EXPONENTIAL FUNCTIONALS OF BROWNIAN MOTION AND DISORDERED SYSTEMS

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Abstract

The paper deals with exponential functionals of the linear Brownian motion which arise in different contexts such as continuous time finance models and one-dimensional disordered models. We study some properties of these exponential functionals in relation with the problem of a particle coupled to a heat bath in a Wiener potential. Explicit expressions for the distribution of the free energy are presented.

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Consider a linear Brownian motion $(B_s, s \geq 0, B_0 = 0)$, and a given drift μ. Exponential functionals of the following form

$$A_t^{(\mu)} = \int_0^t ds\ e^{-2(B_s + \mu s)} \tag{1}$$

have recently been a subject of common interest for mathematicians and for physicists.

Some recent mathematical studies have been partly motivated by continuum-time t finance models in which most stock price dynamics are assumed to be driven by the exponential of a Brownian motion with drift [12] [14]. In such studies, the representation

$$e^{-(B_t + \mu t)} = R\Big(\int_0^t ds\ e^{-2(B_s + \mu s)} \Big) \tag{2}$$

where $(R(u), u \geq 0)$ is a Bessel process, i.e. an element of an important class of diffusions, exhibits the importance of the functional $A_t^{(\mu)}$. Formula (2) is a particular instance of the Lamperti relation which expresses $(\exp(\xi_t), t \geq 0)$ as

$$e^{\xi_t} = X\Big(\int_0^t ds\ e^{2\xi_s} \Big) \tag{3}$$

where $(X(u), u \geq 0)$ is a semi-stable Markov process (see [9] [10] for some applications, partly in mathematical finance).

In physics, these exponential functionals play a central role in the context of one dimensional classical diffusion in a random environment. The random variable $A_\infty^{(\mu)}$ can indeed be interpreted as a trapping time. Its probability distribution controls the anomalous diffusive behaviors of the particle at large time in a infinite sample [5] [6] [15]. The distribution of $A_L^{(\mu)}$ occurs when studying the maximum reached by the Brownian particle in a drifted Brownian potential [17]. The functional $A_L^{(\mu)}$ also arises in the study of the transport properties of disordered samples of finite length L [23] [24] [21].
In fact, $A_L^{(\mu)}$ represents the continuous space analogue of the random series introduced by Kesten *et al.* [19] for the so called "random random walk". This random series is generated by a linear recurrence relation with random coefficients and therefore constitutes a discrete random multiplicative stochastic process. $A_t^{(\mu)}$ also represents a very simple case of a continuous random multiplicative stochastic process [11] which may be related to hyperbolic Brownian motion [29] [11].

In this article, we discuss some properties of these functionals, concentrating ourselves mostly on the mean-value $E(\ln A_t^{(\mu)})$, in relation with another physical interpretation inspired by the statistical mechanics of disordered systems. In these systems, the partition function Z is a functional which depends on a set of "quenched" random couplings. In order to obtain the thermodynamic properties of the system, one has to compute the average over the disorder of the free-energy F

$$E(F) = E(-kT \ln Z) \tag{4}$$

This calculation can rarely be done exactly. The determination of the probability distribution of F is a still more difficult task. One of the very few cases for which such a calculation can be done exactly is the Random Energy Model [13]. It is therefore highly desirable to investigate other explicitly solvable cases [22] [7] where the usual tools of disordered systems such as replica methods and variational techniques can be tested [7].

1 Physical motivation : A toy-model for disordered systems

Let us consider a particle confined on the interval $0 \leq x \leq L$ and submitted to a random force $F(x)$ distributed as a Gaussian white noise around some mean value $-f_0$. The corresponding random potential is then simply a Brownian motion with drift which can be written in terms of the Wiener process B_x as

$$U(x) = -\int_0^x F(y)dy = f_0 x + \sqrt{\sigma} B_x \tag{5}$$

For a given sample, that is for a given realization of the potential $U(x)$, we define the partition function

$$Z_L = \int_0^L dx e^{-\beta U(x)} \tag{6}$$

where as usual β is the inverse temperature of the system. It is convenient to introduce $\alpha = \frac{\beta^2 \sigma}{2}$ and the dimensionless parameter $\mu = \frac{2 f_0}{\beta \sigma}$ to rewrite Z_L as

$$Z_L^{(\mu)} = \int_0^L dx\ e^{-(\alpha \mu x + \sqrt{2\alpha} B_x)} \tag{7}$$

Therefore, for $\alpha = 2$, $Z_L^{(\mu)}$ and $A_L^{(\mu)}$ coincide. For $\alpha \neq 2$, using the scaling properties of the Brownian motion, one obtains

$$Z_L^{(\mu)} \overset{\text{(law)}}{=} \frac{2}{\alpha} \int_0^{\frac{\alpha L}{2}} dx e^{-2(\mu x + B_x)} \tag{8}$$

Hence

$$Z_L^{(\mu)} \overset{\text{(law)}}{=} \frac{2}{\alpha} A_{\frac{\alpha L}{2}}^{(\mu)} \tag{9}$$

The thermal average of any function $g(x)$ of the position of the particle for a given sample will be denoted by an upper-bar

$$\overline{g(x)} \equiv \frac{\int_0^L dx\ g(x)\ e^{-\beta U(x)}}{\int_0^L dx\ e^{-\beta U(x)}} \tag{10}$$

For instance the thermal average and variance of the position read

$$\overline{x} = \frac{1}{Z_L^{(\mu)}} \left(-\frac{1}{\alpha} \frac{\partial}{\partial \mu} Z_L^{(\mu)} \right) = -\frac{1}{\alpha} \frac{\partial}{\partial \mu} \ln(Z_L^{(\mu)}) \tag{11}$$

$$\overline{x^2} - (\overline{x})^2 = \frac{1}{\alpha^2} \frac{\partial^2}{\partial \mu^2} \ln(Z_L^{(\mu)}) \tag{12}$$

More generally, the generating function of the thermal cumulants of the position reads

$$\ln\left(\overline{e^{-px}}\right) = \ln Z_L^{(\mu + \frac{p}{\alpha})} - \ln Z_L^{(\mu)} \tag{13}$$

These relations show that the statistical properties of the position of the particle in the case $\mu = 0$ require in fact the knowledge of the partition function with an arbitrary drift μ.

