Exploring Mathematics through the Analysis of Errors*

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I. Introduction

A review of the mathematics education literature shows that researchers as well as teachers in this field have given serious consideration to students' mathematical errors (see Radatz [1980]). In particular, it has been acknowledged that errors can be a powerful tool to diagnose learning difficulties and consequently direct remediation. Research using this interpretation of the role of errors has provided valuable contributions to mathematics education, such as an increased awareness of individual differences and difficulties in learning mathematics, and the realization of the inefficiency of remediating errors by simply explaining the same topic over again or assigning additional practice exercises. However, most of these studies still share a rather limited conception of errors, one in which errors are seen as signals that something has gone awry in the learning process and that remediation is needed.

On the contrary, the work of philosophers and historians of science such as Kuhn [1970], Lakatos [1976] and M. Kline [1980] can help us realize that errors have a much more fundamental role in the growth of a discipline. The history of mathematics, for example, has shown on more than one occasion that it is possible to capitalize on errors in various ways, regardless of any tendency to repeat or eliminate them. Sometimes, failure to meet an initial goal has led to unexpected and revolutionary results. Think for example of Saccheri's creation of a non-Euclidean geometry, which was generated by an unsuccessful attempt to "prove" the parallel postulate. The problems and contradictions encountered in the early development of the calculus also provide a very interesting example of the role of errors in the history of mathematics. In fact, the errors made initially in this field shook mathematicians' confidence to the point of motivating a revision of the methodology used in the discipline. The search for rigor eventually uncovered profound questions about the foundation of mathematics. Less spectacular, but equally important for the growth of mathematical knowledge, is the use that mathematicians make of their own errors in their everyday research. Incorrect conjectures, unjustified guesses, and partial results are all necessary and invaluable steps in the creation of new mathematical results. Lakatos [1976] has provided very interesting historical examples of this aspect of mathematical discovery.

We suggest, however, that geniuses and professional mathematicians are not the only ones who could, and should, make use of the potential of mathematical errors, both as springboards for problem solving and problem posing and as grist for critical thinking on the nature of mathematics itself. Mathematics students, at their own levels and dealing with more elementary areas of mathematics, could profit from an interpretation of errors as the motivation and means for exploration in mathematics [1]

Before analyzing in more detail the alternative interpretation of the role of errors as "springboards for inquiry," and discussing its implications for mathematics instruction, we would like to briefly discuss an example in order to give the reader a flavour for the educational potential of mathematical errors and of the variety of ways in which they could be exploited to enhance learning.

II. Using errors for exploration versus diagnosis: an exemplification

An error commonly encountered when students learn to operate with fractions is illustrated by the following examples:

$$3/4 + 6/7 = 9/11; \quad 2/3 + 5/7 = 7/10.$$

The extensive body of research on "error analysis" in mathematics education has already made us aware of the value of probing the causes of a mistake of this kind. Indeed, this mistake may contain valuable information about the specific difficulties that the student encountered in learning fractions, and about his/her conceptions of fractions, rules and mathematics. The analysis, or diagnosis of this error could be accompanied by questions such as: What is the alternative rule that the student is applying here to add fractions? Why is s/he doing so?

It seems rather clear, in this case, that the student is adding fractions by simply adding numerators and denominators separately. The reasons behind such behavior, however, could be various. For instance, the student could have just confused the rule for *adding* fractions with the rule for *multiplying* fractions. Another plausible explanation of this error is that its author tried to operate with fractions as s/he did when adding whole numbers. If a fraction is seen as two numbers separated by a line, it is not so unreasonable to perform the addition of two fractions by operating on the "top" numbers and "bottom" numbers separately, and finally by drawing a line between the two answers.

Another explanation is also possible, as a few authors have observed [Meyerson, 1976; Kline, 1980]. There are

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some real-life situations in which such a way of operating seems indeed appropriate, as the following examples illustrate:

a) *baseball batting average:*
If a player gets 3 hits out of 4 times at bat in one game and 6 hits out of 7 times at bat in another, his average is 9 out of 11 and not the traditional sum of 3/4 and 6/7 (45/28)

b) *keeping a record of "game results":*
If you won 2 out of 3 games yesterday, and 5 out of 7 games today, altogether you have won 7 out of 10 games, and not 29/21.

Remediation of this error will certainly be most effective if the teacher is willing and able to hypothesize about its possible causes, and to verify which ones are relevant in the case of each individual making such mistakes.

