On the Structure of Hopf Algebras

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The notion of Hopf algebra[1] has been abstracted from the work of Hopf on manifolds which admit a product operation. The homology $H_*(M; K)$ of such a manifold with coefficients in the field K admits not only a diagonal or co-product

$$H_*(M; K) \longrightarrow H_*(M; K) \otimes H_*(M; K)$$

induced by the diagonal $M \to M \times M$, but also a product

$$H_*(M, K) \otimes H_*(M; K) \longrightarrow H_*(M; K)$$

induced by the product $M \times M \to M$. The structure theorem of Hopf concerning such algebras has been generalized by Borel, Leray, and others.

This paper gives a comprehensive treatment of Hopf algebras and some surrounding topics. New proofs of the classical theorems are given, as well as some new results. The paper is divided into eight sections with the following titles:

1. Algebras and modules.
2. Coalgebras and comodules.
3. Algebras, coalgebras, and duality.
4. Elementary properties of Hopf algebras.
5. Universal algebras of Lie algebras.
6. Lie algebras and restricted Lie algebras.
7. Some classical theorems.
8. Morphisms of connected coalgebras into connected algebras.

The first four sections are introductory in nature. Section 5 shows that, over a field of characteristic zero, the category of graded Lie algebras is isomorphic with the category of primitively generated Hopf algebras. In § 6, a similar result is obtained in the case of characteristic $p \neq 0$, but with graded Lie algebras replaced by graded restricted Lie algebras. Section 7 studies conditions when a Hopf algebra with commutative multiplication splits either as a tensor product of algebras with a single generator or a tensor product of

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[1] The term *hyperalgebra* is used by Cartier, Halpern, and others for the same concept. Our Hopf algebras differ from those defined by Borel in that the complication or diagonal map is considered as part of the structure.

Hopf algebras with a single generator. It is here that one finds the fundamental theorem of Hopf, Leray, and Borel. Section 8 introduces and studies mildly a canonical anti-automorphism defined for connected Hopf algebras. For the Steenrod algebra, this operation was essentially introduced by Thom (Compare Milnor [6]).

In addition to the main body of the paper, there is an appendix on the homology of H-spaces with coefficients in a field of characteristic zero.

1. Algebras and modules

For convenience we assume that we have chosen a fixed commutative ring K. Tensor products will be taken over K, and the tensor product of two K-modules A and B will be denoted by $A \otimes B$. Similarly $\mathrm{Hom}(A, B)$ will denote the morphisms of A into B in the category of K-modules.

A graded K-module A is a family of K-modules $\{A_n\}$ where the indices n run through the non-negative integers. If A, B are graded K-modules, a morphism of graded K-modules $f\colon A \to B$ is a family of morphisms $\{f_n\}$ such that $f_n\colon A_n \to B_n$ is a morphism of K-modules.

If A and B are graded K-modules, then $A \otimes B$ is the graded K-module such that $(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$, and if $f\colon A \to A'$, $g\colon B \to B'$ are morphisms of graded K-modules, then $(f \otimes g)\colon A \otimes B \to A' \otimes B'$ is the morphism of graded K-modules such that $(f \otimes g)_n = \bigoplus_{i+j=n} f_i \otimes g_j$.

If A is a graded K-module, we denote by A^* the graded K-module such that $A^*_n = \mathrm{Hom}(A_n, K)$. If $f\colon A \to B$ is a morphism of graded K-modules, then $f^*\colon B^* \to A^*$ is the morphism of graded K-modules such that $f^*_n = \mathrm{Hom}(f_n, K)$. Here we use the convention that a module and the identity morphism of the module will be denoted by the same symbol.

Sometimes K itself will be considered as a graded K-module which is the 0-module in all degrees except 0, and the ring K in degree 0. Recall that $A \otimes K = A = K \otimes A$ where A and K both denote either K-modules or graded K-modules.

1.1. DEFINITIONS. An *algebra* over K is a graded K-module A together with morphisms of graded K-modules $\varphi\colon A \otimes A \to A$ and $\eta\colon K \to A$ such that the diagrams

(1)
$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "A \otimes \varphi"] \arrow[d, "\varphi \otimes A"] & A \otimes A \arrow[d, "\varphi"] \\ A \otimes A \arrow[r, "\varphi"] & A \end{tikzcd}$$

and

(2)
$$\begin{tikzcd} K\otimes A \arrow[rr, "\eta\otimes A"] & & A\otimes A \arrow[dl, "\varphi"] \\ & A \arrow[ul, "\approx"] \arrow[dl, "\approx"'] & \\ K\otimes A \arrow[rr, "A\otimes\eta"] & & A\otimes A \arrow[ul, "\varphi"'] \end{tikzcd}$$

are commutative. The morphism φ is called the multiplication of the algebra A, and η is called the unit of A.

A multiplication in a graded K-module A is a morphism $\varphi: A\otimes A \to A$. For this to be the multiplication of an algebra condition, (1) says that it must be associative; while the existence of η together with the commutativity of diagram (2) says that this multiplication must have a unit.

For convenience we introduce the *twisting morphism* $T: A\otimes B \to B\otimes A$ where A and B are graded K-modules, and T is the morphism such that $T_n(a\otimes b) = (-1)^{pq} b\otimes a$ for $a\in A_p$, $b\in B_q$ and $p+q=n$. The algebra A is commutative if the diagram

$$\begin{tikzcd} A\otimes A \arrow[dd, "T"] \arrow[dr, "\varphi"] & \\ & A \\ A\otimes A \arrow[ur, "\varphi"'] & \end{tikzcd}$$

is commutative. Classically such an algebra was called anti-commutative.

If A and B are algebras over K, then $A\otimes B$ is the algebra over K with multiplication the composition

$$A\otimes B\otimes A\otimes B \xrightarrow{A\otimes T\otimes B} A\otimes A\otimes B\otimes B \xrightarrow{\varphi_A\otimes\varphi_B} A\otimes B$$

and unit

$$K = K\otimes K \xrightarrow{\eta_A\otimes\eta_B} A\otimes B\ .$$

A morphism of algebras $f: A\to B$ is a morphism of graded K-modules such that the diagrams

$$\begin{tikzcd} A\otimes A \arrow[r, "\varphi_A"] \arrow[d, "f\otimes f"] & A \arrow[d, "f"] \\ B\otimes B \arrow[r, "\varphi_B"] & B \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "\eta_A"] \arrow[d, "f"] & A \arrow[d, "f"] \\ K \arrow[r, "\eta_B"] & B \end{tikzcd}$$

are commutative. Observe that an algebra A is commutative if and only if $\varphi: A\otimes A\to A$ is a morphism of algebras.

An *augmentation* of an algebra A is a morphism of algebras $\varepsilon: A\to K$.

An algebra A together with an augmentation ε_A is called an augmented (or supplemented) algebra. If A is an augmented algebra, we denote by $I(A)$ the kernel of $\varepsilon\colon A \to K$. Observe that $I(A)_q = A_q$ for $q > 0$, and $I(A)_0$ is the kernel of $\varepsilon_0\colon A_0 \to K$. The ideal $I(A)$ in A is called the augmentation ideal of A. Observe that as a graded K-module A may be considered as the direct sum

$$A = \text{image } \eta \oplus \text{kernel } \varepsilon \ ,$$

or identifying K and image η,

$$A = K \oplus I(A) \ .$$

Here we use the fact that $\varepsilon\eta\colon K \to K$ is the identity morphism of K.

1.2. DEFINITION. If A is an algebra over K, a *left A-module* is a graded K-module N together with a morphism $\varphi_N\colon A \otimes N \to N$ such that the diagrams

$$\begin{tikzcd}
A \otimes A \otimes N \arrow[r, "A \otimes \varphi_N"] \arrow[d, "\varphi_A \otimes N"] & A \otimes N \arrow[d, "\varphi_N"] \\
A \otimes N \arrow[r, "\varphi_N"] & N ,
\end{tikzcd}$$

$$\begin{tikzcd}
K \otimes N \arrow[rr, "\eta \otimes N"] & & A \otimes N \arrow[dl, "\varphi_N"] \\
& N \arrow[ul, "\approx"] &
\end{tikzcd}$$

are commutative.

If N, N' are left A-modules, a *morphism $f\colon N \to N'$ of left A-modules* is a morphism of graded K-modules such that the diagram

$$\begin{tikzcd}
A \otimes N \arrow[r, "\varphi_N"] \arrow[d, "A \otimes f"] & N \arrow[d, "f"] \\
A \otimes N' \arrow[r, "\varphi_{N'}"] & N'
\end{tikzcd}$$

is commutative.