The fundamental quantity for the statistical mechanics of this disordered system is the free energy $F_L^{(\mu)}$ related to the logarithm of the partition function

$$F_L^{(\mu)} = -\frac{1}{\beta} \ln Z_L^{(\mu)} \tag{14}$$

This paper deals essentially with the statistical properties of the free energy, and particularly with the mean free energy over the disorder denoted by

$$E\left(F_L^{(\mu)}\right) = -\frac{1}{\beta} E\left(\ln Z_L^{(\mu)}\right) \tag{15}$$

This work also gives us the opportunity to bring together various results which were scattered both in the physics and mathematics literature. A comparison between them often yields some interesting identities with a non trivial probabilistic content.

The paper is organized as follows. In section 2, we consider the case of random potential with zero-drift $\mu = 0$. We first give the probability distribution of the free energy. We then establish various formulae for its mean value, using in particular the Bougerol identity, and show that the replica method gives the correct result. We also discuss the asymptotic behavior of the mean free energy in the limit of a large sample $L \to \infty$. In section 3, we discuss the properties of the free energy in the case of a random potential with a positive drift $\mu > 0$. We also establish some relations between $E\left(\ln Z_L^{(\mu)}\right)$ and $E\left(\frac{1}{Z_L^{(\nu)}}\right)$. In section 4, we discuss the case where the length of the sample is an independent random variable which is exponentially distributed.

For notations and properties of the special functions appearing in this paper, we refer the reader to Lebedev [20].

2 Case of random potential with zero drift $\mu = 0$

2.1 Distribution of the partition function $Z_L^{(0)}$ and associated free energy

The expression of the generating function of $Z_L^{(0)}$ has already been derived in another context through the resummation of the series of moments [23] and by a path-integral approach [21]

$$E\left(e^{-pZ_L^{(0)}}\right) = \frac{2}{\pi} \int_0^\infty dk \ \cosh \frac{\pi k}{2} \ K_{ik}\left(2\sqrt{\frac{p}{\alpha}}\right) \ e^{-k^2 \frac{\alpha L}{4}} \tag{16}$$

We also refer the reader to [1], where it is shown that this result may be derived from the Bougerol formula (see later 31). To invert the Laplace transform (16), it is convenient to start from the integral representation

$$K_{ik}\left(2\sqrt{\frac{p}{\alpha}}\right) = \int_0^\infty dt \ \cos kt \ e^{-2\sqrt{\frac{p}{\alpha}} \cosh t} \tag{17}$$

and to perform the integration over k in (16) to obtain

$$E\left(e^{-pZ_L^{(0)}}\right) = \frac{2}{\sqrt{\pi \alpha L}} \ e^{\frac{\pi^2}{4\alpha L}} \ \int_0^\infty dt \cos\left(\frac{\pi t}{\alpha L}\right) e^{-\frac{t^2}{\alpha L}} \ e^{-2\sqrt{\frac{p}{\alpha}} \cosh t} \tag{18}$$

We may then use the following identity

$$e^{-2\sqrt{\frac{p}{\alpha}}\cosh t} = \frac{\cosh t}{\sqrt{\pi\alpha}} \int_0^\infty \frac{dZ}{Z^{\frac{3}{2}}}\ e^{-pZ}\ e^{-\frac{\cosh^2 t}{\alpha Z}} \tag{19}$$

to cast (18) into the form

$$E\Big(e^{-pZ_L^{(0)}}\Big) = \int_0^\infty dZ\ e^{-pZ}\ \psi_L^{(0)}(Z) \tag{20}$$

where

$$\psi_L^{(0)}(Z) = \frac{2e^{\frac{\pi^2}{4\alpha L}}}{\pi\alpha\sqrt{L}}\ \frac{1}{Z^{\frac{3}{2}}} \int_0^\infty dt \cosh t\ \cos\left(\frac{\pi t}{\alpha L}\right) e^{-\frac{t^2}{\alpha L}}\ e^{-\frac{\cosh^2 t}{\alpha Z}} \tag{21}$$

denotes the probability distribution of the partition function $Z_L^{(0)}$. A simple change of variables then gives the probability distribution $P_L^{(0)}(F)$ of the free energy $F_L^{(0)} = -\frac{1}{\beta}\ln Z_L^{(0)}$

$$P_L^{(0)}(F) = \frac{2\beta e^{\frac{\pi^2}{4\alpha L}}}{\pi\alpha\sqrt{L}}\ e^{\frac{\beta}{2}F} \int_0^\infty dt \cosh t\ \cos\left(\frac{\pi t}{\alpha L}\right) e^{-\frac{t^2}{\alpha L}}\ e^{-\left(\frac{\cosh^2 t}{\alpha}\right)e^{\beta F}} \tag{22}$$

It is interesting to compare with a similar formula given in [7].

In the limit of large L, the probability density $X_L(\xi)$ of the reduced variable $\xi = \frac{\beta F - \ln\alpha}{\sqrt{2\alpha L}}$ tends to the Gaussian

$$X_L(\xi) \underset{L\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} \tag{23}$$

This asymptotic result may in fact be directly obtained since from the definition (7)

$$Z_L^{(0)} = \int_0^L dx\ e^{+\sqrt{2\alpha}B_x} \quad \overset{\text{(law)}}{=} \quad L\int_0^1 ds\ e^{\sqrt{2\alpha L}B_s} \tag{24}$$

hence

$$\frac{1}{\sqrt{L}}\ln Z_L^{(0)} = \frac{\ln L}{\sqrt{L}} + \ln\Big(\int_0^1 ds\ e^{\sqrt{2\alpha L}B_s}\Big)^{\frac{1}{\sqrt{L}}} \tag{25}$$

Since

$$\ln\Big(\int_0^1 ds e^{\sqrt{2\alpha L}B_s}\Big)^{\frac{1}{\sqrt{L}}} \underset{L\to\infty}{\longrightarrow} \ln\ e^{\sqrt{2\alpha}\sup_{s\leq 1} B_s} \quad \overset{\text{(law)}}{=} \quad \sqrt{2\alpha}|N| \tag{26}$$

where N is a normalized Gaussian variable, one has

$$\frac{1}{\sqrt{2\alpha L}}\ln Z_L^{(0)} \quad \underset{L\to\infty}{\overset{\text{(law)}}{\longrightarrow}} \quad |N| \tag{27}$$

