2 - 4)} + \frac{1 + 2m\theta_{Z_i}}{\nu^2 - 4} \right] + \frac{\nu^2_k}{\nu^2 - 2} \left( \frac{1}{m} + \theta_{Z_i} \right) - \theta_i^2.
\]

(11)
The approximate MSE formula for $\hat{\theta}_i^{BCSV-Z}$ is similar to the MSE formula for $\hat{\theta}_i^{SV}$, the two differences being that $m$ is used in place of $n$ and $\theta_{Zj}$ is used in place of $\theta_j$. The squared bias term in (11) suggests a bias corrected estimator of the form $\hat{\theta}_i^{BCSV-Z} = (v^2 - 2)\hat{\theta}_i^{SV-Z} + (1/m)$ from which it follows

$$\text{MSE}(\hat{\theta}_i^{BCSV-Z}) = \frac{2(v^2 - 2)}{m^2} \left[ \frac{(1 + m\theta_{Zj})^2}{(v^2 - 2)(v^2 - 4)} + \frac{1 + 2m\theta_{Zj}}{v^2 - 4} \right] + (\theta_{Zj} - \theta_j)^2. \quad (12)$$

Figure 8 includes the simulated RMSE values of $\hat{\theta}_i^{BCSV-Z}$, labeled as BCSV-Z (S), for the case where $m = 20$ and $r = 0.1$. The simulated RMSE values for $\hat{\theta}_i^{SV-Z}$ are very close to the simulated RMSE values of $\hat{\theta}_i^{BCSV-Z}$, and consequently are not shown in Figure 8. In addition, the approximate RMSE values from (11) and (12) match closely with the simulated values and thus are not shown in Figure 8.

### 5.3. Combined Smoothed Estimators

Estimators based on combining $\hat{\theta}_i^{BCSV}$ and $\hat{\theta}_i^{BCSV-Z}$ can be obtained with constrained or unconstrained weights using the approximate MSE expressions given by (10) and (12), together with the weight formulas given in Section 4. Figure 8 shows the simulated RMSE values of $\hat{\theta}_i^{C3} = \alpha_1^{opt}(\theta_j)\hat{\theta}_i^{BCSV} + \alpha_2^{opt}(\theta_j)\hat{\theta}_i^{BCSV-Z}$, labeled as C3 (S) for the case where $m = 20$ and $r = 0.1$. The analytic approximate RMSE values for $\hat{\theta}_i^{C3}$ are not shown since they match the simulated RMSE values extremely well. We conclude from Figure 8 that the RMSE of $\hat{\theta}_i^{C3}$ is uniformly smaller than the RMSE of any other estimator considered in this paper, and for a majority of the operating region the magnitude of the reduced RMSE is appreciable. Also shown in Figure 8 are the simulated RMSE values of $\hat{\theta}_i^{FC3} = \alpha_1^{opt}(\hat{\theta}_i^{BCSV})\hat{\theta}_i^{BCSV} + \alpha_2^{opt}(\hat{\theta}_i^{BCSV})\hat{\theta}_i^{BCSV-Z}$, where the weights have been estimated using $\hat{\theta}_i^{BCSV}$. The increase in the RMSE due to using estimated weights is quite apparent, however, it is still the case that the RMSE of $\hat{\theta}_i^{FC3}$ is uniformly smaller than any other useable estimator considered in this paper.

For completeness, we note that the RMSE values of the constrained combined estimators $\hat{\theta}_i^{C4} = \alpha_1^{opt}(\theta_j)\hat{\theta}_i^{BCSV} + [1 - \alpha_1^{opt}(\theta_j)]\hat{\theta}_i^{BCSV-Z}$ and $\hat{\theta}_i^{EC4} = \alpha_1^{opt}(\hat{\theta}_i^{BCSV})\hat{\theta}_i^{BCSV} + [1 - \alpha_1^{opt}(\hat{\theta}_i^{BCSV})]\hat{\theta}_i^{BCSV-Z}$ are very close to each other and to the RMSE of $\hat{\theta}_i^{FC3}$. Thus, when estimated weights must be used, it does not matter much whether estimated unconstrained weights or estimated constrained weights are used.

### 6. SUMMARY

Several important control actions in wireless communication systems (e.g., transmit power level determination, handoff decisions, data rate adaptation algorithms) depend on an accurate SINR estimator. In this paper, we have developed SINR estimators using some new approaches that yield estimators with significantly smaller RMSE than traditional estimators. Estimators based on pilot bits only or user bits only were considered first and then estimators that are linear combinations of these two types of estimators were discussed. It was discussed that for estimating the interference variance (the denominator of the SINR), long-term averaging via exponential smoothing is viable and dramatically reduces the RMSE of SINR estimators.

