Multicast Multigroup Beamforming under Per-antenna Power Constraints

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Abstract

Linear precoding exploits the spatial degrees of freedom offered by multi-antenna transmitters to serve multiple users over the same frequency resources. The present work focuses on simultaneously serving multiple groups of users, each with its own channel, by transmitting a stream of common symbols to each group. This scenario is known as physical layer multicasting to multiple co-channel groups. Extending the current state of the art in multigroup multicasting, the practical constraint of a maximum permitted power level radiated by each antenna is tackled herein. The considered per antenna power constrained system is optimized in a maximum fairness sense. In other words, the optimization aims at favoring the worst user by maximizing the minimum rate. This Max-Min Fair criterion is imperative in multicast systems, where the performance of all the receivers listening to the same multicast is dictated by the worst rate in the group. An analytic framework to tackle the Max-Min Fair multigroup multicasting scenario under per antenna power constraints is therefore derived. Numerical results display the accuracy of the proposed solution and provide insights to the performance of a per antenna power constrained system.

I. INTRODUCTION

Multiuser multiple antenna transmitters are the way forward towards achieving the high throughput demands of next generation systems. Advanced transmit signal processing techniques are employed to optimize the performance of the multi-antenna transmitter without compromising the complexity of the receivers. A fundamental requisite for the application of these techniques, namely linear transmit

This work was partially supported by the National Research Fund, Luxembourg under the project “CO²SAT: Cooperative & Cognitive Architectures for Satellite Networks’.
beamforming (also referred to as precoding) is the perfect knowledge of the channel state at the transmitter. Subsequently, the exploitation of the spatial degrees of freedom offered by the antenna array mitigates interferences thus allowing co-channel beams to be made adjacent. In this fashion, a Spatial Division Multiple Access (SDMA) scheme is realized.

Physical layer multicasting has the potential to efficiently address the nature of traffic demand in future systems and has become part of standards such as LTE. When multiple multicast co-channel groups are considered towards trading off between the unicasting and broadcasting system functionalities, then the design can be optimized in a manner described hereafter.

An important application of physical layer multigroup multicasting can be found when the goal is to optimize full frequency reuse multibeam transmitters without changing the framing structure of communication standards. For instance, specific physical layer designs are optimized to cope with noise limited channels with long propagation delay. What is more, the framing of multiple users per transmission is emanated to guarantee scheduling efficiency when long forward error correction codes are employed. Consequently, precoding techniques in such systems cannot be based on the conventional user-by-user design and multigroup multicasting needs to be considered.

II. RELATED WORK

In the multigroup multicasting literature, two fundamental optimization problems have been considered until now; the sum power minimization under specific quality of service (QoS) constraints and the maximization of the minimum SNIR (Max-Min Fair). In the following, a brief literature review is provided

A. Sum Power Minimization under QoS constraints

In multi-antenna systems, the optimal downlink transmission strategy, in the sense of minimum total transmit power that guarantees specific QoS constraints at each user, was given in [1], [2]. Therein, the powerful tool of Semi-Definite Relaxation (SDR) reduced the non-convex quadratically constrained quadratic program (QCQP) into a relaxed convex problem by changing the optimization variables and disregarding the unit-rank constraints over the new variable. One fundamental assumption of this work has been the independence of the data addressed to the multiple users. Under this assumption, the SDR guarantees an optimal solution to the QCQP problem [1].

Based on the principles of SDR, the QoS problem in a physical layer multicasting scenario was

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1 The multi-antenna multicast problem was originally proposed in [3] where the maximization of the average SNIR over all users was the goal, without however guaranteeing some QoS at the user side.
tackled by Sidiropoulos et al. in [4]. The inherent difference between the prior scenario of transmitting independent information to each user, is that in the physical layer multicasting problem, a set of common symbols is addressed to all the users. The general multicasting problem was proven NP-hard [4]. This result suggested that if the polynomial in complexity SDR method could provide the globally optimal solution, then it would be possible to solve a whole class of computationally challenging problems in polynomial time. Consequently and in order to derive a low complexity and accurate approximate solution to the multicast problem, SDR was combined with Gaussian randomization [5]. In more detail, a candidate solution to the original problem is generated as a random Gaussian variable with statistics given by the solution of the relaxed problem. After generating a finite number of random Gaussian instances, depending on the desired accuracy of the solution, the instance that yields the best objective value of the original problem is chosen. A key step in this process remains the simple power rescaling of the random Gaussian solutions which guarantees the feasibility of the original problem without affecting the beamforming directions.