2.2 An expression of $E\Big(\ln Z_L^{(0)}\Big)$ derived from the generating function

The Frullani identity

$$\ln Z_L^{(0)} = \int_0^\infty \frac{dp}{p}\ \Big[e^{-p} - e^{-pZ_L^{(0)}}\Big] \tag{28}$$

may be used to compute the mean of the logarithm of $Z_L^{(0)}$ from the generating function of equation (16). Using the intermediate regularization

$$E\Big(\ln (Z_L^{(0)})\Big) = \lim_{\epsilon\to 0^+}\left[\Gamma(\epsilon) - \int_0^\infty dp\; p^{\epsilon - 1}\; E\Big(e^{-pZ_L^{(0)}}\Big)\right] \tag{29}$$

one obtains

$$E\Big(\ln Z_L^{(0)}\Big) = \frac{2}{\pi}\int_0^\infty \frac{dk}{k^2}\left[1 - e^{-\alpha L k^2}\pi k\coth(\pi k)\right] + C - \ln\alpha \tag{30}$$

where $C = -\Gamma'(1)$ is Euler's constant.

2.3 Bougerol's identity

Bougerol's identity [4] is an identity in law relating two independent Brownian motions $(B_s, s \geq 0)$ and $(\gamma_u, u \geq 0)$, and involving the exponential functional we are interested in. The statement is that for fixed t

$$\sinh(B_t) \overset{\text{(law)}}{=} \gamma_{A_t} \qquad \text{where} \qquad A_t = \int_0^t ds\; e^{-2B_s} = A_t^{(0)} \tag{31}$$

In the Appendix, we present a simple proof of this identity due to L. Alili, and we refer the reader to [1] for further details and possible generalizations for the case of a non-vanishing drift $\mu \neq 0$.

Scaling properties of Brownian motion then give

$$A_t \overset{\text{(law)}}{=} t\int_0^1 du\; e^{-2\sqrt{t}B_u} \tag{32}$$

$$\sinh(B_t) \overset{\text{(law)}}{=} \left(\int_0^1 du\; e^{-2\sqrt{t}B_u}\right)^{\frac{1}{2}}\gamma_t \tag{33}$$

$$\overset{\text{(law)}}{=} \left(\frac{A_t}{t}\right)^{\frac{1}{2}}\gamma_t \tag{34}$$

It follows that

$$E\Big(\ln A_t\Big) = E\left(\ln\left(\frac{\sinh(B_t)}{B_t}\right)^2\right) + \ln t = \int_{-\infty}^{+\infty}\frac{dx}{\sqrt{2\pi t}}\; e^{-\frac{x^2}{2t}}\;\ln\left(\frac{\sinh(x)}{x}\right)^2 + \ln t \tag{35}$$

Starting from the partition function

$$Z_L^{(0)} \overset{\text{(law)}}{=} \frac{2}{\alpha}\, A_{\frac{\alpha L}{2}} \tag{36}$$

we thus get a new expression for the mean value of the logarithm

$$E\Big(\ln Z_L^{(0)}\Big) = \int_{-\infty}^{+\infty}\frac{dx}{\sqrt{\pi\alpha L}}\; e^{-\frac{x^2}{\alpha L}}\;\ln\left(\frac{\sinh(x)}{x}\right)^2 + \ln(L) \tag{37}$$

Comparison with our previous result (30) leads to the identity

$$2\int_{-\infty}^{+\infty}\frac{dx}{\sqrt{\pi\alpha L}}\; e^{-\frac{x^2}{\alpha L}}\;\ln\left(\frac{\sinh(x)}{x}\right) = \frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{dk}{k^2}\left[1 - e^{-\alpha L k^2}\pi k\coth(\pi k)\right] + C - \ln(\alpha L) \tag{38}$$

In order to understand the meaning of this identity, we differentiate both sides with respect to L and use the heat equation

$$\frac{\partial}{\partial L}\Big(\frac{1}{\sqrt{\pi\alpha L}}\ e^{-\dfrac{x^2}{\alpha L}}\Big) = \frac{\alpha}{4}\frac{\partial^2}{\partial^2 x}\Big(\frac{1}{\sqrt{\pi\alpha L}}\ e^{-\dfrac{x^2}{\alpha L}}\Big) \tag{39}$$

After an easy integration by part we obtain

$$\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{\pi\alpha L}}\ e^{-\dfrac{x^2}{\alpha L}}\ \Big[\frac{\partial^2}{\partial^2 x}\ln\Big(\frac{\sinh(x)}{x}\Big)\Big] = \alpha\int_{-\infty}^{+\infty} dk\ e^{-\alpha L k^2}\Big[k\coth(\pi k) - |k|\Big] \tag{40}$$

This identity is a particular instance of Plancherel's formula, since the Fourier transform of the function

$$\frac{\partial^2}{\partial^2 x}\ln\Big(\frac{\sinh(x)}{x}\Big) = \frac{1}{x^2} - \frac{1}{\sinh^2(x)} \tag{41}$$

can be easily obtained after an integration in the complex plane

$$\int_{-\infty}^{+\infty} dx\ e^{ikx}\Big[\frac{1}{x^2} - \frac{1}{\sinh^2(x)}\Big] = \pi\Big[k\coth\Big(\frac{\pi}{2}k\Big) - |k|\Big] \tag{42}$$

This formula is also encountered as follows in the study [3] of the Hilbert transform of Brownian motion

$$H_u = \lim_{\epsilon\to 0^+}\int_0^u \frac{ds}{B_s}1_{(|B_s|\geq\epsilon)} \tag{43}$$

If $\mathrm{n}(d\epsilon)$ denotes the characteristic measure of Brownian excursions, and ϵ the generic excursion with lifetime $V(\epsilon) = \inf\{t > 0 : \epsilon(t) = 0\}$, then the law of $H_V \equiv \int_0^V \frac{ds}{\epsilon_s}$ under n is $\frac{\pi dx}{x^2}$, and formula (42) may be interpreted as

$$\pi\Big(\lambda\coth\Big(\frac{\pi\lambda}{\theta}\Big) - |\lambda|\Big) = \pi\int_{-\infty}^{+\infty}\frac{dx}{x^2}\ e^{i\lambda x}\ \mathrm{n}\Big(1 - e^{\frac{-\theta^2 V}{2}}\Big|H_V = x\Big) \tag{44}$$

with

$$\mathrm{n}\Big(1 - e^{\frac{-\theta^2 V}{2}}\Big|H_V = x\Big) = 1 - \left(\frac{\frac{\theta x}{2}}{\sinh\left(\frac{\theta x}{2}\right)}\right)^2 \tag{45}$$

in the particular case $\theta = 2$.