Approximate analytic expressions for the RMSE of the estimators were derived and their accuracy was validated using simulation methods. The analytic expressions serve two purposes. First, improved estimators based on bias correction and/or variance reduction become apparent
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</tr>
<tr>
<td>$\hat{\theta}^{C3}$</td>
<td>5.3</td>
<td>Unconstrained linear combination of $\hat{\theta}^{BCSV}$ and $\hat{\theta}^{BCSV-Z}$</td>
<td>Smallest RMSE of all the estimators considered in this paper, but weights depend on unknown SINR</td>
</tr>
<tr>
<td>$\hat{\theta}^{EC3}$</td>
<td>5.3</td>
<td>Variation of $\hat{\theta}^{C3}$ where weights are estimated using $\hat{\theta}^{BCSV}$</td>
<td>RMSE of $\hat{\theta}^{C3}$ remains the smallest of all the estimators considered in this paper, even when the weights are estimated</td>
</tr>
<tr>
<td>$\hat{\theta}^{C4}$</td>
<td>5.3</td>
<td>Constrained linear combination of $\hat{\theta}^{BCSV}$ and $\hat{\theta}^{BCSV-Z}$</td>
<td>RMSE values are comparable to those of $\hat{\theta}^{C3}$</td>
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<tr>
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<td>RMSE values are comparable to those of $\hat{\theta}^{EC3}$</td>
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</table>
from the expressions and, second, optimal weights for combining pilot bit and user bit based estimators can be evaluated. Table 1 provides a summary and review of all the different estimators considered in this paper. A brief comment about the RMSE performance of each estimator is also provided. In general, our results show that the best estimators are obtained through optimal combining. Using estimated weights in place of the unknown optimal weights predictably increases the RMSE, but not to the extent of diminishing the value of combining.

**APPENDIX A: SATTERTHWAITE APPROXIMATION CONSTANTS**

The constants \( g_i = g(\theta_i) \) and \( \eta_i = \eta(\theta_i) \) for the Satterthwaite approximation to the distribution of \( W^2 = (m - 1)T_i^2/\sigma_{Z,i}^2 \), which is utilized in Sections 3.1 and 5.2, satisfy the equations \( m - 1 = g_i \eta_i \) and \( (m - 1)^2 \text{Var}(T_i^2) = 2g_i^2 \eta_i \sigma_{Z,i}^4 \). Equivalently, \( g_i = (m - 1) \text{Var}(T_i^2)/2\sigma_{Z,i}^2 \) and \( \eta_i = 2\sigma_{Z,i}^2/\text{Var}(T_i^2) \). We therefore need to evaluate the ratio \( \text{Var}(T_i^2)/\sigma_{Z,i}^2 \). It is well known (see, for example, Cramer [3], Section 27.4) that

\[
\frac{\text{Var}(T_i^2)}{\sigma_{Z,i}^2} = \frac{(m - 1)\mu_i - (m - 3)\mu_i^2}{m(m - 1)}.
\]

where \( \mu_4 = E(Z_{ij} - \mu_{Z,i})^4 \) and \( \mu_2 = E(Z_{ij} - \mu_{Z,i})^2 = \sigma_{Z,i}^2 \). Clearly, \( \mu_2^2/\sigma_{Z,i}^2 = 1 \) and hence

\[
\frac{\text{Var}(T_i^2)}{\sigma_{Z,i}^2} = \frac{\mu_4}{m\sigma_{Z,i}^2} - \frac{m - 3}{m(m - 1)}.
\] (13)

Expanding \( \mu_4 \) gives

\[
\mu_4 = -3\mu_2^2 + 6\mu_2^3 E(Z_{ij}^2) - 4\mu_2 E(Z_{ij}^3) + E(Z_{ij}^4).
\] (14)

Clearly,

\[
E(Z_{ij}^4) = \mu_4 + \sigma_i^4.
\] (15)

Writing \( Z_{ij}^4 = (\alpha_{ij}\mu_i + \epsilon_{ij})^4 \) and expanding the right-hand side, it is straightforward to show that

\[
E(Z_{ij}^4) = 3\mu_i^4 + 6\mu_i^3\sigma_i^2 + \mu_i^4.
\] (16)

It remains to find \( E(Z_{ij}^3) \) which we can evaluate as

\[
E(Z_{ij}^3) = E(\alpha_{ij}\mu_i + \epsilon_{ij})^3 = E(\alpha_{ij}\mu_i + \epsilon_{ij})^3|\alpha_{ij} = 1 \times p + E(\alpha_{ij}\mu_i + \epsilon_{ij})^3|\alpha_{ij} = -1 \times (1 - p),
\] (17)

where \( p = \text{Pr}(\alpha_{ij} = 1) \). Let \( R \) be a random variable with a Gaussian distribution having mean \( \mu_r \) and variance \( \sigma_r^2 \). It follows from (16) that

\[
E(Z_{ij}^3) = E[|R|^3 \times p + E[|R|^3 \times (1 - p)] = E[|R|^3].
\]