A unified framework for physical layer multicasting to multiple interfering groups was given in [6], [7]. In this case, independent sets of common data are transmitted to multiple interfering groups of users by the multiple antennas. Subsequently, common information is now restricted to users that belong to the same group while independent information is sent to different groups. In the extreme cases of either one user per group or a single group that includes all the users, the model reduces to the independent data or the multicast scenarios respectively. This general problem was also shown to be NP-hard to solve, since it includes an NP-hard problem as a special case [7]. Despite the existence of an optimal solution for the single user per group case of the multicast multigroup scenario via SDR, a general solution to the multicast multigroup beamforming optimization problem is more complicated. The combination of SDR and Gaussian randomization is also not straightforward. The difficulty lies in the coupling between the multicast groups. More specifically, the issue of intergroup interference arises due to the independent data being transmitted to different groups over the same channels. Hence, in contrast to the multicast case, after obtaining candidate solutions from the Gaussian randomization process the feasibility of the initial QCQP cannot be guaranteed by a simple rescaling of the randomized precoding vectors. Since the the co-channel groups are coupled by interferences, rescaling the power of one precoder invalidates the solution. To the end of solving the elaborate multicast multigroup problem, Karipidis et al [7] proposed an additional step following the Gaussian randomization. This step consists of a new optimization problem, namely the multigroup multicast power control (MMPC), that converts the candidate solutions into feasible solutions of the original problem [6], [7]. This power control is a linear program that guarantees the feasibility of
the original problem.

For the sake of completeness, in parallel to [6], the independent work presented in [8] is pointed out, where the multicast multigroup problem is also tackled. Despite the use of the SDR method combined with Gaussian randomization to solve the QoS problem, Gao and Schubert [8] rely on dirty paper coding DPC to successively precancel intergroup interferences, thus simplifying the power allocation procedure. Nevertheless, the design of such a transmitter suffers from the non-linear implementation complexity.

B. Max-Min Fair beamforming under Sum-Power constraints

The maximization of the minimum signal to noise plus interference ratio (SNIR) received by any of the available users in the coverage area, subject to a sum power constraint (SPC) at the transmitter is a problem closely related to the QoS problem. The goal of this problem is to maximize the fairness of the system by boosting the SNIR of the worst user. However, this is accomplished at the expense of the total system sum-rate, otherwise achievable by favouring the well conditioned users. As proven in [4], a QoS problem with equal SNIR constraints is equivalent to the Max-Min-Fair problem up to scaling. Thus, after establishing that the latter is also NP-hard to solve, a customized low complexity algorithm for an approximate solution to the fairness problem was derived [4]. In the same direction, the maximum fairness problem was also formulated, proven NP-hard and solved in a multicast multigroup scenario in [7]. In more detail, the equivalence the QoS and the Max-Min-Fair problems is used to solve the latter. A bisection search method over the relaxed power minimization problem provides candidate solutions to the initial fairness problem. However, feasibility of the original problem with these solutions is not guaranteed and an additional power allocation needs to come in play. The complication in the fairness scenario is that the multigroup multicast power control program does not admit a linear program reformulation and thus its solution is not trivial. To overcome this, a bisection search algorithm is again performed, this time over the multigroup multicast power control of the power minimization problem.

In optimization terminology, the fact that the SDR can provide a global optimum for the original unicast problem shows that strong duality holds. In this light, the authors of [9], tackled the multicast multigroup problem under SPC, based on the framework of uplink-downlink duality [10]. Since in the multigroup multicast case one precoder needs to apply for multiple users and thus the principles of duality cannot be straightforwardly applied, approximations where employed. Subsequently, low complexity multicast-aware solutions where proposed in [9]. These solutions where shown to tradeoff the low complexity with

\[2\text{It can however be reformulated as a geometric problem (GP) and solved efficiently with interior point methods [7].}\]
the inferior performance in terms of BER (i.e. minimum SNIR) compared to [7], under numerically exhibited convergence.