If $f, g\colon N \to N'$ are morphisms of left A-modules, then $(f + g)\colon N \to N'$ is the morphism of left A-modules such that $(f + g)_q = f_q + g_q\colon M_q \to M'_q$, a morphism of K-modules.

The *kernel of f* is the left A-module such that as a graded K-module $(\operatorname{Ker} f)_q = \operatorname{Ker}(f_q)$, and the *cokernel of f* is the left A-module such that as a graded K-module $(\operatorname{Coker} f)_q = \operatorname{Coker}(f_q)$. Making a few routine verifications, one sees that the category of left A-modules is an abelian category.

The notion of right A-module M is defined similarly using a morphism of graded K-modules $\varphi_M\colon M \otimes A \to M$. The right A-modules also form an abelian category.

If M is a right A-module, and N is a left A-module the *tensor product* of M and N is the graded K-module $M \otimes_A N$ such that the sequence

$$M \otimes A \otimes N \xrightarrow{\varphi_M \otimes N - M \otimes \varphi_N} M \otimes N \longrightarrow M \otimes_A N \longrightarrow 0$$

is an exact sequence of graded K-modules.

If $M' \to M \to M' \to 0$ is an exact sequence of right A-modules, and N is a left A-module, then

$$M' \otimes_A N \longrightarrow M \otimes_A N \longrightarrow M'' \otimes_A N \longrightarrow 0$$

is an exact sequence of graded K-modules.

Similarly if M is a right A-module, and $N' \to N \to N'' \to 0$ is an exact sequence of left A-modules, then

$$M \otimes_A N' \longrightarrow M \otimes_A N \longrightarrow M \otimes_A N'' \longrightarrow 0$$

is an exact sequence of graded K-modules.

In particular if A is an augmented algebra, we have that K is a right A-module *via* the augmentation $\varepsilon\colon A \to K$. Thus if N is a left A-module, we have defined $K \otimes_A N$ and $K \otimes_A N = N/I(A)N$ where $I(A)N$ is the image of the composition

$$I(A) \otimes N \longrightarrow A \otimes N \xrightarrow{\varphi_N} N\,.$$

1.3. Definition. The algebra A over K is *connected* if $\eta\colon K \to A_0$ is an isomorphism.

Notice that any connected algebra A has a unique augmentation $\varepsilon\colon A \to K$, and that $K \xrightarrow{\approx} A_0 \to K$ where $\varepsilon_0\eta_0 = K$.

1.4. Proposition. *If A is a connected algebra over K, and N is a left A-module, then $N = 0$ if and only if $K \otimes_A N = 0$.*

Proof. Certainly if $N = 0$, then $K \otimes_A N = 0$. Suppose, on the other hand, that $K \otimes_A N = 0$. This is equivalent to saying that $I(A) \otimes N \to N$ is an epimorphism. Now since A is connected, $I(A)_0 = 0$; and then if $N_q = 0$ for $q \leqq k$, we have $\big(I(A) \otimes N\big)_q = 0$ for $q \leqq k + 1$, and it follows that $N_q = 0$ for $q \leqq k + 1$. Since $\big(I(A) \otimes N\big)_0 = 0 = N_0$, we have proved the proposition.

1.5. Corollary. *If A is a connected algebra over K and $f\colon N' \to N$ is a morphism of left A-modules, then f is an epimorphism if and only if $K \otimes_A f\colon K \otimes_A N' \to K \otimes_A N$ is an epimorphism.*

Proof. Certainly if f is an epimorphism, so is $K \otimes_A f$. Suppose $K \otimes_A f$ is an epimorphism. Let N'' be the cokernel of f. We have exact sequences $N' \to N \to N'' \to 0$, and $K \otimes_A N' \to K \otimes_A N \to K \otimes_A N'' \to 0$. Since $K \otimes_A f$

is an epimorphism, $K \otimes_A N'' = 0$, and it follows from the preceding proposition that $N'' = 0$ which proves the corollary.

If C is a graded K-module, then $A \otimes C$ is a left A-module called the *extended A-module of C* with $\varphi_{A\otimes C}\colon A \otimes A \otimes C \to A \otimes C$ the morphism

$$A \otimes A \otimes C \xrightarrow{\varphi \otimes C} A \otimes C .$$

If N is a left A-module, and $f\colon C \to N$ is a morphism of graded K-modules, then the composition $A \otimes C \xrightarrow{A \otimes f} A \otimes N \xrightarrow{\varphi_N} N$ is a morphism of left A-modules.

Observe that $K \otimes_A (A \otimes C) = C$, and that the morphism $K \otimes_A (A \otimes C) \to K \otimes_A N$ under the conditions above is just the composition of morphisms of graded K-modules

$$C \xrightarrow{f} N \longrightarrow K \otimes_A N .$$

1.6. PROPOSITION. *If A is a connected algebra over K, C is a graded K-module, N is a left A-module, and $f\colon C \to N$ is a morphism of graded K-modules, then the composition*

$$A \otimes C \xrightarrow{A \otimes f} A \otimes N \xrightarrow{\varphi_N} N$$

is an epimorphism of left A-modules if and only if the composition

$$C \xrightarrow{f} N \longrightarrow K \otimes_A N$$

is an epimorphism of graded K-modules.

PROOF. The proposition follows at once from Corollary 1.5.

In addition to speaking of algebras over K being connected, it is possible to speak of graded K-modules being connected. If N is a graded K-module, it is connected if $N_0 \approx K$. If A is an algebra over K, and N is a left A-module, we say that N is connected if the underlying graded K-module of N is connected. In this case, given an isomorphism $\eta\colon K \xrightarrow{\approx} N_0$, there is defined a unique morphism $i\colon A \to N$ which is the composition

$$A = A \otimes K \xrightarrow{A \otimes \eta} A \otimes N \xrightarrow{\varphi_N} N .$$

1.7. PROPOSITION. *Suppose the following conditions are satisfied:*

(1) *A is a connected algebra over K;*

(2) *N is a connected left A-module;*

(3) *$C = K \otimes_A N$;*

(4) *$\Delta\colon N \to N \otimes C$ is a morphism of left A-modules where*

$$\varphi_{N \otimes_K C}\colon A \otimes N \otimes C \longrightarrow N \otimes C$$

is the morphism $\varphi_N \otimes C$;

(5) $\Pi: N \to C$ *is the canonical epimorphism, and* $f: C \to N$ *is a morphism of graded K-modules such that* $\Pi f = C$;

(6) $(\varepsilon \otimes C)\Delta = \Pi: N \to C$;

(7) $(N \otimes \varepsilon)\Delta = N: N \to N$; *and*

(8) *the sequence* $0 \to A \otimes C \xrightarrow{i \otimes C} N \otimes C$ *is an exact sequence of left A-modules.*

If $\tilde{f}: A \otimes C \to N$ *is the composition* $A \otimes C \xrightarrow{A \otimes f} A \otimes N \xrightarrow{\varphi_N} N$, *then* $\tilde{f}$ *is an isomorphism of left A-modules.*

PROOF. By Proposition 1.6, we have that $\tilde{f}$ is an epimorphism. Now define filtrations on $A \otimes C$ and $N \otimes C$ as follows: $F_p(A \otimes C) = \sum_{q \leqq p} A \otimes C_q$, and $F_p(N \otimes C) = \sum_{q \leqq p} N \otimes C_q$. Let $E^0(A \otimes C)$ and $E^0(N \otimes C)$ denote the corresponding associated bigraded modules. We now have $E^0_{p,q}(A \otimes C) = A_q \otimes C_p$, and $E^0_{p,q}(N \otimes C) = N_q \otimes C_p$. Moreover $\Delta\tilde{f}\left(\sum_{q \leqq p} K \otimes C_q\right) \subset F_p(N \otimes C)$ since $\sum_{q \leqq p} (N \otimes C)_q \subseteqq F_p(N \otimes C)$. Since $\Delta\tilde{f}$ is a morphism of left A-modules, it follows that $\Delta\tilde{f}\left(F_p(A \otimes C)\right) \subset F_p(N \otimes C)$ and thus Δf induces

$$E^0(\Delta\tilde{f}): E^0(A \otimes C) \longrightarrow E^0(N \otimes C)\ .$$

Moreover identifying $E^0(A \otimes C)$ with $A \otimes C$ and $E^0(N \otimes C)$ with $N \otimes C$, we have that $E^0(\Delta\tilde{f})$ is just the morphism $i \otimes C$ which is a monomorphism by hypothesis 8. This implies that $\Delta\tilde{f}$ is a monomorphism, and proves the proposition, since now $\tilde{f}$ is a monomorphism.