2.4 Mean free energy through Replica method

Let $X = Z_L^{(0)}$. The replica method is based on the identity

$$E[\ln X] = \lim_{n\to 0}\frac{E[X^n] - 1}{n} \tag{46}$$

In many applications, one proceeds by looking for an analytic continuation $n \to 0$ of the expressions of integer moments of X. This procedure is in general mathematically ill-founded, but in the present case non-integer moments can be computed.

The integer moments of the partition function are well-known [12] [29] [23] [21]

$$E\Big[(Z_L^{(0)})^n\Big] = \frac{1}{\alpha^n}\sum_{k=1}^{n} e^{\alpha L k^2}(-1)^{n-k}\frac{\Gamma(n)}{\Gamma(2n)}C_n^k + \frac{(-1)^n}{n!} \tag{47}$$

Since this expression contains a sum of n terms, it has a priori no meaning for non-integer n. However, the following integral representation for the moment of order n [23]

$$E\left[(Z_L^{(0)})^n\right] = 2\frac{\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\int_0^{\infty}\frac{dx}{\sqrt{\pi\alpha L}}\; e^{-\dfrac{x^2}{\alpha L}}\;\left(\frac{\sinh^2 x}{\alpha}\right)^n \tag{48}$$

is valid for any positive real $n \geq 0$, as can be shown using the consequence (34) of Bougerol's identity (31).

The following expansions in n, as $n \to 0$

$$\left(\frac{\sinh^2 x}{\alpha}\right)^n = 1 + n\ln\left(\frac{\sinh^2 x}{\alpha}\right) + o(n) \tag{49}$$

$$\frac{\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} = 1 - n\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + o(n) = 1 + n(C + 2\ln 2) + o(n) \tag{50}$$

where $C = -\Gamma'(1)$ denotes Euler constant, lead to the expression

$$E\left[\ln(Z_L^{(0)})\right] = \lim_{n\to 0}\frac{E\left((Z_L^{(0)})^n\right) - 1}{n} \tag{51}$$

$$= C + 2\ln 2 - \ln\alpha + \int_{-\infty}^{+\infty}\frac{dx}{\sqrt{\pi\alpha L}}\; e^{-\dfrac{x^2}{\alpha L}}\;\ln(\sinh^2 x) \tag{52}$$

This formula is of course directly related to the previous expression (37) obtained using Bougerol's identity since

$$-\int_{-\infty}^{+\infty}\frac{dx}{\sqrt{\pi\alpha L}}\; e^{-\dfrac{x^2}{\alpha L}}\;\ln(x^2) + \ln(L) = C + 2\ln 2 - \ln\alpha \tag{53}$$

2.5 Asymptotic expression of the mean free energy for large L

The various expressions obtained previously (30-37-52) yield the asymptotic behavior

$$E\left[\ln(Z_L^{(0)})\right] = 2\sqrt{\frac{\alpha L}{\pi}} + C - \ln\alpha - \frac{\pi^{\frac{3}{2}}}{3\sqrt{\pi\alpha L}} + O\left(\frac{1}{(\alpha L)^{\frac{3}{2}}}\right) \qquad \text{as} \quad L\to\infty \tag{54}$$

which agrees with [7].

We now show how the first two terms may in fact be recovered through a direct asymptotic analysis of

$$Z_L^{(0)} = \int_0^L dx\; e^{\sqrt{2\alpha}B_x} \tag{55}$$

Using the scaling properties of Brownian motion, one has

$$Z_L^{(0)} \overset{(\text{law})}{=} \frac{1}{2\alpha}\int_0^{2\alpha L} dx\; e^{B_x} \tag{56}$$

It is convenient to set $\Lambda = 2\alpha L$ and to define

$$I_\Lambda = E\Big[\ln \int_0^\Lambda dx\ e^{B_x}\Big] \tag{57}$$

Introducing the one-sided supremum $S_\Lambda = \sup_{x\le\Lambda} B_x$, one has

$$I_\Lambda = E(S_\Lambda) + E\Big[\ln \int_0^\Lambda dx\ e^{-(S_\Lambda - B_x)}\Big] \tag{58}$$

It follows from the scaling properties of Brownian motion that

$$I_\Lambda = \sqrt{\Lambda}E(S_1) + E\Big[\ln\big(\Lambda \int_0^1 dx\ e^{-\sqrt{\Lambda}(S_1 - B_x)}\big)\Big] \tag{59}$$

Recall that from the reflection principle $S_1 \overset{(\text{law})}{=} |B_1|$. This implies

$$E(S_1) = E(|B_1|) = \sqrt{\frac{2}{\pi}} \tag{60}$$

For the remaining part of I_Λ, theorem 1 of Pitman and Yor [25] gives the convergence in law

$$\Lambda \int_0^1 dx\ e^{-\sqrt{\Lambda}(S_1 - B_x)} \underset{\Lambda\to\infty}{\overset{(\text{law})}{\longrightarrow}} 4(T_1 + \hat{T}_1) \tag{61}$$

where T_1 and $\hat{T}_1$ are two independent copies of the first hitting time of 1 by a two-dimensional Bessel process starting from 0. Hence

$$E\Big[\ln\big(\Lambda \int_0^1 dx\ e^{-\sqrt{\Lambda}(S_1 - B_x)}\big)\Big] \underset{\Lambda\to\infty}{\longrightarrow} E\Big[\ln 4(T_1 + \hat{T}_1)\Big] \tag{62}$$