Under a different system model, the solution of the Max-Min Fair problem is given in [11], for coordinated multicast multi-cell systems. In this scenario, a sum power constraint is no longer applicable hence the the QoS problem is no longer related to the max-min-fair problem with per base-station (BS) constraints. Nevertheless, in each BS a single group of receivers is assumed. Hence, a power constraint over each precoder is imposed. A solution of the new optimization problem was given following the well established framework of bisection by a modification of the related QoS problem.

Despite the extensive literature on the topic of multigroup multicasting, no existing work has tackled the multigroup multicast problem under per antenna power constraints (PAC). The per-antenna constraints are commonly introduced by the practical limitations of transmitters. When power sharing of the per antenna dedicated RF chains is not possible, then the SPC is no longer applicable. For the sake of clarity, the difference between coordinated and cooperative multicell networks is pointed out. In cooperative multicast multicell systems, all BSs jointly transmit to multicast groups. This varies from the coordinated case of [11] where each BS transmits to a single multicast group.

The remaining of the paper is structured as follows. The multigroup multicast system model is presented in Sec. III. The Max-Min Fair under PAC problem formulation is given in Sec. IV along with a detailed solution. Numerical results are provided in Sec. V while the conclusions are drawn in Sec. VI.

III. System Model

The focus is on a multi-user (MU) multiple input single output (MISO) Multicast system. Assuming a single transmitter, let $N_t$ denote the number of transmitting elements and by $N_u$ the number of users. The input-output analytical expression will read as $y_i = h_i^\dagger x + n_i$, where $h_i^\dagger$ is a $1 \times N_t$ vector composed of the channel coefficients (i.e. channel gains and phases) between the $i$-th user and the $N_t$ antennas of the transmitter, $x$ is the $N_t \times 1$ vector of the transmitted signal and $n_i$ is the independent complex circular symmetric (c.c.s.) independent identically distributed (i.i.d) zero mean Additive White Gaussian Noise (AWGN) measured at the $i$-th user’s receive antenna.

Focusing in a multigroup multicasting scenario, let there be a total of $1 \leq G \leq N_u$ multicast groups with $\mathcal{I} = \{G_1, G_2, \ldots, G_G\}$ the collection of index sets and $G_k$ the set of users that belong to the $k$-th

$^3$Notation: Throughout the paper, bold face lower case characters denote column vectors and upper case denote matrices. The operators $(\cdot)^\dagger$, $|\cdot|$ and $||\cdot||_2$, denote the conjugate transpose, the absolute value and the Frobenius norm operations, respectively, while $[\cdot]_{ij}$ denotes the $i, j$-th element of a matrix. The principal eigenvalue of a matrix $X$ are denoted as $\lambda_{max}(X)$. 

July 2, 2014 DRAFT
multicast group, \( k \in \{1 \ldots G \} \). Each user belongs to only one group, thus \( G_i \cap G_j = \emptyset, \forall i, j \in \{1 \ldots G \} \).

Let \( w_k \in \mathbb{C}^{N_t \times 1} \) denote the precoding weight vector applied to the transmit antennas to beamform towards the \( k \)-th group. By assuming independent information transmitted to different groups, the symbol streams \( \{s_k\}_{k=1}^{G} \) are mutually uncorrelated and the total power radiated from the antenna array is equal to \( P_{\text{tot}} = \sum_{k=1}^{G} ||w_k||^2 \). The power radiated by each antenna element is also a linear combination of all precoders and reads as

\[
P_n = \left[ \sum_{k=1}^{G} w_k w_k^\dagger \right]_{nn},
\]

where \( n \in \{1 \ldots N_t \} \). In (1) the fundamental difference between the SPC of [7] and the proposed PAC is pointed out. Herein, instead of one, \( N_t \) constraints are imposed, each one involving all the precoders. The main complication of this is that the QoS problem is no longer related to the Max-Min problem and the straightforward application of bisection is no longer possible. A more general constraint formulation to model power flexibility amongst groups of antennas as well as to apply non-linear with respect to the power constraints can be found in [13] and this generalization is part of future work.