2. Coalgebras and comodules

2.1. DEFINITIONS. A *coalgebra* over K is a graded K-module A together with morphisms of graded K-modules

$$\Delta: A \longrightarrow A \otimes A \quad \text{and} \quad \varepsilon: A \longrightarrow K$$

such that the diagrams

(1)
$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \otimes A \arrow[d, "\Delta \otimes A"] \\ A \otimes A \arrow[r, "A \otimes \Delta"] & A \otimes A \otimes A \end{tikzcd}$$

and

$$\begin{tikzcd} A \otimes A \arrow[rr, "\varepsilon \otimes A"] & & K \otimes A \arrow[dl, "\approx"] \\ & A \arrow[ul, "\Delta"] \arrow[dl, "\Delta"'] & \\ A \otimes A \arrow[rr, "A \otimes \varepsilon"] & & A \otimes K \arrow[ul, "\approx"'] \end{tikzcd}$$

are commutative. The morphism Δ is called the comultiplication of the coalgebra A, and ε is called the unit of A.

The coalgebra A is commutative if the diagram

$$\begin{tikzcd} & A \otimes A \arrow[dd, "T"] \\ A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \\ & A \otimes A \end{tikzcd}$$

is commutative.

If A and B are coalgebras over K, then $A \otimes B$ is the coalgebra over K with comultiplication the composition

$$A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{A \otimes T \otimes B} A \otimes B \otimes A \otimes B ,$$

and unit

$$A \otimes B \xrightarrow{\varepsilon_A \otimes \varepsilon_B} K \otimes K = K .$$

A morphism of coalgebras $f: A \to B$ is a morphism of graded K-modules such that the diagrams

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[d, "f"] & A \otimes A \arrow[d, "f \otimes f"] \\ B \arrow[r, "\Delta_B"] & B \otimes B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\varepsilon_A"] \arrow[d, "f"] & K \arrow[d, "k"] \\ B \arrow[r, "\varepsilon_B"] & K \end{tikzcd}$$

are commutative. One can verify without difficulty that a coalgebra A is commutative if and only if $\Delta: A \to A \otimes A$ is a morphism of coalgebras.

Noticing that we may consider K to be a coalgebra in a canonical way, an *augmentation* of a coalgebra is a morphism of coalgebras $\eta: K \to A$. If A is an augmented coalgebra, i.e., a coalgebra together with an augmentation η, we denote by $J(A)$ the cokernel of η. Considering A as a graded K-module we have that

$$A = K \oplus J(A) .$$

2.2. DEFINITIONS. If A is a coalgebra over K, a left A-comodule is a graded K-module N together with a morphism $\Delta_N: N \to A \otimes N$ such that the diagrams

$$\begin{array}{ccc} N & \xrightarrow{\Delta_N} & A \otimes N \\ \downarrow{\scriptstyle \Delta_N} & & \downarrow{\scriptstyle \Delta_A \otimes N} \\ A \otimes N & \xrightarrow{A \otimes \Delta_N} & A \otimes A \otimes N \end{array}$$

$$\begin{tikzcd} A \otimes N \arrow[rr, "\varepsilon \otimes N"] & & K \otimes N \arrow[dl, "\approx"] \\ & N \arrow[ul, "\Delta_N"] & \end{tikzcd}$$

are commutative.

If N, N' are left A-comodules, a *morphism* f*:* $N \to N'$ *of left* A*-comodules* is a morphism of graded K-modules such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{\Delta_N} & A \otimes N \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle A \otimes f} \\ N' & \xrightarrow{\Delta_{N'}} & A \otimes N' \end{array}$$

is commutative.

The category of left A-comodules is immediately seen to be an additive category. However, due to the fact that $A \otimes_K (\quad)$ is right exact and not in general left exact, it is not an abelian category in general. If A is a flat graded K-module, then the category of left A-comodules is abelian.

The notion of right A-comodules is defined using a morphism of graded K-modules $\Delta_M \colon M \to M \otimes A$. The category of right A-comodules has similar general properties to the category of left A-comodules.

If M is a right A-comodule and N is a left A-comodule the *cotensor product* of M and N is the graded K-module $M \,\square_A\, N$ such that the sequence

$$0 \longrightarrow M \,\square_A\, N \longrightarrow M \otimes N \xrightarrow{\Delta_M \otimes N - M \otimes \Delta_N} M \otimes A \otimes N$$

is an exact sequence of graded K-modules.

A sequence of graded K-modules

$$\longrightarrow C(q) \xrightarrow{f(q)} C(q-1) \xrightarrow{f(q-1)} C(q-2) \longrightarrow$$

is *split-exact* if it is exact, and $\operatorname{Ker}(f(q))$ is a direct summand of $C(q)$ for each q. Sequences of modules over a K-algebra or a K-coalgebra are split exact if the underlying sequences of graded K-modules are split exact.

If $0 \to M' \to M \to M''$ is a split-exact sequence of right A-comodules, and N is a left A-comodule, then

$$0 \longrightarrow M' \,\square_A\, N \longrightarrow M \,\square_A\, N \longrightarrow M'' \,\square_A\, N$$

is an exact sequence of graded K-modules.

Similarly if M is a right A-comodule and $0 \to N' \to N \to N''$ is a split-exact sequence of left A-comodules, then

$$0 \longrightarrow M \square_A N' \longrightarrow M \square_A N \longrightarrow M \square_A N''$$

is an exact sequence of graded K-modules.

2.3. DEFINITION. The coalgebra A over K is *connected* if $\varepsilon_0 \colon A_0 \to K$ is an isomorphism.

Notice that any connected coalgebra A has a unique augmentation $\eta \colon K \to A$, and that $\varepsilon_0 \eta_0 = K$.

2.4. PROPOSITION. *If A is a connected coalgebra over K, and N is a left A-comodule, then $N = 0$ if and only if $K \square_A N = 0$.*

PROOF. Certainly if $N = 0$, then $K \square_A N = 0$. Suppose that $K \square_A N = 0$. This is equivalent to saying that $N \to J(A) \otimes N$ is a monomorphism. Since A is connected, $J(A)_0 = 0$; and thus, if $N_q = 0$ for $q \leqq k$, we have $\big(J(A) \otimes N\big)_q = 0$ for $q \leqq k + 1$; and it follows that $N_q = 0$ for $q \leqq k + 1$. Thus $N = 0$ and the proposition is proved.

Having proved the preceding proposition we cannot, as in the preceding section, draw a simple corollary since the category of left A-modules is not necessarily abelian.

2.5. PROPOSITION. *If A is a connected coalgebra over K, $f \colon N \to N''$ is a morphism of left A-comodules, $\iota \colon N' \to N$ is the kernel of f considered as a graded K-module, and the sequence*

$$J(A) \otimes N' \longrightarrow J(A) \otimes N \longrightarrow J(A) \otimes N''$$

is an exact sequence of graded K-modules, then f is a monomorphism if and only if $K \square_A f \colon K \square_A N \to K \square_A N'$ is a monomorphism.

PROOF. We have a commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] \\
& K \square_A N \arrow[r, "K \square_A f"] \arrow[d] & K \square_A N'' \arrow[d] \\
N' \arrow[r, "\iota"] & N \arrow[r, "f"] \arrow[d] & N'' \arrow[d] \\
J(A) \otimes N' \arrow[r] & J(A) \otimes N \arrow[r, "J(A) \otimes f"] & J(A) \otimes N''
\end{tikzcd}$$

with exact rows and columns. If f is a monomorphism, it follows at once that $K \square_A f$ is a monomorphism. Suppose $K \square_A f$ is a monomorphism, and that

$N'_q = 0$ for $q \leqq k$. Then since A is connected, $\big(J(A) \otimes N'\big)_q = 0$ for $q \leqq k + 1$; and $\big(J(A) \otimes f\big)_q$ is a monomorphism for $q \leqq k + 1$. Thus f_q is a monomorphism for $q \leqq k + 1$, and $N'_q = 0$ for $q \leqq k + 1$ which shows that $N' = 0$ and proves the proposition.

Observe that the condition that $J(A) \otimes N' \to J(A) \otimes N \to J(A) \otimes N''$ be exact is immediate if either A is a flat K-module or f is an epimorphism.

If C is a graded K-module, then $A \otimes C$ is a left A-comodule with

$$\Delta_{A \otimes C}\colon A \otimes C \longrightarrow A \otimes A \otimes C$$

the morphism $\Delta \otimes C$. If N is a left A-comodule and $f\colon N \to C$ is a morphism of graded K-modules, then the composition

$$N \xrightarrow{\Delta_N} A \otimes N \xrightarrow{A \otimes f} A \otimes C$$

is a morphism of left A-comodules.