Evaluating \( E|R|^3 \) directly gives

\[
E(Z_{ij}^3) = (\mu_i^3 + 3\sigma_i^2\mu_i) \left[ 2\Phi \left( \frac{\mu_i}{\sigma_i} \right) - 1 \right] + \frac{1}{\sqrt{2\pi}} e^{-|\mu_r|/\sigma_r^2} (2\sigma_r^2 + 4\sigma_i^2).
\] (18)

Combining (14)–(18) gives

\[
\frac{\mu_4}{\sigma_{Z,i}^4} = -3 \left( \frac{\mu_2}{\sigma_{Z,i}} \right)^4 + 6 \left( \frac{\mu_2}{\sigma_{Z,i}} \right)^2 \left[ \left( \frac{\mu_1}{\sigma_{Z,i}} \right)^2 + 3 \left( \frac{\sigma_{Z,i}}{\sigma_r} \right)^2 + \left( \frac{\mu_1}{\sigma_{Z,i}} \right)^2 \right] + 4 \left( \frac{\mu_2}{\sigma_{Z,i}} \right)^3 \left[ \left( \frac{\mu_1}{\sigma_{Z,i}} \right)^2 + \left( \frac{\sigma_{Z,i}}{\sigma_r} \right)^2 + \left( \frac{\mu_1}{\sigma_{Z,i}} \right)^2 \right] \left[ 2\Phi \left( \frac{\mu_i}{\sigma_i} \right) - 1 \right]
\]

\[
+ \frac{1}{\sqrt{2\pi}} e^{-|\mu_r|/\sigma_r^2} \left[ 2 \left( \frac{\sigma_{Z,i}}{\sigma_r} \right)^2 + 4 \left( \frac{\sigma_{Z,i}}{\sigma_r} \right)^4 \right] + 3 \left( \frac{\sigma_{Z,i}}{\sigma_r} \right)^4 + 6 \left( \frac{\mu_2}{\sigma_{Z,i}} \right)^2 \left( \frac{\sigma_{Z,i}}{\sigma_r} \right) + \left( \frac{\mu_4}{\sigma_{Z,i}} \right)^4.
\]

It is easy to verify that the ratios \( \mu_{Z,i}/\sigma_{Z,i}, \mu_2/\sigma_{Z,i}, \) and \( \sigma_2/\sigma_{Z,i} \) all depend on \( \mu_i \) and \( \sigma_i \) through the ratio \( \theta_i \). Thus, the ratio \( \mu_{Z,i}/\sigma_{Z,i} \) is a function of \( \theta_i \). It therefore follows from (13) that \( Var(T_i^2)/\sigma_{Z,i}^2 \) is a function of \( \theta_i \) and, consequently, \( g_i \) and \( \eta_i \) are as well.
APPENDIX B: SIMULATION DETAILS

Simulations were run to obtain RMSE estimates that could be compared to the analytic RMSE approximations. An overview of the simulation details is provided in this appendix. For each \( i \) in the set \( \{ -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \} \), each estimator was computed for 50,000 datasets of demodulator outputs and the simulation estimate of the RMSE was obtained from sample RMSE. The sample size of 50,000 was deemed sufficient based on observed accuracy of the simulation results in cases where exact analytic RMSE formulas are available (e.g., \( \hat{\theta}^{PI} \) and \( \hat{\theta}^{BC} \)).

For the estimators based on pilot bits (Section 2), each dataset of demodulator outputs consisted of \( n = 8 P_{ij} \) observations that were simulated from a Gaussian distribution with mean \( \sqrt{\theta_i} \) and variance 1. In the case of the smoothed pilot bit-based estimators (Section 5.1), a warm-up period of 2000 datasets was used to allow the initial value associated with the exponential smoothing of the variance term to dissipate.

For the absolute value estimators based on user bits (Section 3.1), each dataset of demodulator outputs consisted of \( m = 20 U_{ij} \) observations that were simulated from a symmetric mixture of two Gaussian distributions that have respective means \( \pm \sqrt{\theta_i} \) and variance 1. Again, a warm-up period of 2000 datasets was used when computing the smoothed user bit-based absolute value estimator (Section 5.2).

The datasets used for the feedback estimator (Section 3.2) were the same as those used for the absolute value estimator, but in this case the \( U_{ij} \) observations were used to obtain user bit predictions (\( \hat{a}_{ij} \)) rather than the \( Z_{ij} \) observations. The user bit predictions were then used to obtain the \( D_{ij} \) observations from which the feedback estimator was computed.

For the combined estimators in Section 4, each dataset consist of both \( n = 8 P_{ij} \) observations (used to compute the estimator based on pilot bits) and \( m = 20 U_{ij} \) observations (used to compute the estimator based on user bits). For the combined smoothed estimators in Section 5.3, each dataset similarly consisted of both \( P_{ij} \) observations and \( U_{ij} \) observations, but in addition 2000 preliminary datasets were used as a warm-up period.

ACKNOWLEDGMENTS

We would like to thank Yasuo Amemiya for a helpful question pertaining to the justification for our approximation to the MSE of the feedback estimator. We thank the referees for comments and suggestions that greatly improved the readability of our paper.

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