### IV. Multicast Multigroup Beamforming with Per Antenna Power Constraints

The focus is on the Max-Min Fair problem with per antenna constraints which reads as

\[
\mathcal{F}: \quad \max_{t, \{w_k\}_{k=1}^{G}} t
\]

subject to

\[
\sum_{l \neq k} ||w_k w_l^\dagger||^2 + \sigma_i^2 \geq t,
\]

and to

\[
\left[ \sum_{k=1}^{G} w_k w_k^\dagger \right]_{nn} \leq P_n,
\]

\( \forall i \in G_k, k, l \in \{1 \ldots G \}, \forall n \in \{1 \ldots N_t \} \),

where \( w_k \in \mathbb{C}^{N_t} \) and \( t \in \mathbb{R} \). Problem \( \mathcal{F} \), with input \( p = [P_1, P_2 \ldots P_{N_t}] \), optimal value denoted as \( t^* = \mathcal{F}(p) \) and optimal point \( \{w_k^F\}_{k=1}^{G} \), is non-convex due to (2). The difference of the present formulation with respect to the Max-Min Fair problem with SPC presented in [4], [7] lies in (3), where there exist \( N_t \) power constraints over each individual radiating element. This equation also differentiates the present formulation from the coordinated multicell multicasting Max-Min problem since the per antenna constraint is imposed on the \( n \)-th diagonal element of the summation of the correlation matrices of all precoders. On the contrary, in [11], the imposed per base station constraints are translated to one power constraint per each precoder.
The sum transmit power minimization problem under QoS constraints [1], is related to the Max-Min problem [14] and this relation was generalized for the multicast multigroup case in [7]. Hence by bisecting the solution of the QoS problem, a solution to the fairness problem can be derived. Nevertheless, two fundamental differences between the existing formulations and problem \( \mathcal{F} \) complicate the solution. Firstly, the power constraints are not necessarily met with equality. Rescaling the power of one precoding vector affects the power inequality constraints of all antennas. Consequently, if only one constraint is satisfied with equality and some power budget is still left, a rescaling to approach the rest \( N_t - 1 \) constraints will over-satisfy the first constraint and render the problem infeasible. Secondly, the absence of a related, solvable problem prohibits the immediate application of bisection.

To the end of providing a framework for the per antenna constrained Max-Min Fair precoding similar to [7] and by elaborating on the insights of [11], a novel per antenna power utilization minimization problem is defined as

\[
\mathcal{Q} : \min_{r, \{w_k\}_{k=1}^G} \quad r
\]

subject to

\[
\frac{|w_k^h h_i|^2}{\sum_{l \neq k} |w_l^h h_i|^2 + \sigma_i^2} \geq t \quad (4)
\]

and to

\[
\frac{1}{P_n} \left[ \sum_{k=1}^G w_k w_k^* \right]_{nn} \leq r \quad (5)
\]

\( \forall i \in G, k, l \in \{1 \ldots G\}, \forall n \in \{1 \ldots N_t\} \).

Problem \( \mathcal{Q} \) receives as input a common QoS constraint to all users, namely \( t \), and the per antenna power constraint vector \( \mathbf{p} = [P_1, P_2 \ldots P_{N_t}] \). Subsequently, the maximum power consumption out of all antennas is minimized and this solution can be denoted as \( r^* = \mathcal{Q}(t, \mathbf{p}) \).

**Claim 1:** Problems \( \mathcal{F} \) and \( \mathcal{Q} \) are related as follows

\[
1 = \mathcal{Q}(\mathcal{F}(\mathbf{p}), \mathbf{p}) \quad (6)
\]

\[
t = \mathcal{F}(\mathcal{Q}(t, \mathbf{p}) \cdot \mathbf{p}) \quad (7)
\]