Observe that $K \,\square_A\, (A \otimes C) = C$, and the morphism

$$K \,\square_A\, N \longrightarrow K \,\square_A\, (A \otimes C)$$

under the conditions above is just the composition

$$K \,\square_A\, N \longrightarrow N \xrightarrow{f} C\ .$$

We may define the notion of connected left A-comodules, just as we defined the notion of connected modules over an algebra in the preceding section. Thus a left A comodule N is connected if it is connected as a graded K-module, i.e., $N_0 \approx K$. We then have $j\colon N \to A$ defined as the composition

$$N \xrightarrow{\Delta_N} A \otimes N \xrightarrow{A \otimes \varepsilon} A \otimes K = A\ ,$$

where $\varepsilon\colon N \to K$, $\varepsilon \mid N_0$ is an isomorphism.

2.6. Proposition. *Suppose the following conditions are satisfied:*

(1) *A is a connected coalgebra over K,*

(2) *N is a connected left A-comodule,*

(3) *$C = K \,\square_A\, N$,*

(4) *$\varphi\colon N \otimes C \to N$ is a morphism of left A-comodules where $\Delta_{N \otimes C} = \Delta_N \otimes C$,*

(5) *$j\colon C \to N$ is the canonical monomorphism, and $f\colon N \to C$ is a morphism of graded K-modules such that $fj = C$,*

(6) *$\varphi(\eta \otimes C) = j\colon C \to N$,*

(7) *$\varphi(N \otimes \eta) = N\colon N \to N$, and*

(8) *the sequence $N \otimes C \xrightarrow{f \otimes C} A \otimes C \longrightarrow 0$ is an exact sequence of left A-comodules.*

If $\tilde{f}\colon N \to A \otimes C$ is the composition $N \xrightarrow{\Delta_N} A \otimes N \xrightarrow{A \otimes f} A \otimes C$, then

$\tilde{f}$ is an isomorphism of left A-comodules.

PROOF. Define filtrations on $A \otimes C$ and $N \otimes C$ as follows: $F^p(A \otimes C) = \sum_{q \geqq p} A \otimes C_q$, $F^p(N \otimes C) = \sum_{q \geqq p} N \otimes C_q$, and let $E_0(A \otimes C)$ and $E_0(N \otimes C)$ denote the corresponding associated bigraded modules. We now have $E_0^{p,q}(A \otimes C) = A_q \otimes C_p$ and $E_0^{p,q}(N \otimes C) = N_q \otimes C_p$. Moreover

$$\tilde{f}\varphi\big(F^p(N \otimes C)\big) \subset F^p(A \otimes C)$$

and hence induces $E_0(\tilde{f}\varphi)$: $E_0(N \otimes C) \to E_0(A \otimes C)$; identifying $E_0(N \otimes C)$ with $N \otimes C$, and $E_0(A \otimes C)$ with $A \otimes C$, we have $E_0(\tilde{f}\varphi) = j \otimes C$ which is an epimorphism by hypothesis 8. Thus $\tilde{f}\varphi$ is an epimorphism and $\tilde{f}$ is an epimorphism.

Since $K \square_A \tilde{f} = C: C \to C$ and we are now in a position to apply either Proposition 2.4 or 2.5, it follows that $\tilde{f}$ is a monomorphism, and hence an isomorphism, which proves the proposition.

3. Algebras, coalgebras, and duality

A graded K-module A is of *finite type* if each A_n is a finitely generated K-module. It is projective if each A_n is projective. We recall a few facts concerning graded K-modules which are projective of finite type. If A and B are such K-modules, then

(1) the morphism of graded K-modules

$$\lambda: A \longrightarrow A^{**}$$

defined by $\lambda(x)a^* = a^*(x)$ for $x \in A_n$, $a^* \in A_n^*$ is an isomorphism,

(2) the morphism of graded K-modules

$$\alpha: A^* \otimes B^* \longrightarrow (A \otimes B)^*$$

defined by $\alpha(a^* \otimes b^*)(x \otimes y) = a^*(x)b^*(y)$ for $a^* \in A_p^*$, $b^* \in B_q^*$, $x \in A_p$, $y \in B_q$ is an isomorphism.

Thus we write $A = A^{**}$, and $A^* \otimes B^* = (A \otimes B)^*$. Notice that A^* is projective of finite type when A is projective of finite type.

3.1. PROPOSITION. *Suppose that A is a graded K-module which is projective of finite type, then*

(1) *$\varphi: A \otimes A \to A$ is a multiplication in A if and only if $\varphi^*: A^* \to A^* \otimes A^*$ is a comultiplication in A^*,*

(2) *φ is associative if and only if φ^* is associative,*

(3) *$\eta: K \to A$ is a unit for the multiplication φ if and only if $\eta^*: A^* \to K^* = K$ is a unit for the comultiplication φ^*,*

(4) *(A, φ, η) is an algebra if and only if (A^*, φ^*, η^*) is a coalgebra,*

(5) *$\varepsilon: A \to K$ is an augmentation of the algebra (A, η, φ) if and only if*

$\varepsilon^*: K \to A^*$ is an augmentation of the coalgebra (A^*, φ^*, η^*), and

(6) *the algebra (A, φ, η) is commutative if and only if the coalgebra (A^*, φ^*, η^*) is commutative.*

The proof of the preceding proposition follows at once from the definitions and standard properties of projective modules of finite type.

3.2. PROPOSITION. *Suppose (A, φ, η) is an algebra over K such that the graded K-module A is projective of finite type. If N is a graded K-module which is projective of finite type, then*

(1) *$\varphi_N: A \otimes N \to N$ defines the structure of a left A-module on N if and only if $\varphi_{N^*}: N^* \to A^* \otimes N^*$ defines the structure of a left A^*-comodule on N^*, and*

(2) *under the preceding conditions, if $K \otimes_A N$ is projective of finite type, then $(K \otimes_A N)^* = K \square_{A^*} N^*$.*

Once more the proof of the proposition follows at once from the definitions.

3.3. DEFINITIONS. Suppose that A and B are augmented algebras. A morphism $f: A \to B$ of augmented algebras is *left normal* if

(1) the sequence of graded K-modules $I(A) \otimes B \to B \xrightarrow{\pi} C$ is split exact where $C = K \otimes_A B$ and π is the natural epimorphism, and

(2) the composition

$$B \otimes I(A) \longrightarrow B \xrightarrow{\pi} C$$

is zero.

The morphism f is *right normal* if

(1) the sequence of graded K-modules $B \otimes I(A) \to B \xrightarrow{\pi} C$ is split exact where $C = B \otimes_A K$ and π is the natural epimorphism, and

(2) the composition

$$I(A) \otimes B \longrightarrow B \xrightarrow{\pi} C$$

is zero.

We say that f is *normal* if it is both right and left normal, and in this case we write $B//f$ for $K \otimes_A B = B \otimes_A K$. Moreover if f is a monomorphism we write $B//A$ for $B//f$ when f is clear from the context.

3.4. PROPOSITION. *If $f: A \to B$ is a left normal morphism of augmented algebras, $C = K \otimes_A B$, and $\pi: B \to C$ is the natural morphism, there are unique morphisms of graded K-modules $\varphi_C: C \otimes C \to C$, $\eta_C: K \to C$, and $\varepsilon_C: C \to K$ such that π is a morphism of augmented algebras.*

The proof is immediate from the definitions.

3.5. DEFINITIONS. Suppose that A and B are augmented coalgebras. A morphism $f\colon B \to A$ is *left normal* if

(1) the sequence of graded K-modules

$$C \xrightarrow{i} B \longrightarrow J(A) \otimes B$$

is split exact where $C = K \square_A B$ and i is the natural monomorphism, and

(2) the composition

$$C \xrightarrow{i} B \longrightarrow B \otimes J(A)$$

is zero.

The morphism f is *right normal* if

(1) the sequence of graded K-modules

$$C \xrightarrow{i} B \longrightarrow B \otimes J(A)$$

is split exact where $C = B \square_A K$ and i is the natural monomorphism, and

(2) the composition

$$C \longrightarrow B \longrightarrow J(A) \otimes B$$

is zero.

We say that f is *normal* if it is both right and left normal. If f is a normal epimorphism we write $A\backslash\backslash B$ for $K \square_A B = B \square_A K$ when f is clear from the context.