The right-hand side of last equation can be evaluated either by direct calculation or by appealing to recent results related to the "agreement formula" for Bessel processes [26]. The direct calculation proceeds as follows : one starts again with the identity

$$E\Big[\ln(T_1 + \hat{T}_1)\Big] = \int_0^\infty \frac{du}{u}\Big[e^{-u} - E\Big(e^{-u(T_1 + \hat{T}_1)}\Big)\Big] \tag{63}$$

and uses the expression of the Laplace transform [18]

$$E\Big(e^{-uT_1}\Big) = \frac{1}{I_0(\sqrt{2u})} \tag{64}$$

After an intermediate regularization, one obtains

$$E\Big[\ln(T_1 + \hat{T}_1)\Big] = \lim_{\epsilon\to 0^+} \int_\epsilon^\infty \frac{du}{u}\Big[e^{-u} - \frac{1}{I_0^2(\sqrt{2u})}\Big] \tag{65}$$

$$= \lim_{\epsilon\to 0^+}\Big[-C - \ln\frac{\epsilon^2}{2} - 2\frac{K_0(\epsilon)}{I_0(\epsilon)}\Big] = C - \ln 2 \tag{66}$$

The other method relies on the following consequence of the "agreement formula" for Bessel processes (equation (37) of [26])

$$E_\delta\Big[(T_1 + \hat{T}_1)^{\frac{\delta}{2}-1}\Big] = \frac{1}{2^\mu \Gamma(1+\mu)} \tag{67}$$

Here $\delta = 2(1+\mu)$ is the dimension of the Bessel process, and E_δ denotes the expectation with respect to the law of this process, starting at 0. (For $\delta = 2$, or equivalently $\mu = 0$, we simply note E instead of E_2). By differentiating both sides of (67) with respect to μ at $\mu = 0$, one recovers (66)

$$E\Big[\ln\big(T_1 + \hat{T}_1\big)\Big] = C - \ln 2 \tag{68}$$

Finally, combining (62) and (68), we obtain

$$I_\Lambda - \sqrt{\frac{2\Lambda}{\pi}} \underset{\Lambda\to\infty}{\longrightarrow} C + \ln 2 \tag{69}$$

We therefore recover the first two terms of (54)

$$E\Big[\ln\big(Z_L^{(0)}\big)\Big] - \Big(2\sqrt{\frac{\alpha L}{\pi}} + C - \ln\alpha\Big) \underset{L\to\infty}{\longrightarrow} 0 \tag{70}$$

3 Case of random potential with drift $\mu > 0$

3.1 Distribution of the partition function $Z_L^{(\mu)}$ and associated free energy

The probability distribution $\psi_L^{(\mu)}(Z)$ of the partition function $Z_L^{(\mu)}$ has already been obtained in another context as the solution of the Fokker-Planck equation [21] [11]

$$\frac{\partial \psi_L^{(\mu)}(Z)}{\partial L} = \frac{\partial}{\partial Z}\left[\alpha Z^2 \frac{\partial \psi_L^{(\mu)}(Z)}{\partial Z} + \Big((\mu+1)\alpha Z - 1\Big)\psi_L^{(\mu)}(Z)\right] \tag{71}$$

satisfying the initial condition $\psi_{L=0}^{(\mu)}(Z) = \delta(Z)$. The eigenfunction expansion of the solution exhibits the following relaxation spectrum with the length L

$$\psi_L^{(\mu)}(Z) = \alpha \sum_{0\leq n<\frac{\mu}{2}} e^{-\alpha L n(\mu-n)} \frac{(-1)^n(\mu-2n)}{\Gamma(1+\mu-n)} \left(\frac{1}{\alpha Z}\right)^{1+\mu-n} L_n^{\mu-2n}\left(\frac{1}{\alpha Z}\right) e^{-\frac{1}{\alpha Z}} \tag{72}$$

$$+\frac{\alpha}{4\pi^2}\int_0^\infty ds\; e^{-\frac{\alpha L}{4}(\mu^2+s^2)} s \sinh \pi s \left|\Gamma\left(-\frac{\mu}{2} + i\frac{s}{2}\right)\right|^2 \left(\frac{1}{\alpha Z}\right)^{\frac{1+\mu}{2}} W_{\frac{1+\mu}{2}, i\frac{s}{2}}\left(\frac{1}{\alpha Z}\right) e^{-\frac{1}{2\alpha Z}} \tag{73}$$

where L_n^α are Laguerre's polynomials and $W_{p,\nu}$ are Whittaker's functions. Contrary to the case $\mu = 0$, there exists for $\mu > 0$ a limit distribution as $L \to \infty$

$$\psi_\infty^{(\mu)}(Z) = \frac{\alpha}{\Gamma(\mu)}\left(\frac{1}{\alpha Z}\right)^{1+\mu} e^{-\frac{1}{\alpha Z}} \tag{74}$$

In the physics literature, this limit distribution was first obtained in the context of one dimensional Brownian diffusion in a Brownian drifted potential, where $Z_\infty^{(\mu)}$ plays the role of an effective trapping time [6], and was recently rediscovered in another context [22]. In the mathematics literature, this limit distribution

$$A_\infty^{(\mu)} \overset{(\text{law})}{=} \frac{1}{2Y_\mu} \tag{75}$$

where Y_μ is a Gamma variable with parameter μ

$$P\Big(Y_\mu \in (y, y+dy)\Big) = \frac{dy}{\Gamma(\mu)}\ y^{\mu-1}\ e^{-y} \tag{76}$$

was first obtained by Dufresne [12], then shown in [31] to be another expression of the law of last passage times for transient Bessel processes. It is interesting to point out that equation (71) first appeared in a paper of Wong [28] (see also [16]). However, the precise connection with our work is beyond the scope of this paper.