**Proof:** The above claim will be proven by contradiction. Let \( r^* = \mathcal{Q}(t, \mathbf{p}) \) denote the optimal value of \( \mathcal{Q} \) with associated variable \( \{w_k^Q\}_{k=1}^G \). Assuming that the optimal value of \( \mathcal{F} \) under constraints scaled by the solution of \( \mathcal{Q} \) is different, i.e. \( \hat{t} = \mathcal{F}(\mathcal{Q}(t, \mathbf{p}) \cdot \mathbf{p}) \) with \( \{w_k^F\}_{k=1}^G \), the following contradictions arise. In the case where \( \hat{t} < t \), then the precoders \( \{w_k^Q\}_{k=1}^G \) are feasible solutions to \( \mathcal{F}(\mathcal{Q}(t, \mathbf{p}) \cdot \mathbf{p}) \) which lead to a higher minimum SNIR, thus contradicting the optimality of \( \hat{t} \). Alternatively, if \( \hat{t} > t \) then the solution set \( \{w_k^F\}_{k=1}^G \) can be scaled by a positive constant \( c = \frac{t}{\hat{t}} < 1 \). The new solution \( \{cw_k^F\}_{k=1}^G \) respects the
feasibility conditions of \( Q \) and provides a lower optimal value, i.e. \( c \cdot r^* \), thus again contradicting the optimality hypothesis of \( \{ w^G_{Q k} \}_{k=1}^G \). The proof of (6) follows an identical line of reasoning and is omitted for shortness. □

Having defined an equivalent problem that admits a solution via the well known SDR technique, the original Max-Min Fair problem with PAC can be solved via a one dimensional bisection search over problem \( Q \). More details on bisection are given in Sec. [IV-D]

A. Semidefinite Relaxation

Problem \( Q \) belongs to the general class of non-convex QCQPs for which the SDR technique has proven a powerful and computationally efficient approximation technique. In the spirit of [II], \( Q \) can be rewritten by using the change of variables \( X_i = w_i w_i^\dagger \) and introducing two additional constraints. Hence, the new variable is constrained to be symmetric positive semi-definite and unit-rank. However, the latter is a non-convex constraint. The SDR method consists of dropping the unit-rank constraint and reducing the original problem into

\[
Q_r : \min_{r, \{X_k\}_{k=1}^G} r \\
\text{subject to } \frac{\text{Tr} \left( Q_i^\dagger X_k \right)}{\sum_{l \neq k}^G \text{Tr} \left( Q_l^\dagger X_k \right) + \sigma_i^2} \geq t \\
\text{and to } \frac{1}{P_n} \left[ \sum_{k=1}^G X_k \right]_{nn} \leq r \\
\text{and to } X_k \succeq 0, \\
\forall i \in G_k, k, l \in \{1 \ldots G\}, \forall n \in \{1 \ldots N_i\},
\]

(8)

(9)

(10)
\[
\begin{align*}
\mathcal{Q} & : \min_{r, \{X_k\}_{k=1}^G} r \\
\text{subject to} & \quad \frac{\text{Tr} \left( Q_i X_k \right)}{\sum_{l \neq k}^G \text{Tr} \left( Q_l X_k \right) + \sigma_i^2} \geq t \\
\text{and to} & \quad \frac{1}{P_n} \left[ \sum_{k=1}^G X_k \right]_{nn} \leq r \\
\text{and to} & \quad X_k = w_k w_k^\dagger \geq 0, \\
\text{and to} & \quad \text{rank} \left( X_k^* \right) \neq 1
\end{align*}
\]

where \( Q_i = h_i h_i^\dagger \). \( Q_r \) is convex and can therefore be solved to an arbitrary accuracy [15]. In the same direction, the Max-Min Fair optimization can be also relaxed as

\[
\begin{align*}
\mathcal{F}_r & : \max_{t, \{X_k\}_{k=1}^G} t \\
\text{subject to} & \quad \text{Tr} \left( Q_i X_k \right) \frac{\text{Tr} \left( Q_i X_k \right)}{\sum_{l \neq k}^G \text{Tr} \left( Q_l X_k \right) + \sigma_i^2} \geq t \\
\text{and to} & \quad \left[ \sum_{k=1}^G X_k \right]_{nn} \leq P_n \\
\text{and to} & \quad X_k \succeq 0,
\end{align*}
\]

which, however, remains non-convex due to the constraint (15); an obstacle that will be overcome in the remaining by acknowledging that the relaxed problems are also related by (6) and (7).

The problems described so far belong to the general class of multigroup multicasting problems. Mathematically, the main difference is that the number of precoding vectors is equal to the number of groups and thus less than the number of users. This fact, renders these problems NP-hard and the semidefinite relaxation cannot provide a globally optimal solution to the original problem. Therefore, the approximations described in the following section need to be employed.