3.6. PROPOSITION. *If $f\colon B \to A$ is a left normal morphism of augmented coalgebras, $C = K \square_A B$, and $i\colon C \to B$ is the natural morphism, there are unique morphisms of graded K-modules $\Delta_c\colon C \to C \otimes C$, $\varepsilon_c\colon C \to K$, $\eta_c\colon K \to C$ such that i is a morphism of augmented coalgebras.*

The proof is immediate from the definitions.

Observe that condition 1 in the definition of either normal morphism of algebras or of normal morphism of coalgebras is always satisfied if the ground ring K is a field; while condition 2 is always satisfied in the commutative case.

3.7. DEFINITIONS AND NOTATION. If A is an augmented algebra over K, let $Q(A) = K \otimes_A I(A)$. The elements of the graded K-module $Q(A)$ are called the *indecomposable elements of A*. If A is an augmented coalgebra over K, let $P(A) = K \square_A J(A)$. The elements of the graded K-module $P(A)$ are called the *primitive elements of A*.

Note that there is a natural exact sequence

$$I(A) \otimes I(A) \longrightarrow I(A) \longrightarrow Q(A) \longrightarrow 0$$

for an augmented algebra A. Thus $Q(A) = I(A) \otimes_A K$.

Similarly if A is an augmented coalgebra, there is a natural exact sequence

$$0 \longrightarrow P(A) \longrightarrow J(A) \longrightarrow J(A) \otimes J(A) ,$$

and thus $P(A) = J(A) \,\square_A\, K$.

If $f\colon A \to B$ is a morphism of augmented algebras, then f induces

$$Q(f)\colon Q(A) \longrightarrow Q(B) .$$

Similarly if $f\colon A \to B$ is a morphism of augmented coalgebras, then f induces $P(f)\colon P(A) \to P(B)$.

3.8. PROPOSITION. *If $f\colon A \to B$ is a morphism of augmented algebras and B is connected, then f is an epimorphism if and only if $Q(f)\colon Q(A) \to Q(B)$ is an epimorphism.*

PROOF. Certainly if f is an epimorphism, then $Q(f)$ is an epimorphism. Suppose now that $Q(f)$ is an epimorphism. We have a commutative diagram

$$\begin{array}{ccccccc} I(A) \otimes I(A) & \longrightarrow & I(A) & \longrightarrow & Q(A) & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle I(f)\otimes I(f)} & & \big\downarrow{\scriptstyle I(f)} & & \big\downarrow{\scriptstyle Q(f)} & & \\ I(B) \otimes I(B) & \longrightarrow & I(B) & \longrightarrow & Q(B) & \longrightarrow & 0 \end{array}$$

with exact rows. Since $I(B)_0 = 0$, it follows that $I(f)_0$ is an epimorphism. Suppose $I(f)_q$ is an epimorphism for $q < n$, then $\big(I(f) \otimes I(f)\big)_q$ is an epimorphism for $q \leqq n$; and thus by the 5-lemma, $I(f)_q$ is an epimorphism for $q \leqq n$, and the proposition follows.

3.9. PROPOSITION. *If $f\colon A \to B$ is a morphism of augmented coalgebras, the underlying graded K-modules of A and B are flat, and A is connected, then f is a monomorphism if and only if $P(f)\colon P(A) \to P(B)$ is a monomorphism.*

PROOF. Certainly if f is a monomorphism, then $P(f)$ is a monomorphism. Suppose that $P(f)$ is a monomorphism. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P(A) & \longrightarrow & J(A) & \longrightarrow & J(A) \otimes J(A) \\ & & \big\downarrow{\scriptstyle P(f)} & & \big\downarrow{\scriptstyle J(f)} & & \big\downarrow{\scriptstyle J(f)\otimes J(f)} \\ 0 & \longrightarrow & P(B) & \longrightarrow & J(B) & \longrightarrow & J(B) \otimes J(B) \end{array}$$

with exact rows. Since $J(A)_0 = 0$ it follows that $J(f)_0$ is monomophism. Suppose $J(f)_q$ is a monomorphism for $q < n$, then since A and B and hence $J(A)$ and $J(B)$ are flat we have that $\big(J(f) \otimes J(f)\big)_q$ is a monomorphism for $q \leqq n$; and thus by the 5-lemma, $J(f)_q$ is a monomorphism for $q \leqq n$, and the proposition follows.

Note that the difference between Propositions 3.8 and 3.9 comes from the fact that the functor tensor product is right exact, but not left exact. Thus a hypothesis is needed to guarantee its left exactness under certain conditions.

3.10. PROPOSITION. *If A is an augmented algebra, and the underlying graded K-module of A is projective of finite type, then*

(1) *the sequence*

$$I(A) \otimes I(A) \rightarrow I(A) \rightarrow Q(A) \rightarrow 0$$

is split exact if and only if the sequence

$$0 \rightarrow P(A^*) \rightarrow J(A^*) \rightarrow J(A^*) \otimes J(A^*)$$

is split exact, and

(2) *if the sequences of* (1) *are split exact, then* $P(A^*) = Q(A)^*$, *and* $Q(A) = P(A^*)^*$.

The proof of the proposition is immediate from the definitions.

3.11. PROPOSITION. *If $f\colon A \rightarrow B$ is a left normal morphism of augmented algebras, $C = K \otimes_A B$, and $\pi\colon B \rightarrow C$ is the natural morphism of augmented algebras, then the sequence*

$$Q(A) \xrightarrow{Q(f)} Q(B) \xrightarrow{Q(\pi)} Q(C) \longrightarrow 0$$

is an exact sequence of graded K-modules.

PROOF. We have that the cokernel of $Q(f)$ is the cokernel of

$$I(A) \oplus I(B)^2 \longrightarrow I(B) ,$$

and that $Q(C)$ is the cokernel of $I(B)^2 \rightarrow I(B)/I(A)B$, and since these two cokernels are naturally isomorphic the proposition follows.

3.12. PROPOSITION. *If $f\colon B \rightarrow A$ is a left normal morphism of augmented coalgebras, $C = K \square_A B$, and $i\colon C \rightarrow B$ is the natural morphism of augmented coalgebras, then the sequence*

$$0 \longrightarrow P(C) \xrightarrow{P(i)} P(B) \xrightarrow{P(f)} P(A)$$

is an exact sequence of graded K-modules.

PROOF. Suppose $x \in P(B)_n$ and $f(x) = 0$, then $\Delta_B(x) = x \otimes 1 + 1 \otimes x$, $(f \otimes B)\Delta_B(x) = 1 \otimes x$, and so $x \in P(B)_n \cap C_n$. Thus $x \in P(C)_n$. The rest of the proof of the proposition is immediate.

4. Elementary properties of Hopf algebras

4.1. DEFINITION. A *Hopf algebra* over K is a graded K-module A together with morphisms of graded K-modules

$$\varphi\colon A \otimes A \longrightarrow A, \qquad \eta\colon K \longrightarrow A$$
$$\Delta\colon A \longrightarrow A \otimes A, \qquad \varepsilon\colon A \longrightarrow K$$

such that

(1) (A, φ, η) is an algebra over K with augmentation ε,

(2) (A, Δ, ε) is a coalgebra over K with augmentation η, and

(3) the diagram

$$\begin{tikzcd} A\otimes A \arrow[r, "\varphi"] \arrow[d, "\Delta\otimes\Delta"] & A \arrow[r, "\Delta"] & A\otimes A \\ A\otimes A\otimes A\otimes A \arrow[rr, "A\otimes T\otimes A"] & & A\otimes A\otimes A\otimes A \arrow[u, "\varphi\otimes\varphi"] \end{tikzcd}$$

is commutative.

Notice that condition 3 is equivalent to saying either that Δ is a morphism of algebras over K, or that φ is a morphism of coalgebras over K except for the condition involving preservation of units. However since η is an augmentation of the coalgebra in question, the diagram

$$\begin{tikzcd} K \arrow[r, "\approx"] \arrow[d, "\eta"] & K\otimes K \arrow[d, "\eta\otimes\eta"] \\ A \arrow[r] & A\otimes A \end{tikzcd}$$

is commutative and thus Δ is a morphism of algebras. Similarly φ is a morphism of coalgebras.

In the diagram

$$\begin{tikzcd} A \arrow[rr, "\Delta"] \arrow[d, "\varepsilon"] \arrow[ddrr, "\approx"] & & A\otimes A \arrow[dd, "\varepsilon\otimes A"] \\ K \arrow[d, "\approx"] & & \\ K\otimes K & & K\otimes A \arrow[ll, "K\otimes\varepsilon"'] \end{tikzcd}$$

the upper right hand triangle is commutative if ε is a unit of the coalgebra. The lower part of the diagram being clearly commutative, we have Δ is a morphism of augmented algebras. Similarly φ is a morphism of augmented coalgebras.