An alternative approach, consistent with some existing models of remediation, would make use of the "diagnosis" of this error to shed light on the way students view fractions. Educators may thus become aware of the difficulties students often encounter while studying such a topic, and consequently they may prepare more suitable curricula in which fractions are presented in ways that minimize these difficulties. The expectation would be that this kind of analysis could prevent a similar error from occurring with other students in the future.

What else could be done besides trying to "eliminate" this error? What kinds of questions could be raised and pursued, if we look at this error in a different light, and if we want to involve the students themselves in its analysis?

We will briefly sketch some possible avenues of exploration that can be stimulated quite naturally by this erroneous method of adding fractions (which hereafter will be symbolized by a/b/&/c/d, to distinguish it from the standard addition of fractions).

1) *Can the two rules of addition be equivalent, at least in some cases?*
Are there circumstances under which our error would go undetected, i.e. is it ever the case that a/b & $c/d = a/b + c/d$? Using elementary algebra we can show that the equality:

$$(a + c)/(b + d) = (ad + bc)/bd$$

holds if and only if $c = (-a)d^2/b^2$ [see Carmory, 1978]. However, it may still be interesting to compare the results obtained by adding fractions according to the two alternative rules and searching for patterns, within each structure independently, and between the two structures, even when equality does not prevail.

2) *In what cases does it seem to make sense to use the rule* a/b & c/d?
A closer analysis of situations such as a the baseball batting average or the score keeping exemplified above, may make us aware that in such situations we are not really dealing with *fractions*, but rather with *ratios*. Because we generally symbolize them in the same way (*"n"* over *"m"*, or n/m) we may tend to believe that ratios and fractions are essentially the same mathematical objects. However, since they operate differently at least with respect to addition, they must constitute two distinct systems of numbers. This realization may in turn stimulate further inquiry into the nature of ratios.

3) *Can we create a mathematical system to operate with ratios, in the same way as we have systems of natural numbers, integers or fractions?* (To better distinguish ratios from fractions, hereafter we will use the notation a/b solely to indicate a fraction, while $a{:}b$ will indicate a ratio.)
Attempting to answer this question can involve one in very challenging and creative activities, while requiring a minimal mathematical background. By looking at the more standard systems of numbers as a reference and inspiration, we could explore more specific questions such as:

3.1) *Can we "reduce" ratios, i.e. is* ka:kb = a:b?
This question must be answered negatively, since "equivalent" ratios do not behave in the same way with addition (for example: 1:3 & 2:3 = 3:3, but 10:30 & 2:3 = 12:33)

3.2) *Can ratios have an additive identity? What about 0:0?*
Notice that, while 0/0 is indeterminate if interpreted as a fraction, 0:0 might make sense as a ratio. For instance, such a ratio may be used to record a situation in which one plays no game and wins none. This illustrates another interesting difference between fractions and ratios

3.3) *Could we establish some reasonable rules and meanings for other operations among ratios, besides addition?*
For example, what about subtraction and multiplication? While we can find some interpretation and motivation for a subtraction rule defined as:

$$a{:}b - c{:}d = (a - c){:}(b - d)$$

we have not yet been able to find a reasonable rule and meaningful application for multiplication or division of ratios.

3.4) *Is there an order relation among ratios?*
Again, such an apparently innocent question has no easy answer. Though we would feel comfortable saying that 2:5 < 3:5, what about a comparison between 1:3 and 10:30, or even 1:3 and 9:30?

3.5) *How could we extend the concept of ratio to allow for more operations or properties?*
For instance, what sense could be given to "improper" ratios, such as 7:5? How should "negative" ratios be defined to allow for additive inverses in the system?

4) The last three questions may invite a closer analysis of more standard systems of numbers in the attempt to learn how mathematicians have dealt with similar problems when "creating" the mathematical systems of integers, rational numbers or complex numbers.

For example:

4.1) *What is the meaning of "multiplication" in other numerical systems?*

An answer to this question may provide surprises, since multiplication among complex numbers, or even among fractions, has little to do with the intuitive idea of multiplication as repeated addition.