### B. Gaussian Randomization

For specific optimization problems, the SDR provides globally optimum solutions. This implies that the relaxed solution \( X^* \) has a unit rank. The most prominent example of this case is the optimal downlink
beamforming under independent data transmission to all users [1]. Nevertheless, due to the NP-hardness of the multicast problem, the relaxed problems do not necessarily yield unit rank matrices. Consequently, one can apply a rank-1 approximation over $X^*$ and use the principal eigenvalue and eigenvector as an approximate solution to the original problem. Other types of rank-1 approximations are also possible depending on the nature of the original problem.

Despite the effectiveness and intuitive simplicity of any rank-1 approximation, the solution with the highest provable accuracy for the multicast case is given by the Gaussian randomization method [5]. Let $X^*$ be a symmetric positive semidefinite solution of the relaxed problem. Then, a candidate solution to the original problem can be generated as a Gaussian random variable with zero mean and covariance equal to $X^*$, i.e. $\hat{w}_k \sim \mathcal{CN}(0, X^*_k)$. Nonetheless, an intermediate step between generating a Gaussian instance with the statistics obtained from the relaxed solution and creating a feasible candidate instance of the original problem still remains, since the feasibility of the original problem is not yet guaranteed. This step is described in the following section.

Finally, after generating a predetermined number of candidate solutions, the one that yields the highest objective value of the original problem can be chosen. The accuracy of this approximate solution is measured by the distance of the approximate objective value and the optimal value of the relaxed problem. This accuracy increases with the predetermined number of Gaussian randomizations.

C. Feasibility Power Control

Despite the wide applicability of the Gaussian randomization method, one has to bare in mind that it is a problem dependent procedure. After generating a random instance of a Gaussian variable with statistics defined by the relaxed problem, an additional step comes in play to guarantee the feasibility of the original problem. In [4], a simple power rescaling of the candidate solutions which follows the Gaussian randomization is sufficient to guarantee feasibility. Nevertheless, baring in mind that in the multigroup case an interference scenario is dealt with, different than in [4], a simple rescaling does not guarantee feasibility. Therefore, an additional optimization step is proposed in [7] that distributes the power amongst the candidate precoders whilst guaranteeing the feasibility of the original problem. In the same direction, a novel power control problem with per antenna power constraints is defined herein. Given a set of Gaussian instances, denoted as $\{\hat{w}_k\}_{k=1}^G$, the Multigroup Multicast Per Antenna power
Control (MMPAC) problem is defined as

\[
S^F : \max_{t, \{p_k\}_{k=1}^G} t \\
\text{subject to } \frac{|w_h^k|^2 p_k}{\sum_{l \neq k}^G |w_l^k|^2 p_l + \sigma_i^2} \geq t \\
\text{and to } \sum_{k=1}^G \hat{w}_k \hat{w}_k^\dagger p_k \leq P_n
\]

\forall i \in G, k, l \in \{1 \ldots G\}, \forall n \in \{1 \ldots N_t\},

with \(\{p_k\}_{k=1}^G > 0\). Problem \(S^F\) receives as input the per antenna power constraints and returns the maximum worst SNIR \(t^* = S(p)\) and is also non convex in the fashion of \(F\).

A very important remark is clear in the formulation of the power control problem. The optimization variable \(p\) is of size \(G\), i.e. equal to the number of groups, while the power constraints are equal to the number of antennas, \(N_t\). In each constraint, all the optimization variables contribute. This fact prohibits the total exploitation of the available power at the transmitter. Once at least one of the \(N_t\) constraints is satisfied with equality and remaining power budget, then the rest can not be scaled up since this would lead to at least one constraint exceeding the maximum allowable value.