4.2. DEFINITIONS. If A is a Hopf algebra, a *left module over* A is a left module over the underlying algebra of A. If M and N are left modules over A, then $M\otimes N$ is the left module over A with $\varphi_{M\otimes N}\colon A\otimes M\otimes N\to M\otimes N$ defined as the composition

$$A\otimes M\otimes N \xrightarrow{\Delta\otimes M\otimes N} A\otimes A\otimes M\otimes N \xrightarrow{A\otimes T\otimes N}$$
$$A\otimes M\otimes A\otimes N \xrightarrow{\varphi_M\otimes\varphi_N} M\otimes N\,.$$

A *left module coalgebra over* A is a left A-module B together with morphisms of left A-modules $\Delta_B\colon B\to B\otimes B$ and $\varepsilon_B\colon B\to K$ such that $(B, \Delta_B, \varepsilon_B)$ is a coalgebra over K.

Similar considerations apply to right A-modules and A-coalgebras.

4.3. PROPOSITION. *If A is a Hopf algebra over K, B is a left A-module coalgebra $C = K \otimes_A B$ and $\pi: B \to C$ is the natural morphism of graded K-modules, then there are unique morphisms $\Delta_C: C \to C \otimes C$ and $\varepsilon_C: C \to K$ such that π is a morphism of coalgebras over K.*

PROOF. Since $\Delta_B: B \to B \otimes B$ is a morphism of left A-modules, it induces

$$K \otimes_A \Delta_B: K \otimes_A B \longrightarrow K \otimes_A (B \otimes B) .$$

However there is a natural morphism

$$\theta: K \otimes_A (B \otimes B) \longrightarrow (K \otimes_A B) \otimes (K \otimes_A B) .$$

We let $\Delta_C = \theta \circ (K \otimes_A \Delta_B)$. Since $\varepsilon_B: B \to K$ is a morphism of left A-modules, it induces

$$K \otimes_A \varepsilon_B: K \otimes_A B \longrightarrow K \otimes_A K .$$

Noting that $K \otimes_A K = K$ where A acts on K *via* ε, we see easily that $(C, \Delta_C, \varepsilon_C)$ fulfill the required conditions.

Note that C could be considered as a coalgebra over A on which $I(A)$ acts trivially. Observe further that, if N is a left A-module on which A acts *via* ε, and we consider A itself as a left A-module, then $A \otimes N$ is a left A-module, and is in fact the extended A-module of N as defined in paragraph 1.

Recall further that if B was a connected A-module, we defined a canonical morphism $i: A \to B$ in paragraph 1.

4.4. THEOREM. *If A is a connected Hopf algebra over K, B is a connected left A-module coalgebra, $C = K \otimes_A B$, $i: A \to B$, and $\pi: B \to C$ are the canonical morphisms, and the sequences $0 \to A \xrightarrow{i} B$, $B \xrightarrow{\pi} C \to 0$ are split exact as sequences of graded K-modules, then there exists $h: B \to A \otimes C$ which is simultaneously an isomorphism of left A-modules and right C-comodules.*

PROOF. Let $f: C \to B$ be a morphism of graded K-module such that $\pi f = C$. Note that the composition

$$B \xrightarrow{\Delta_B} B \otimes B \xrightarrow{B \otimes \pi} B \otimes C$$

is a morphism of left A-modules and the conditions of Proposition 1.7 are satisfied. Thus if $\tilde{f}$ is the composition

$$A \otimes C \xrightarrow{A \otimes f} A \otimes B \longrightarrow B$$

we have that $\tilde{f}$ is an isomorphism of left A-modules. Hence there is a morphism of left A-modules $g: B \to A$ such that $gi = A$. Now let h be the composition

$$B \xrightarrow{\Delta_B} B \otimes B \xrightarrow{g \otimes \pi} A \otimes C .$$

We now have that h is a morphism of left A-modules and of right C-comodules. Filtering $A \otimes C$ by setting $F_p(A \otimes C) = \sum_{q \leq p} A \otimes C_q$, we have that $h\tilde{f}$ is filtration preserving and that $E_0(h\tilde{f})$ is just the identity morphism of $E^0(A \otimes C)$. Thus $h\tilde{f}$ is an isomorphism of left A-modules, and hence so is h. Clearly h fulfills the required conditions and the theorem is proved.

4.5. DEFINITIONS. If A is a Hopf algebra, a *left comodule over A* is a left comodule over the underlying coalgebra of A. If M and N are left comodules over A with $\Delta_{M \otimes N}$: $M \otimes N$ defined as the composition

$$M \otimes N \xrightarrow{\Delta_M \otimes \Delta_N} A \otimes M \otimes A \otimes N \xrightarrow{A \otimes T \otimes N}$$
$$A \otimes A \otimes M \otimes N \xrightarrow{\varphi \otimes M \otimes N} A \otimes M \otimes N .$$

A *left comodule algebra over A* is an A-comodule B together with morphisms of left A-comodules φ_B: $B \otimes B \to B$ and η_B: $K \to B$ such that (B, φ_B, η_B) is an algebra over K.

4.6. PROPOSITION. *If A is a Hopf algebra over K, B is a left A-comodule algebra, $C = K \square_A B$, and i: $C \to B$ is the natural morphism of graded K-modules, then there are unique morphisms φ_C: $C \otimes C \to C$ and η_C: $K \to C$ such that i is a morphism of algebras over K.*

The proof of the proposition is similar to that of 4.3.

Recall that if B was a connected A-comodule a canonical morphism j: $B \to A$ was defined in paragraph 2.

4.7. THEOREM. *If A is a connected Hopf algebra over K, B is a connected left A-comodule algebra, $C = K \square_A B$, j: $B \to A$, and i: $C \to B$ are the canonical morphisms, and the sequences $B \xrightarrow{\partial} A \to 0$, $0 \to C \xrightarrow{i} B$ are split exact as sequences of graded K-modules, then there exists h: $A \otimes C \to B$ which is simultaneously an isomorphism of left A-comodules and right C-modules.*

The proof is similar to that of 4.4. One uses Proposition 2.6 instead of Proposition 1.7.

4.8. PROPOSITION. *If A is a graded projective K-module of finite type, then $(A, \varphi, \eta, \Delta, \varepsilon)$ is a Hopf algebra with multiplication φ, comultiplication Δ, unit η, and counit ε if and only if $(A^*, \Delta^*, \varepsilon^*, \varphi^*, \eta^*)$ is a Hopf algebra with multiplication Δ^*, comultiplication φ^*, unit ε^*, and counit η^*.*

The proof is immediate from the definitions, and what has been done earlier.

4.9. PROPOSITION. *If A, B, C are connected Hopf algebras, and i: $A \to B$,*

π: $B \to C$ are morphisms of Hopf algebras, then the following are equivalent:

(1) *i is a left normal monomorphism of algebras, $C = K \otimes_A B$, and the sequences $0 \to A \xrightarrow{i} B$, $B \xrightarrow{\pi} C \to 0$ are split exact as sequences of graded K-modules,*

(2) *π is a right normal epimorphism of coalgebras, $A = B \,\square_C\, K$, and the sequences $0 \to A \xrightarrow{i} B$, $B \xrightarrow{\pi} C \to 0$ are split exact as sequences of graded K-modules,*

(3) *there exists $f: A \otimes C \to B$ which is an isomorphism of left A-modules and right C-comodules such that $i = f(A \otimes \eta)$, and $\pi f = \varepsilon \otimes C$.*

This proposition is a resumé of some part of what has already been proved. Notice that exactness and split exactness are equivalent if K is a field. Thus if K is a field and the multiplication in B is commutative, any sub Hopf algebra of B is normal as a subalgebra. Similarly if K is a field and the comultiplication in B is commutative, then any quotient Hopf algebra of B is normal as a coalgebra.

4.10. PROPOSITION. *If i: $A \to B$, π: $B \to C$ are morphisms of connected Hopf algebras satisfying the conditions of the preceding proposition, then there is a commutative diagram*

$$\begin{array}{ccccccc} 0 \longrightarrow & P(A) & \longrightarrow & P(B) & \longrightarrow & P(C) & \\ & \downarrow & & \downarrow & & \downarrow & \\ & Q(A) & \longrightarrow & Q(B) & \longrightarrow & Q(C) & \longrightarrow 0 \end{array}$$

with exact rows.

PROOF. The proposition follows at once from 3.11, 3.12 and the fact that, for any Hopf algebra D, there is a natural morphism $P(D) \to Q(D)$.