4 2) *How is "order" determined in numerical systems? How do the standard numerical systems differ with respect to the properties of their internal order relation?*

We may remember, for instance, the problems raised by attempting to introduce an order relation among complex numbers

4.3) *How were systems of numbers successively extended by mathematicians. and why?*

We may be so familiar now with objects such as negative numbers that we do not fully appreciate the creativity and labor behind their creation A look at the history of mathematics can be very instructive in this sense

The difficulties encountered while attempting to explore and extend the "system of ratios" may also generate more general questions:

5) As our analysis shows, though "2/3 + 5/7 = 7/10" will generally be considered a mistake, there are some contexts in which this way of adding seems reasonable. This realization can generate the following questions:

5 1) *Can something be right and wrong at the same time in mathematics?*

5 2) *How can we decide whether a rule is right or wrong in mathematics?*

5.3) *Is it always possible to determine it?*

6) While attempting to define multiplication of ratios, we may be able to come up with a few alternative rules, without, however, finding a satisfactory meaning for any one of them.

6 1) How should we choose among them?

6.2) *How do mathematicians choose rules and definitions when they create new systems?*

7) We have pointed out before that the use of the same "words" and "symbols" (such as *"n"* over *"m"*, or *n/m*) to talk about fractions and ratios could be one of the causes of confusion between these two mathematical objects.

7 1) *What is the effect of symbolization on the learning of mathematics?*

7 2) *What is the effect of choosing different symbolizations on the development of a mathematical topic?*

These last questions may indeed lead us into deep inquiry about the development of mathematical ideas and about the nature of mathematics itself. Let us now move from this specific example, to a more general analysis of the potential of errors as springboards for exploration in mathematics.

III. An interpretation of errors as springboards for exploration

It might at first seem surprising that even a simple error, like the one discussed in the previous section, could motivate such a variety of questions and avenues of exploration.

It is interesting to note that an interpretation of errors solely as tools for diagnosis and remediation would have only partially exploited the educational potential of the error discussed First of all, in such a framework, only teachers and researchers would be involved in the process of analyzing the error. Thus the students themselves would be deprived of the opportunity of engaging in the activity of attempting to "explain" and "fix up" their own errors — an activity that could prove to be highly motivating and challenging. In addition, the creativity of the researchers themselves when analyzing the error would be constrained by their limited focus on finding the causes of the students' error so that they could eliminate it Thus they see the error necessarily as a deviation from an established body of knowledge, and do not even allow themselves to consider it as a possible challenge to the standard results — as we have done instead in questions 2 and 3, when we recognized the difference between fractions and ratios and attempted to develop a mathematical system for the latter. It may be worth remarking, in light of our example, that even simple mathematical errors can lead to such a challenge, without requiring sophisticated mathematical knowledge, nor a very high level of mathematical ability.

Let us now try to identify and discuss *how* and *why* errors could provide the motivation and means for various kinds of exploration in mathemtics

First of all, we must realize that errors present a natural stimulus to action, since they provide evidence that the expected result has not been reached, and that something else has to be done Errors may also provide very valuable information about the causes of such failure, and thus suggest alternatives However, in order to fully appreciate the potential of errors, this must be interpreted in a very broad way Errors not only allow one to identify shortcomings in the strategy chosen to reach a predetermined goal. More importantly, errors also point to strengths and limitation of available strategies Errors may also help us identify specific characteristics of the context, and thus show that the original goal was inadequate and it needs to be redefined.

How this approach to the analysis of errors can go beyond pure remediation, is shown by questions 1 and 2 of our example. There, instead of looking for reasons why the students might have erroneously added fractions, we questioned whether there were particular cases or contexts in which their operation might indeed be considered correct. This analysis of the error can often involve challenging problem-posing and problem-solving activities. As a result, we may not only overcome our original difficulties, but also gain a better understanding of the problem in question, of the situation in which it is embedded, of alternative strategies of solution, and even of ourselves as learners of mathematics.

An even more radical way of making use of the potential of errors to motivate explorations will occur if, instead of analyzing the causes of our "failure," we challenge the context of the error itself by questioning:

— What would be the consequences in mathemat-

ics, if the result of our error were accepted as correct?
— Could a mathematical system be created so that our result would be correct in suich context?

Question 3 in our example, where we explored the behavior of ratios with respect to order, equivalence, and a few arithmetic operations, was developed very much along these lines

To explain this potential of errors for problem generation, we can first of all observe that the mere presence of errors naturally generates a *contrast,* since errors are recognized as such because they do not respond to our original expectations. As we are all aware, even for creative people it is very difficult to generate questions for exploration about a given situation without considerable stimulus. Brown and Walter [1969, 1970, 1983] have suggested an approach to problem generation in mathematics that makes use of the power of contrast for inquiry Though there are five distinct phases of their "What-If-Not" strategy for problem generation, its essence is captured by an identification of the attributes of a given situation, the conscious modification of some of them, and the exploration of the consequences of such changes We suggest that error can sometimes be interpreted as the result of an involuntary change of attributes or assumption, and thus they may naturally provide the stimulus and starting point for inquiry.