3.2 Mean free energy $E\Big(\ln Z_L^{(\mu)}\Big)$

The Laplace transform of $Z_L^{(\mu)}$ corresponding to the probability distribution (72-73) reads [21]

$$E\Big(e^{-pZ_L^{(\mu)}}\Big) = \sum_{0\le n<\frac{\mu}{2}} e^{-\alpha L n(\mu-n)} \frac{2(\mu-2n)}{n!\Gamma(1+\mu-n)} \left(\frac{p}{\alpha}\right)^{\mu/2} K_{\mu-2n}\left(2\sqrt{\frac{p}{\alpha}}\right) \tag{77}$$

$$+\frac{1}{2\pi^2}\int_0^\infty ds\ e^{-\frac{\alpha L}{4}(\mu^2+s^2)} s\sinh\pi s\left|\Gamma\left(-\frac{\mu}{2}+i\frac{s}{2}\right)\right|^2 \left(\frac{p}{\alpha}\right)^{\mu/2} K_{is}\left(2\sqrt{\frac{p}{\alpha}}\right) \tag{78}$$

We use again the Frullani identity

$$E\Big(\ln Z_L^{(\mu)}\Big) = \int_0^\infty \frac{dp}{p}\left[e^{-p} - E\left(e^{-pZ_L^{(\mu)}}\right)\right] \tag{79}$$

to obtain after an intermediate regularisation as in (29)

$$E\Big(\ln Z_L^{(\mu)}\Big) = -\ln\alpha - \frac{\Gamma'(\mu)}{\Gamma(\mu)} - \sum_{1\le n<\frac{\mu}{2}} \frac{\mu-2n}{n(\mu-n)}\ e^{-\alpha L n(\mu-n)} \tag{80}$$

$$-2\int_0^\infty ds\ \frac{s}{\mu^2+s^2}\ \frac{\sinh\pi s}{\cosh\pi s-\cos\pi\mu}\ e^{-\frac{\alpha L}{4}(\mu^2+s^2)} \tag{81}$$

Obviously, the two first terms correspond to the contribution of the limit distribution (74)

$$E\Big(\ln Z_\infty^{(\mu)}\Big) = -\ln\alpha - \frac{\Gamma'(\mu)}{\Gamma(\mu)} \tag{82}$$

Now, it is easily deduced from (3.2) and (82) that

$$E\Big(\ln Z_\infty^{(\mu)}\Big) - E\Big(\ln Z_L^{(\mu)}\Big) \underset{L\to\infty}{\simeq} \begin{cases} \left(\frac{\mu-2}{\mu-1}\right)\ e^{-\alpha(\mu-1)L} & \text{if } \mu>2 \\ \frac{1}{\sqrt{\pi\alpha L}}\ e^{-\alpha L} & \text{if } \mu=2 \\ \frac{1}{\mu^2(1-\cos\pi\mu)}\left(\frac{\pi}{\alpha L}\right)^{\frac{3}{2}}\ e^{-\frac{\alpha\mu^2}{4}L} & \text{if } \mu<2 \end{cases} \tag{83}$$

It is interesting to compare these results to those obtained in [17] for the limiting distribution of the maximum of a diffusion process in a Brownian drifted environment. The latter may be stated as

$$E\left(\frac{A_\infty^{(\mu)}}{A_\infty^{(\mu)}+\tilde{A}_L^{(-\mu)}}\right) \underset{L\to\infty}{\simeq} \begin{cases} \left(\frac{\mu-2}{\mu-1}\right) e^{-2(\mu-1)L} & \text{if } \mu>2 \\ \frac{1}{\sqrt{2\pi L}} e^{-2L} & \text{if } \mu=2 \\ \frac{K}{(2L)^{\frac{3}{2}}} e^{-\frac{\mu^2}{2}L} & \text{if } \mu<2 \end{cases} \tag{84}$$

where $A_\infty^{(\mu)}$ and $\tilde{A}_L^{(-\mu)}$ are two independant functionals of type $A_t^{(\nu)}$ defined in (1). The constant K given in [17] [1] as the following 4-fold integral

$$K=\frac{2^{1-\mu}}{\sqrt{\pi}}\frac{1}{\Gamma(\mu)}\int_0^\infty dy\ y^\mu \int_0^\infty dz\ e^{-\frac{z}{2}(1+y^2)}\int_0^\infty da\ a^{\mu-1}\frac{z}{z+a}\ e^{-\frac{a}{2}}\int_0^\infty du\ u\sinh u\ e^{-zy\cosh u} \tag{85}$$

may in fact be explicitely computed to give in the end the simple result

$$K=\pi^{\frac{3}{2}}\frac{\Gamma(\frac{\mu}{2})^2}{\Gamma(\mu)}\frac{1}{(1-\cos\pi\mu)} \tag{86}$$

For $\mu\geq 2$, the result (84) therefore coincide with the result (83), where for the particular value $\alpha=2$, $Z_L^{(\mu)}$ reduces to $A_L^{(\mu)}$ (see (7) and (1)). To understand this coincidence, we write

$$A_\infty^{(\mu)}=A_L^{(\mu)}+e^{-2(B_L+\mu L)}\ \tilde{A}_\infty^{(\mu)} \tag{87}$$

where $\tilde{A}_\infty^{(\mu)}$ is a variable distributed as $A_\infty^{(\mu)}$ and independent of $A_L^{(\mu)}$. Therefore

$$E\left(\ln A_\infty^{(\mu)}\right)-E\left(\ln A_L^{(\mu)}\right)=E\left(\ln\left[1+e^{-2(B_L+\mu L)}\frac{\tilde{A}_\infty^{(\mu)}}{A_L^{(\mu)}}\right]\right)=E\left(\ln\left[1+\frac{\tilde{A}_\infty^{(\mu)}}{A_L^{(-\mu)}}\right]\right) \tag{88}$$

The comparison between (83) and (84) therefore leads to

$$E\left(\ln\left[1+\frac{\tilde{A}_\infty^{(\mu)}}{A_L^{(-\mu)}}\right]\right)\underset{L\to\infty}{\simeq}E\left(\frac{\tilde{A}_\infty^{(\mu)}}{\tilde{A}_\infty^{(\mu)}+A_L^{(-\mu)}}\right)\qquad \text{for}\quad \mu\geq 2 \tag{89}$$

This is likely to be understood as a consequence of the following plausible statement

$$\text{If}\qquad X_n \overset{\text{(a.s.)}}{\longrightarrow} 0,\quad \text{then}\quad E\Big[\ln(1+X_n)\Big]\sim E\left[\frac{X_n}{1+X_n}\right] \tag{90}$$

which presumably holds for a large class of random variables $\{X_n\}$, but the precise conditions of validity of (90) elude us. However, (89) does not hold for $\mu<2$ since, in this case, the prefactors in (83) and (84) differ.