4.11. PROPOSITION. *Suppose i: $A \to B$, j: $B \to C$ are morphisms of Hopf algebras such that i, j are split monomorphisms, and left normal morphisms of algebras. Let $B' = K \otimes_A B$, $C' = K \otimes_A C$, and j': $B' \to C'$ be the morphism of Hopf algebras induced by j. Now j' is a split monomorphism which is left normal as a morphism of algebras, and*

$$K \otimes_B C = K \otimes_{B'} C' .$$

PROOF. Recall that f: $E \to E'$ is a split monomorphism if the sequence $0 \to E \to E'$ is split exact. Now applying 4.9, we have that $C \simeq B \otimes D$ where $D = K \otimes_B C$. Thus $C' \simeq K \otimes_A (B \otimes D) \simeq B' \otimes D$, and $K \otimes_{B'} C' \simeq D$ which proves the proposition.

4.12. DEFINITION. A set I is *directly ordered* if there is given a partial ordering of I such that

(1) $i \leqq j$, $j \leqq k$ implies $i \leqq k$,

(2) $i \leqq j$ and $j \leqq i$ if and only if $i = j$,

(3) for $i, j \in I$, there exists $k \in I$ such that $i \leqq k$ and $j \leqq k$.

If for each $i \in I$, $A(i)$ is a graded K-module, and if for $i \leqq j$, $f(j, i)$: $A(i) \rightarrow A(j)$ is a morphism of graded K-modules such that if $i \leqq j, j \leqq k$, then $f(k, j) f(j, i) = f(k, i)$, and $f(i, i) = A(i)$, we say that we have a *direct system of graded K-modules*; and we denote by $\lim_I A(i)$ the direct limit of this direct system of graded K-modules. If each $A(i)$ is an algebra, a coalgebra, or a Hopf algebra, and each $f(i, j)$ is a morphism of algebras, coalgebras, or Hopf algebras as the case may be, then $\lim_I A(i)$ is an algebra, coalgebra, or Hopf algebra according to the type of direct system with which we are dealing. Moreover in each case the underlying graded K-module is the direct limit of the underlying graded K-modules of the direct system.

More general direct limits exist in any of the categories being considered, but we will not concern ourselves with that here.

Recall that a Prüfer ring is an integral domain K such that every ideal in K is flat, or equivalently every finitely generated ideal in K is projective. Such a ring has the property that every submodule of a flat module is flat, or equivalently that every finitely generated submodule of a flat module is projective. Thus for such a ring we have that every submodule of finite type of a flat graded K-module is projective of finite type.

4.13. PROPOSITION. *If B is a connected Hopf algebra over the Prüfer ring K such that the underlying graded K-module of B is flat, then B is a direct limit of sub Hopf algebras of B whose underlying graded K-module is projective of finite type.*

PROOF. Let $\mathfrak{B}$ be the set of sub Hopf algebras of B which are direct limits of sub Hopf algebras of B of finite type. Clearly $\mathfrak{B}$ is closed under limits, and thus has maximal elements. Let A be such a maximal element of B. We propose to show that $A = B$.

Suppose that x is an element of least degree of $B - A$. Now

$$\Delta_B(x) = x \otimes 1 + 1 \otimes x + y$$

where $y \in A \otimes A$. Without loss of generality we may assume that $y \in A(i)$ for $i \in I$ where $A = \lim_I A(i)$, and $A(i)$ is a sub Hopf algebra of B of finite type. Now if we let $B(i)$ be the subalgebra of B generated by $A(i)$ and x, we have that $B(i)$ is a sub Hopf algebra of B which is of finite type $\lim_I B(i) \ni x$ and contains A. This is impossible. Therefore $A = B$ and the proposition follows.

4.14. *Notation*. If B is an algebra over K, we denote by $B(n)$ the subalgebra of B generated by elements of degree less than or equal to n.

4.15. Proposition. *If B is a connected Hopf algebra over the Prüfer ring K, and the underlying graded K-module of B is projective of finite type, then*

(1) *there is a unique comultiplication in $B(n)$ such that $B(n)$ is a sub Hopf algebra of B;*

(2) *the underlying graded K-module of $B(n)$ is projective of finite type,*

(3) *$B(n)$ is finitely generated as an algebra;*

(4) *$Q(B(n))_r = Q(B)$ for $r \leqq n$, and*

(5) *$Q(B(n))_r = 0$ for $r > n$.*

The proof of this proposition is immediate from the definitions.

4.16. Definitions. A *quasi Hopf algebra* over K is a graded K-module A together with morphisms of graded K-modules

$$\varphi\colon A \otimes A \longrightarrow A\,, \qquad \eta\colon K \longrightarrow A$$
$$\Delta\colon A \longrightarrow A \otimes A\,, \qquad \varepsilon\colon A \longrightarrow K$$

such that

(1) η is a unit for the multiplication φ,

(2) ε is a counit for the comultiplication Δ,

(3) the diagram

$$\begin{tikzcd}
A \otimes A \arrow[r, "\varphi"] \arrow[d, "\Delta \otimes \Delta"] & A \arrow[r, "A"] & A \otimes \Delta \arrow[d, "\varphi \otimes \varphi"] \\
A \otimes A \otimes A \otimes A \arrow[rr, "A \otimes T \otimes A"] & & A \otimes A \otimes A \otimes A
\end{tikzcd}$$

is commutative, and

(4) $\varepsilon\eta = K\colon K \to K$.

One sees easily that a quasi Hopf algebra A over K is a Hopf algebra if and only if both φ and Δ are associative.

Moreover if A is a quasi Hopf algebra, then $P(A)$ is the kernel of $J(A) \to J(A) \otimes J(A)$, and $Q(A)$ is the cokernel of $I(A) \otimes I(A) \to I(A)$, just as for Hopf algebras, there is a natural morphism of graded K-modules $P(A) \to Q(A)$. Propositions 4.13 and 4.15 follow for quasi Hopf algebras, just as for Hopf algebras.

4.17. Proposition. *If A is a connected quasi Hopf algebra over the field K of characteristic zero, then the natural morphism $P(A) \to Q(A)$ is a monomorphism if and only if the multiplication φ is commutative and associative.*

PROOF. Suppose that $P(A) \to Q(A)$ is a monomorphism. For $x \in A_p$, $y \in A_q$, $z \in A_r$, let $a(x, y, z) = (xy)z - x(yz)$. If x, y, z are primitive $a(x, y, z)$ is primitive and its image in $Q(A)$ is zero. Thus $a(x, y, z) = 0$. Suppose now $a(x, y, z) = 0$ for $p \leqq u$, $q \leqq v$, $r \leqq w$, and consider the case $p \leqq u$, $q \leqq v$, $r = w + 1$. Now $a(x, y, z)$ is primitive and has zero image in $Q(A)$ and hence is zero. Proceeding by induction we have that $a\colon A \otimes A \otimes A \to A$ is the zero morphism, or that the multiplication φ is associative. For $x \in A_p$, $y \in A_q$, let $[x, y] = xy - (-1)^{pq}yx$. Notice that if x, y are primitive, $[x, y]$ is primitive and has image zero in $Q(A)$. Proceeding by induction as before we obtain that $[\,,\,]\colon A \otimes A \to A$ is zero, and then φ is commutative.

Suppose now that the algebra A is commutative and has one generator $x \in A_n$, $n > 0$. If n is odd then $x^2 = (-1)^{n^2}x^2 = -x^2$ and $x^2 = 0$, that $P(A) = I(A) = Q(A)$. If n is even, $\Delta(x) = x \otimes 1 + 1 \otimes x$,

$$\Delta(x^k) = \textstyle\sum_{i+j=k}(i, j)x^i \otimes x^j$$

where (i, j) denotes the appropriate binomial coefficient, and $1 = x^0$. We now see that $P(A) \to Q(A)$ is an isomorphism.

Suppose that the converse part of the proposition is proved for algebras with less than or equal to m generators, and that A is an algebra such that $x_i \in A_{n_i}$ for $i = 1, \cdots, m + 1$, $n_i \leqq n_{i+1}$, and the images of the elements $x_1, \cdots, x_{m+1}$ form a basis of $\bigoplus_r Q(A)_r$. Let A' be the subalgebra of A generated by $x_1, \cdots, x_m$, and note that A' is a commutative algebra with comultiplication. Let $A'' = K \otimes_{A'} A = A//A'$ and observe that A'' is a commutative algebra with comultiplication and one generator as an algebra. We now have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P(A') & \longrightarrow & P(A) & \longrightarrow & P(A'') & & \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\approx} & & \\
0 & \longrightarrow & Q(A') & \longrightarrow & Q(A) & \longrightarrow & Q(A'') & \longrightarrow & 0
\end{array}$$

with exact rows. Since $P(A') \to Q(A')$ is a monomorphism it follows that $P(A) \to Q(A)$ is a monomorphism. Thus by induction the converse part of the proposition is proved for finitely generated algebras. As in Proposition 4.13 any connected coalgebra with comultiplication is a direct limit of connected algebras with comultiplication which are of finite type, and those of finite type are direct limits of finitely generated ones as in Proposition 4.14. Thus since the functors P and Q commute with direct limits, and the direct limit of monomorphisms is a monomorphism in the category of graded K-modules, the proposition follows.