Besides involving the learner in the original exploration of a non-standard mathematical situation, challenging an error can eventually help one to gain a deeper understanding of more standard elements of mathematics as well For example, our analysis of the system of ratios invited a closer look at other traditional systems of numbers (as exemplified in our questions 1, 4.2, 4 3). Such analysis can help us realize the existence and importance of certain properties in these systems that we otherwise tend to take for granted. For the first time we could also appreciate the creativity that was necesary to come up with systems such as the integers and the rational numbers, which now constitute such an integral part of elementary mathematics

So far we have discussed ways in which errors could provide the stimulus for explorations that are mostly technical in nature — regarding a specific mathematical object, problem or procedure. Our study of the error *a/b & c/d,* however, showed also the possibility of conducting inquiry at a different level of abstraction.

Our questions 5 to 7, for example, dealt with the possibility of having mathematical rules that are wrong in one context but right in another, and led us to discuss the role of symbolism in mathematics. They almost totally disregarded the specific "mathematical content" of the error studied, and rather used it as an illustration of more abstract and general issues regarding the nature of mathematics In this case, the error motivated an historical and philosophical search, rather than a technical exploration.

We can notice a similar distinction in the positive use of errors which occurred in the development of mathematics The discovery of non-Euclidean geometries provides one of the best examples in this sense The original error (the failure to prove the parallel postulate from the other axioms of Euclidean geometry) certainly helped in the accumulation of more knowledge about Euclidean geometry and alternative geometric systems However, the biggest effect of this error on mathematics occurred at a different level Its analysis, in fact, challenged some very basic assumptions about the nature of geometry. Mathematicians could no longer justify their geometrical results on the ground that they were rigorously derived from self-evident axioms, since alternative theories to Euclidean geometry were also possible and justified. This realization forced the mathematics community to seriously reconsider fundamental questions about proof and truth in mathematics. As a consequence, we can say that our view of mathematics nowadays is quite different from what it was a few centuries ago

There are at least two major directions along which errors can be used to motivate reflection and inquiry about the nature of mathematics:

1) Errors can be used to investigate the nature of fundamental mathematical notions such as "proof," "algorith," or "definition " Since we continually use algorithms, proofs, and definitions when studying mathematics, it may seen that we must know pretty well what these notions are and whether we are making an error in dealing with them On the contrary, it is very difficult to explain what characterizes a "good" mathematical proof (or algorithm or definition) and it is even more difficult to become really aware of their limitation or their roles It may be easier to point out why a certain proof does not seem correct, to attempt to fix it up, and from this concrete process to try to abstract what properties we wish a mathematical proof to have.
2) An analysis of the variety of "degree of wrongness" among mathematical errors can help clarify the nature of "truth" in mathematics When mathematics students, or even teachers, are asked to state the characterizing properties of mathematical results, the following adjectives are recurring: rigorous, exact, complete, unique, non-contradictory Consequently, most people believe they are in the presence of errors if any of these expectations are not met in their solution to a mathematical problem. They will also likely feel that the error in question was caused by their own ignorance and shortcomings, or at best is a temporary deficiency of the present status of mathematical knowledge, which hopefully mathematicians in the future will learn how to take care of An analysis of some "borderline" cases of errors may challenge these unrealistic expectations about mathematical results, and have important consequences for people's conceptions of mathematics and mathematical errors.

Errors can thus help us investigate abstract issues regarding the nature of mathematics that would otherwise be difficult to approach. Again, this makes use of the contrast provided by the error, as well as of its informational content, though with a different focus and at a higher level of

Table 1
Questions generated by some common types of errors

Incorrect definitions

Math Content:

What properties can be derived from this definition?
Which ones fit our image of the concept?
Which ones don t?

What other mathematical objects could be described by this definition?

What instances of the concept are not described by this definition?

Are all the properties stated essential?
Could any be eliminated?

Could we modify the definition and turn it into a correct one?

What if this were the correct definition for the concept?
What would the concept itself be?
How would it compare with the standard one?

What would be the consequences of accepting this definition in mathematics?

How could this definition be further modified?
What other alternative notions could be created?

Nature of Mathematics:

What characteristics do we want a mathematical definition *not* to have?

What properties should a mathematical definition have?

How can we evaluate and choose among alternative definitions for a given concept?