D. Bisection

As already mentioned, a solution to non-convex problems can be obtained by iteratively solving a related problem at the midpoint of an interval that includes the original optimal value. To be more precise, let us consider problems \(Q_r\) and \(F_r\), which are related as given in (6) and (7). Since \(t\) represents the SNIR and is actually a fraction of two positive quantities, it will always be positive or zero. Also, if the system was interference free while all the users had the channel of the best user, then the maximum worst SNIR would be \(\max_i \{P_{tot} Q_i / \sigma_i\}\). These bounds do not affect the optimal solution as long as they include it. Further analytic investigation of a tighter upper bound can reduce the bisection iterations but is not included in the present work for brevity. By defining the interval \([L, U]\) with the minimum and maximum values of SNIR, the solution of \(r^* = Q_r ((L + U) / 2, p)\) is obtained. If this solution is lower than 1 then the lower bound of the interval is updated with this value. Otherwise the value assigned to the upper bound of the interval. Bisection is iteratively continued until an the interval size is reduced to a pre-specified value \(\epsilon\). After a finite number of iterations the optimal value of \(F_r\) is given as the resulting value for which \(L\) and \(U\) become almost identical. This procedure provides an accurate solution to the relaxed non-convex Max-Min Fair problem. Actually, this value is the upper bound of
the minimum SNIR and it can be used to evaluate how close to the optimal solution the approximate solutions generated by Gaussian randomization are.

The precoder correlation matrices for which the optimal value of the relaxed problem is achieved are possible solutions to the original problem. In the case that these solutions are unit rank, then the original problem is globally solved by the non zero eigenvalue and the corresponding eigenvector of the relaxed solution. Nevertheless, when this is not the case, the Gaussian randomization described in Sec. [V-B] is employed. Hence a predetermined number of random Gaussian solutions are generated. Following this, for each and every solution \( \{ \hat{w}_k \}^{G}_{k=1} \), the power of the precoders needs to be controlled so that feasible candidate solutions can be obtained. Once the MMPAC is solved, the candidate precoder can be calculated as

\[
\{ w^*_k \}^{G}_{k=1} = \sqrt{\{ p_k \}^{G}_{k=1}} \cdot \{ \hat{w}_k \}^{G}_{k=1}.
\]  

The final complication of the solution process lies in the non-convexity of \( S^F \). Hence, bisection needs to be employed\(^4\) again over its convex equivalent

\[
\begin{align*}
S^Q & : \min_{r, \{ p_k \}^{G}_{k=1}} r \\
\text{subject to} & \quad \frac{|\hat{w}_k^\dagger h_i|^2 p_k}{\sum_{l \neq k} |\hat{w}_l^\dagger h_i|^2 p_l + \sigma^2_i} \geq t \\
\text{and to} & \quad \frac{1}{P_n} \left[ \sum_{k=1}^{G} \hat{w}_k \hat{w}_k^\dagger p_k \right]_{nn} \leq r \\
\forall i \in G_k, k, l \in \{1 \ldots G\}, \forall n \in \{1 \ldots N_t\}.
\end{align*}
\]  

It should be noted that in this case, the bisection interval can be further reduced and thus the efficiency of the algorithm greatly improved by exploiting the fact that the maximum optimal value of \( S^F \) cannot be greater than the optimal value of \( F_r \) which has already been calculated. Thus the bisection interval is constrained between zero and \( t^* \) of the relaxed problem and the number of iterations is reduced.

\[4\]A possible reformulation as a GP is not considered herein for the sake of brevity.

E. Complexity

The complexity of the SDR and Gaussian randomization technique has been exhaustively discussed in \[5\] and the references therein. In brief, the complexity can be considered as follows. The interior point methods that solve the SDR problems require at most \( O(\sqrt{LN_t} \log(1/\epsilon)) \), where \( \epsilon \) is the desired accuracy,
while the arithmetic operations for each iteration are not more than \( O(L^3 N_t^6 + KLN_t^2) \). Modern solvers such as the CVX tool [15] which calls numerical solvers such as SeDuMi for SD programs, also exploit the specific structure of matrices and the actual running time is reduced. Furthermore, the bisection technique typically runs for \( N_{\text{iter}} = \lceil \log_2 (U - L) \rceil < \epsilon \). Finally, the Gaussian randomization can be executed for an arbitrary number of iterations of course with increasing accuracy. Typically, 100 randomizations are performed [4], [7]. The general complexity of the proposed approach is increased due to the \( N_t \) constraints but still in a polynomial and thus computationally efficient manner.