To go further into the comparison, we have computed exactly the quantity occuring in (84) for arbitrary L. We start from the identity

$$E\left(\frac{Z_\infty^{(\mu)}}{Z_\infty^{(\mu)}+\tilde{Z}_L^{(-\mu)}}\right)=\int_0^\infty dp\ E\left(Z_\infty^{(\mu)}\ e^{-pZ_\infty^{(\mu)}}\right)\ E\left(e^{-p\tilde{Z}_L^{(-\mu)}}\right) \tag{91}$$

[1] In fact (85) corrects the formula for the constant K found in [17]. The need to divide by $\sqrt{\pi}$ is due to a misprint in [29], where on page 528, just after formula (6.e), one should read $\frac{1}{(2\pi^3 t)^{1/2}}\cdots$ instead of $\frac{1}{(2\pi^2 t)^{1/2}}\cdots$

Using the following consequences of (78)

$$E\left(Z_\infty^{(\mu)} e^{-pZ_\infty^{(\mu)}} \right) = \frac{2}{\alpha\Gamma(\mu)} \left(\frac{p}{\alpha}\right)^{\frac{\mu-1}{2}} K_{\mu-1}\left(2\sqrt{\frac{p}{\alpha}}\right) \tag{92}$$

and

$$E\left(e^{-p\tilde{Z}_L^{(-\mu)}} \right) = \frac{1}{2\pi^2}\int_0^\infty ds\ e^{-\frac{\alpha L}{4}(\mu^2+s^2)} s\sinh \pi s \left|\Gamma\left(\frac{\mu}{2}+i\frac{s}{2}\right)\right|^2 \left(\frac{p}{\alpha}\right)^{-\mu/2} K_{is}\left(2\sqrt{\frac{p}{\alpha}}\right) \tag{93}$$

we get

$$E\left(\frac{Z_\infty^{(\mu)}}{Z_\infty^{(\mu)}+\tilde{Z}_L^{(-\mu)}} \right) = \frac{1}{\pi^2\Gamma(\mu)}\int_0^\infty dx K_{\mu-1}(x)\int_0^\infty ds\ e^{-\frac{\alpha L}{4}(\mu^2+s^2)} s\sinh \pi s \left|\Gamma\left(\frac{\mu}{2}+i\frac{s}{2}\right)\right|^2 K_{is}(x) \tag{94}$$

For $\mu \leq 2$, the order of integrations may be exchanged to give

$$E\left(\frac{Z_\infty^{(\mu)}}{Z_\infty^{(\mu)}+\tilde{Z}_L^{(-\mu)}} \right) = \frac{1}{2\Gamma(\mu)}\int_0^\infty ds\ e^{-\frac{\alpha L}{4}(\mu^2+s^2)} \frac{s\sinh \pi s}{\cosh \pi s - \cos \pi\mu} \left|\Gamma\left(\frac{\mu}{2}+i\frac{s}{2}\right)\right|^2 \tag{95}$$

which reduces for $\mu = 2$ to

$$E\left(\frac{Z_\infty^{(2)}}{Z_\infty^{(2)}+\tilde{Z}_L^{(-2)}} \right) = \frac{1}{2}\int_0^\infty ds\ e^{-\frac{\alpha L}{4}(4+s^2)} \frac{\pi s^2 \cosh\frac{\pi s}{2}}{\sinh^2\frac{\pi s}{2}} \tag{96}$$

For $\mu > 2$, one cannot exchange the order of integrations in (94). However, one may start from a series representation (eq (5.9) in [21]) of the generating function (93) to obtain, after some algebra involving deformation of a contour integral in the complex plane (see [21] for a similar approach) the general result for arbitrary $\mu > 0$

$$E\left(\frac{Z_\infty^{(\mu)}}{Z_\infty^{(\mu)}+\tilde{Z}_L^{(-\mu)}} \right) = \sum_{1\leq n<\frac{\mu}{2}} (\mu-2n)\frac{\Gamma(n)\Gamma(\mu-n)}{\Gamma(\mu)}\ e^{-\alpha L n(\mu-n)} \tag{97}$$

$$+\frac{1}{2\Gamma(\mu)}\int_0^\infty ds\ e^{-\frac{\alpha L}{4}(\mu^2+s^2)} \frac{s\sinh \pi s}{\cosh \pi s - \cos \pi\mu} \left|\Gamma\left(\frac{\mu}{2}+i\frac{s}{2}\right)\right|^2 \tag{98}$$

From (95-96-98), one easily recovers the corresponding asymptotic results of (84), which were obtained in [17] by a quite different method, relying on the computations made in [29]. The presence of discrete terms for $\mu > 2$ again explains the transition at $\mu = 2$ of the asymptotic behavior.

3.3 Some relations between $E\left(\ln Z_L^{(\mu)}\right)$ and mean inverse $E\left(\frac{1}{Z_L^{(\nu)}}\right)$

The first negative moment can be obtained from the generating function (78) written above ([21])

$$E\left(\frac{1}{Z_L^{(\mu)}}\right) = \int_0^\infty dp\ E\left(e^{-pZ_L^{(\mu)}}\right) \tag{99}$$

$$= \alpha \sum_{0\leq n<\frac{\mu}{2}} (\mu-2n)\ e^{-\alpha L n(\mu-n)} + \frac{\alpha}{2}\int_0^\infty ds\ \frac{s\sinh \pi s}{\cosh \pi s - \cos \pi\mu}\ e^{-\frac{\alpha L}{4}(\mu^2+s^2)} \tag{100}$$

The previous explicit expressions therefore lead to the very simple identity for any $\mu \geq 0$

$$\frac{\partial}{\partial L} E\left(\ln Z_L^{(\mu)} \right) = E\left(\frac{1}{Z_L^{(\mu)}} \right) - \alpha\mu \tag{101}$$