What should a definition accomplish?
What do we use definitions for?

How do mathematical definitions differ from those in other fields?

Wrong results

Math Content:

In what sense is the result wrong?

Where did the procedure fail?
Could it be fixed up and thus lead to different results?

What were our assumptions and are they justified?
In what cases?

What are the consequences of accepting this alternative result?

In what circumstances could such a result be considered right?

Nature of Mathematics:

How can we test whether we used a mathematical procedure correctly?

How can we decide whether it is appropriate to apply a certain procedure in a given situation?

How can we determine the domain of application of a given procedure?

Right results reached by an unsatisfactory procedure

Math Content:

Why do we get right results in this case?

Could the procedure be slightly modified and be made more rigorous?

Does the procedure "work in this specific case because of specific properties pertaining to it?
In such case what are these properties?
In what cases would it work?
In what cases would it fail?
What assumptions are necessary to be sure it will work?

Nature of Mathematics:

Is the difference between being rigorous or not rigorous a difference in degree?

Who decides whether a procedure is sufficiently rigorous?
On what basis?
Were the criteria used the same throughout the history of mathematics?

Unsatisfactory models

Math Content:

In what sense does the model 'work' and in what sense does it not?

How does the model compare with another "good" model of the same concept if there are any such models available?

Why does the model fail to represent some aspects of the concept?

How could we try to modify the model so that it "fits" the concept better?

Is the real problem a limitation in the specific model or in the concept itself?

Nature of Mathematics:

How can we determine the aspects for which a model "fits" the original object, and the aspects for which it does not fit?

How "different" from the actual object could an acceptable model be?

What is the value of alternative yet impartial models?
How could we evaluate which one is better?

Approximate results

Math Content

Can you evaluate how "big an error you are making by using the approximate result instead of the exact one?

What would be the consequences of your "error", once you use the approximate result in other applications?

Are other approximate results available?
How do they compare with yours?

Could you further "improve" your result, and obtain a "closer" approximate result?
Would such an activity be worth it?

Nature of Mathematics:

Can we always get exact results in mathematical problems?
If not, why?

What could the role and value of approximate results be when exact results are available?

What if the exact results are not available?

How can alternative approximate results be evaluated?

abstraction than when we employ errors as springboards for technical explorations of a mathematical topic.

Using errors as motivation and means for inquiry into the nature of mathematics could improve students' understanding of mathematics *as a discipline*. As Petrie [1970] and Martin [1970] pointed out, understanding a subject involves much more than simply "learning with understanding" its basic content. It includes understanding its philosophy, the methodology employed, the scope and limitations of the discipline. It may also involve developing positive attitudes towards the discipline. This kind of understanding is unfortunately not very common, especially in mathematics, and gaining it would be extremely important for both students and teachers of any level and subject.

To fully appreciate the educational potential of errors as springboards for inquiry, at both the levels identified, it is also important to realize the variety of questions and explorations that can be motivated by different kinds of mathematical errors. In fact, though most people seem to identify mathematical errors with the misuse or misunderstanding of a rule, mathematical errors can present quite different characteristics at least with respect to:

- *The degree of wrongness* – besides "wrong" results, we can in fact have partial or approximate results, right results obtained through unacceptable or inefficient procedures, results that can be considered correct in certain contexts but not in others, problems for which no solution has yet been reached, and so on
- *the mathematical context* – i.e., whether we are dealing with problems, algorithms, theorems, definitions, models, etc.

In Table 1, we have tried to unravel some of the challenging questions that a few common types of errors can naturally stimulate. The role of Table 1 is that of further illustrating the educational potential of errors for inquiry as well as of providing some concrete directions to readers interested in pursuing an in-depth study of a specific error and/or in organizing classroom activities using errors as springboards.

IV. Concluding observations

In this paper we have tried to open new vistas about the roles that errors can play in students' learning of mathematics, beyond diagnosis and remediation. To summarize, we have suggested that errors can be used as a motivational device and as a starting point for creative mathematical explorations, involving valuable problem solving and problem posing activities. We have also suggested that errors can foster a deeper and more complete understanding of mathematical content, as well as of the nature of mathematics itself.

Realizing the educational potential of errors as springboards for explorations, however, is only one step in the implementation of a use of mathematical errors as motivation and means for inquiry in educational settings. Similarly to what happens when we want to introduce a focus on problem solving in instruction, several fundamental issues have to be addressed in order to do so. Let us briefly present a few important ones to the reader's attention.