F. Summary

The overall procedure to acquire an approximate solution to the original multigroup multicast NP-hard Max-Min Fair problem under PAC is summarized in Alg. 1.

\begin{algorithm}
\begin{enumerate}
  \item \textbf{Step 1}: Solve \( \mathcal{F}_r \) by bisecting \( Q_r \), (see Sec. IV-D). Let this solution be \( \{w_{k}^{\text{opt}}\}_{k=1}^{G} \) and the associated SNIR as \( t_{\text{opt}} \). This solution will be the upper bound for any solution of \( \mathcal{F} \).
  \item if \( \text{rank}(X_{k}^{\text{opt}}) = 1, \forall k \in \{1 \ldots G\} \) then
     \item the output is the dominant eigenvector and the max eigenvalue \( \lambda_{\text{max}}(X^{\text{opt}}) \).
  \item else \textbf{Step 2}: Gaussian randomization: generate \( N_{\text{rand}} \) precoding vectors \( \{\hat{w}_k\}_{k=1}^{G} \), (see Sec. IV-B).
  \item \( t_{(0)}^* = 0; \)
  \item for \( i = 1 \ldots N_{\text{rand}} \) do
     \item \textbf{Step 3}: Solve \( \mathcal{S}^\mathcal{F} \) by bisecting the related \( S_{\mathcal{Q}} \). The corresponding solution \( \{w_{k}^{\text{can}}\}_{k=1}^{G} \) reads as in (20) with associated optimum value \( t_{(i)}^* \).
     \item if \( t_{(i)}^* > t_{(i-1)} \) then
       \item the current solution becomes the output;
     \item end
  \item end
\end{enumerate}
\end{algorithm}

Algorithm 1: Max-Min-Fair Multigroup Multicasting under Per Antenna power Constraints.
V. Numerical Results

The performance of Linear Multicast Multigroup beamforming under per antenna power constraints is presented herein. A system with $N_t = 5$ transmit antennas, $G = 2$ groups and $N_u = 4$ users is assumed unless stated otherwise. Rayleigh fading is considered, thus the channel instances are generated as Gaussian complex variable instances with unit variance and zero mean. For every channel instance, the solutions of the Max-Min Fair SPC [7] and the proposed PAC problems are evaluated using $N_{rand} = 50$ Gaussian randomizations. Noise variance is normalized to one for all receivers.

Firstly, the accuracy of the approximate solution is numerically shown. The Max-Min Fair minimum SNIR under SPC and PAC constraints is plotted in Fig. 1 with respect to the total transmit power to receive noise ratio (SNR) in dB. For fair comparison, the total power constraint $P_{tot}$ [Watts] is equally distributed amongst the transmit antennas when PAC is considered, hence each antenna can radiate at most $P_{tot}/N_t$ [Watts]. The accuracy of the approximate solutions for both problems is clear across a wide range of on board power. Nevertheless, the accuracy due to the per antenna constraints is insignificantly reduced. This is justified by the fact that a Gaussian randomization instance is less likely to approach the optimal point when the number of constraints is increased. It is reminded that both problems are solved under the same number of Gaussian randomizations ($N_{rand} = 50$).

A significant discussion over the SDR techniques in multicast applications is the scaling of the approximate solution to the NP-hard problem versus an increasing number of receivers per multicast. In the extreme case of one user per group, it was proven in [1] that the relaxation provides an optimal solution. Thus the solution is no longer approximate but exact. However, the increasing number of users per group degrades the solution, as depicted in Fig. 2 for both problems. It is especially noticed that the PAC system suffers equivalently with respect to the SPC of [7] as the number of users per multicast group increases.

VI. Conclusions

The problem of optimizing the linear precoding design under per antenna power constraints, when common data is addressed to multiple co-channel groups is tackled in the present work. A novel framework to find an approximate solution to the NP-hard multigroup multicast problem with PAC is proposed. Under this framework, the linear precoders for the multi-antenna systems with limited power in each transmitting element, can be accurately approximated in polynomial time. Consequently, an important practical constraint towards the implementation of physical layer multigroup multicasting is alleviated.
Minimum SNIR vs Power

Fig. 1. Minimum SNIR with SPC and PAC versus increasing total transmit power over the noise level (SNR [dB]).

REFERENCES


Fig. 2. Minimum SNIR with SPC and PAC versus increasing ratio of users per group $\rho = N_u/G$. 