Can it be derived directly using only basic properties of Brownian motion ? Trying to do so gives in fact two other identities of the same kind, but not (101). The first one relates the exponential functionals for two opposite drifts $(+\mu)$ and $(-\mu)$

$$\frac{\partial}{\partial L} E\left(\ln Z_L^{(\mu)} \right) = E\left(\frac{1}{Z_L^{(-\mu)}} \right) \tag{102}$$

This identity can be obtained from a simple reparametrisation $x' = L - x$ in the denominator of the left handside of expression

$$\frac{\partial}{\partial L} E\left(\ln \int_0^L dx\ e^{-(\alpha\mu x + \sqrt{2\alpha} B_x)} \right) = E\left(\frac{e^{-(\alpha\mu L + \sqrt{2\alpha} B_L)}}{\int_0^L dx\ e^{-(\alpha\mu x + \sqrt{2\alpha} B_x)}} \right) \tag{103}$$

The second one relates the exponential functionals for two dimensionless drifts differing by two

$$\frac{\partial}{\partial L} E\left(\ln Z_L^{(\mu)} \right) = e^{-\alpha L(\mu - 1)}\ E\left(\frac{1}{Z_L^{(\mu-2)}} \right) \tag{104}$$

This identity follows from Cameron-Martin or Girsanov relations. In fact these relations lead to a more general relation between the two characteristic functions of the exponential functionals for (μ) and $(\mu - 2)$

$$\frac{\partial}{\partial L} E\left(e^{-pZ_L^{(\mu)}} \right) = -p\ e^{-\alpha L(\mu - 1)}\ E\left(e^{-pZ_L^{(\mu-2)}} \right) \tag{105}$$

4 Expressions of $E\left(\ln (Z_{L_\lambda}^{(\mu)}) \right)$ with an independent exponential length L_λ

It is well known that the laws of additive functionals of Brownian motion with drift μ

$$\mathcal{A}_t^f \overset{\text{(def)}}{=} \int_0^t ds\ f(B_s + \mu s) \tag{106}$$

may be easier to compute when the fixed time t is replaced by an independent exponential time T_λ of parameter λ

$$P\left(T_\lambda \in [t, t + dt] \right) = \lambda e^{-\lambda t} dt \tag{107}$$

This is indeed the case for $\mathcal{A}_t^f = A_t^{(\mu)}$ i.e. $f(x) = e^{-2x}$. It was shown in [30] [33] that

$$A_{T_\lambda}^{(\mu)} \overset{\text{(law)}}{=} \frac{X_{1,a}}{2Y_b} \qquad \text{where} \qquad a = \frac{\sqrt{2\lambda + \mu^2} - \mu}{2}, \qquad b = \frac{\sqrt{2\lambda + \mu^2} + \mu}{2}, \tag{108}$$

$X_{\alpha,\beta}$ denotes a Beta-variable with parameters (α, β)

$$P\Big(\ X_{\alpha,\beta} \in [x, x+dx] \ \Big) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} dx \qquad (0 < x < 1) \tag{109}$$

and Y_γ denotes a Gamma-variable of parameter (γ) (76). Consequently, one has

$$E\Big(\left(A^{(\mu)}_{T_\lambda}\right)^n \Big) = \frac{\Gamma(1+n)\Gamma(1+a)\Gamma(b-n)}{2^n\Gamma(1+a+n)\Gamma(b)} \tag{110}$$

and

$$-E\Big(\ln A^{(\mu)}_{T_\lambda} \Big) = C + \ln 2 + \psi(1+a) + \psi(b) \tag{111}$$

where $C = -\Gamma'(1)$ is Euler's constant, and $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. Classical integral representation of the function ψ allows to invert the Laplace transform in λ implicit in (111), hence to recover $E\Big(\ln A^{(\mu)}_t \Big)$; however the formulae we have obtained in this way are not simple.

To transpose the result (111) for the partition function $Z^{(\mu)}_{L_\lambda}$ describing the case where the length of the disordered sample is exponentially distributed, one only needs to use the identity derived from (9)

$$Z^{(\mu)}_{L_\lambda} \overset{\text{(law)}}{=} \frac{2}{\alpha} A^{(\mu)}_{T_{\frac{2}{\alpha}\lambda}} \tag{112}$$

Appendix : A simple proof of Bougerol's identity

As Bougerol's identity (31) may appear quite mysterious at first sight, we find useful to reproduce here a simple proof of this identity due to L. Alili and D. Dufresne. We refer the reader to [1] and [2] for further details and possible generalizations for the case of a non-vanishing drift $\mu \neq 0$.

Consider the Markov process

$$X_t = e^{B_t} \int_0^t e^{-B_s} \, d\gamma_s \tag{113}$$

where B_t and γ_t are two independent Brownian motions. Itô formula yields the stochastic differential equation

$$dX_t = \frac{1}{2} X_t \, dt + (X_t \, dB_t + d\gamma_t) \tag{114}$$

We introduce a new Brownian motion β_t by setting

$$X_t \, dB_t + d\gamma_t = \sqrt{X_t^2 + 1} \, d\beta_t \tag{115}$$

from which it follows that

$$dX_t = \frac{1}{2} X_t \, dt + \Big(X_t^2 + 1 \Big)^{\frac{1}{2}} \, d\beta_t \tag{116}$$

The comparison with

$$d[\sinh(\beta_t)] = \frac{1}{2}[\sinh(\beta_t)]dt + \Big(\sinh^2(\beta_t) + 1 \Big)^{\frac{1}{2}} \, d\beta_t \tag{117}$$

shows that

$$\sinh(\beta_t) \stackrel{\text{(law)}}{=} X_t = e^{B_t} \int_0^t e^{-B_s}\, d\gamma_s \tag{118}$$

The use of scaling properties of Brownian motion finally gives

$$\sinh(\beta_t) \stackrel{\text{(law)}}{=} \hat{\gamma}_{A_t^{(0)}} \qquad \text{with} \quad A_t^{(0)} = \int_0^t e^{2B_s}\, ds \tag{119}$$

where $\hat{\gamma}$ denotes a Brownian motion, which is independent of B_s.

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