First of all, it seems important to choose suitable errors to use as starting points for reflection and exploration. All the errors we have studied so far [see Borasi, 1986a, and Brown/Callahan, 1985] have proved to be very stimulating, even in cases when we doubted of their potential at the beginning. However, it was also true that some errors turned out to be more thought-provoking and challenging than others. In addition, in the choice of appropriate errors for a specific learning situation, the interest and preparation of the audience will also have to be taken into serious consideration.

More importantly, we should also recognize that introducing this use of errors in school goes much beyond adding a new topic in the curriculum, and will require appropriate teaching strategies. First of all, since both students and teachers have strong pre-existing conceptions of errors, as well as of mathematics, knowledge and learning, we may expect that these conceptions will influence their behavior with regard to activities involving errors. Consequently, the implementation of a use of errors as "springboards" cannot be successful unless such conceptions are taken into serious account. On the one hand, since becoming aware of one's own beliefs is the first and main step towards questioning and eventually modifying such beliefs, it will be important that teachers help their students reach such awareness about their own conceptions of errors as well as of mathematics, learning and teaching. In order to really challenge the negative feeling people hold towards errors, it will also be important to enable students to personally experience some rewarding activities stimulated by the consideration of specific errors.

We suggest that a valuable way to do this is to involve students in the in-depth study of a specific error, or group of errors, along the lines of the one sketched in section II. An error, or collection of errors on a certain topic, is taken as a starting point for a very open-ended exploration. Suitable questions among those elaborated in Table 1 could be pursued, along with others that could be stimulated by the specific error considered. A project of this kind could allow for the variety of activities that errors can motivate, and demonstrate how far one can be led by the analysis even of apparently trivial errors. We believe in the educational value of projects in general, and suggest that errors can provide the motivation, starting point and direction to organize meaningful ones in mathematics.

It is an open question whether the use of mathematical errors as "springboards" for exploration is possible or appropriate for all audiences. Indeed it is evident that the depth to which one can analyze an error, as well as the choice of errors to study, is influenced at least by the level of technical knowledge, i.e. of mathematical preparation of the audience, and the level of "maturity" of the audience. Philosophical and empirical research would be necessary to better determine what level of these requisites is necessary to pursue certain questions, and what influence they could have on the use of specific errors. Though we think that the benefit of using errors as springboards should not be restricted to certain levels of preparation and maturity, we can expect that different backgrounds might lead to different uses of errors, and accomplish different educational goals

While the value of future research with respect to the specifics of this approach is clear, the principal goal of this paper has been to provide the background and impetus for such effort In this regard, we hope that we have provided sufficient evidence to question the current role of "errors" in mathematics instruction, and to recognize the need for reconstructing this role in order to make full use of the educational potential of errors.

Note

The author developed this thesis in her doctoral dissertation [see Borasi, 1986a] Such study involved three complementary research strategies Conceptual analysis was employed to discuss the motivation values and implications of an interpretation of errors as springboards for inquiry in mathematics instruction. The ideas thus developed were then applied to the study of a few specific errors, in order to provide further illustration and evidence of the potential of various kinds of errors as "springboards" Finally, a course for mathematics teachers, which focussed on the use of errors as "springboards," was designed and implemented. As a result of this course, an additional collection of in-depth studies of errors was generated; a selection of these studies recently appeared in a special issue of the journal *Focus on Learning Problems in Mathematics* [see Brown/Callahan 1985]

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It is important at this point to recall that we are discussing man-made systems constructed for the purpose of making sense of mathematics Euclid's axiomatic method was a way of comprehending and systematizing knowledge about a subject matter: geometry. Euclid's conception had the following two characteristics:

1 The definitions of the axioms were meaningful. To Euclid, "line" meant line not great circle.

2. The axioms were precisely characterized in such a way that consequences would follow by reason alone.

Geometry for Euclid was something to be done in natural language — with terms that are meaningful and understandable. ( ) The advent of noneuclidean geometry showed that these two characteristics were in conflict. Hilbert "saved" the axiomatic method by splitting off half of it — the half to do with meaningfulness — and consigning it to metamathematics. ( ) This division has, in recent years, been attributed to natural language and human reason by the professionals whose job it is to study such things: linguists, philosophers, artifical intelligence researchers, and cognitive psychologists. It has seemed natural to them precisely *because* they have been trained in mathematical logic. By now, it has largely been forgotten just why the division into formal syntax and formal semantics was made in the first place — and what an alien division it is relative to human language and thought

George Lakoff