4.18. COROLLARY. *If A is a connected quasi Hopf algebra over the field K of characteristic zero, then*

(1) *the comultiplication Δ is commutative and associative if and only if $P(A) \to Q(A)$ is an epimorphism, and*

(2) *A is a Hopf algebra with both commutative multiplication and commutative comultiplication if and only if $P(A) \to Q(A)$ is an isomorphism.*

4.19. DEFINITION OR NOTATION. Suppose that K is an integral domain of characteristic $p \neq 0$, and A is a commutative algebra over K, define $\xi\colon A_n \to A_{pn}$ by $\xi(x) = x^p$.

Notice that $\xi(x+y) = \xi(x) + \xi(y)$, and $\xi(xz) = \xi(x)\xi(z)$ for $x, y \in A_n$, $z \in A_m$. In particular $\xi(kx) = k^p\xi(x)$ for $x \in A_n$, $k \in K$. Observe also that if p and n are odd, then $\xi(x) = 0$.

4.20. PROPOSITION. *If A is a connected quasi Hopf algebra over the field K of characteristic $p \neq 0$, then the natural morphism $P(A) \to Q(A)$ is a monomorphism if and only if the multiplication φ is commutative and associative, and $\xi(A_n) = 0$ for $n > 0$.*

PROOF. Suppose $P(A) \to Q(A)$ is a monomorphism. The first part of the proof of 4.17 shows that φ is associative and commutative. Notice that if $x \in P(A)_n$ then $\xi(x) \in P(A)_{pn}$, and the image of $\xi(x)$ in $Q(A)_{pn}$ is zero. Thus $\xi(x) = 0$. If $\xi(A_m) = 0$ for $0 < m < n$, then $\xi(A_n) \subset P(A)_{pn}$; and since the composition $A_n \xrightarrow{\xi} P(A)_{pn} \to Q(A)_{pn}$ is zero, it follows by induction that $\xi(A_n) = 0$ for all n, and the first half of the proposition is proved.

Suppose now that φ is commutative and associative, and $\xi(A_n) = 0$ for all $n > 0$. As in the second half of 4.17, in order to show that $P(A) \to Q(A)$ is a monomorphism, it suffices to do so in case the algebra A has one generator $x \in A_n$, $n > 0$. If $p = 2$, then $x^2 = 0$, and $P(A) = I(A) = Q(A)$. The situation is the same if p is odd and n is odd. If $p \neq 2$, and n is even, then $A_q = 0$ for $q \neq kn$ $k = 0, 1, \cdots, (p-1)$ and x^k is a basic element for A_{kn}. Since

$$\Delta(x^k) = \sum\nolimits_{i+j=k}(i, j)x^i \otimes x^j$$

and $(i, j) \neq 0$, we have $P(A)_n = A_n = Q(A)_n$, $P(A)_q = 0 = Q(A)_q$ for $q \neq n$, and the proposition follows.

If S is a subset of the K-module M, we denote by $K(S)$ the K-submodule generated by S.

4.21. PROPOSITION. *If A is a connected quasi Hopf algebra with associative commutative multiplication over the field K of characteristic $p \neq 0$, there is an exact sequence*

$$0 \longrightarrow P\big(K(\xi A)\big) \longrightarrow P(A) \longrightarrow Q(A) .$$

PROOF. Let $A' = K(\xi(A))$, and $A'' = K \otimes_{A'} A$. Now $\xi(A''_n) = 0$ for $n > 0$, and we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P(A') & \longrightarrow & P(A) & \longrightarrow & P(A'') \\ & & \downarrow & & \downarrow & & \downarrow \\ & & Q(A') & \longrightarrow & Q(A) & \longrightarrow & Q(A'') \end{array}$$

with exact rows. The proposition now follows from 4.20.

4.22. DEFINITION. Suppose that K is a field of characteristic $p \neq 0$, and A is a commutative coalgebra of finite type over K. Define $K(\lambda(A))$ to be $K(\xi(A^*))^*$, and observe that $K\lambda(A)$ is a quotient coalgebra of A. If A is a direct limit of commutative coalgebras of finite type $A = \lim_I A(i)$, define $K\lambda(A) = \lim_I K\lambda(A(i))$.

Observe that if A is of finite type, then $K(\lambda(A))^* = K\xi(A^*)$.

4.23. PROPOSITION. *If A is a connected Hopf algebra with both commutative multiplication and commutative comultiplication over the field K of characteristic $p \neq 0$, there is an exact sequence*

$$0 \longrightarrow P\big(K\xi(A)\big) \longrightarrow P(A) \longrightarrow Q(A) \longrightarrow Q\big(K\lambda(A)\big) \longrightarrow 0 .$$

The proof of the proposition is immediate from the definitions. Observe that if n is odd we have $K\xi(A)_n = 0 = K\lambda(A)_n$; and thus for n odd, if follows that $P(A)_n \xrightarrow{\approx} Q(A)_n$ for any Hopf algebra A satisfying the conditions of 4.23.

5. Universal algebras of Lie algebras

5.1. DEFINITIONS. If A is an algebra over K, define $[\ ,\ ]\colon A \otimes A \to A$ by $[x, y] = xy - (-1)^{pq}yx$ for $x \in A_p$, $y \in A_q$. The morphism of graded K-modules $[\ ,\ ]$ is called the Lie product of A. The graded K-module A, together with its Lie product, is called the *associated Lie algebra* of the algebra A.

A *Lie algebra* over K is a graded K-module L together with a morphism $[\ ,\ ]\colon L \otimes L \to L$ such that for some algebra A, there is a monomorphism of graded K-modules $f\colon L \to A$ such that the diagram

$$\begin{array}{ccc} L \otimes L & \xrightarrow{[\ ,\ ]} & L \\ \Big\downarrow{\scriptstyle f \otimes f} & & \Big\downarrow{\scriptstyle f} \\ A \otimes A & \xrightarrow{[\ ,\ ]} & A \end{array}$$

is commutative.

If L and L' are Lie algebras over K, a morphism $f\colon L \to L'$ of Lie algebras is a morphism of graded K-modules such that the diagram

$$\begin{array}{ccc} L \otimes L & \xrightarrow{[\ ,\ ]} & L \\ \downarrow{\scriptstyle f \otimes f} & & \downarrow{\scriptstyle f} \\ L' \otimes L' & \xrightarrow{[\ ,\ ]} & L' \end{array}$$

is commutative.

5.2. PROPOSITION. *If L is a Lie algebra over K, $x \in L_p$, $y \in L_q$, and $z \in L_r$, then*

(1) $[x, y] = (-1)^{pq+1}[x, y]$, *and*

(2) $(-1)^{pr}[x, [y, z]] + (-1)^{qp}[y, [z, x]] + (-1)^{rq}[z, [x, y]] = 0$.

The proof of the proposition is immediate from the definitions. If K is a field of characteristic $\neq 2$ or 3, it is possible to characterize Lie algebras by the identities (1) and (2). Notice that (1) implies that $2[x, x] = 0$ for p even. To characterize Lie algebras over a field of characteristic 2, it is necessary to assume, in addition to (1) and (2), that for any integer p, $[x, x] = 0$. Identity (2) implies that $3[x, [x, x]] = 0$ for p odd. To characterize Lie algebras over a field of characteristic 3, it is necessary to assume that $[x, [x, x]] = 0$ for p odd.

It is perhaps worth remarking that a Lie algebra is not an algebra since it does not have a unit, and its multiplication or product is not associative.

5.3. DEFINITION. If L is a Lie algebra, the *universal enveloping algebra of L* is an algebra $U(L)$ together with a morphism of Lie algebras $i_L \colon L \to U(L)$ such that if A is an algebra and $f \colon L \to A$ is a morphism of Lie algebras, there is a unique morphism of algebras $\tilde{f} \colon U(L) \to A$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "i_L"] \arrow[dr, "f"'] & U(L) \arrow[d, "\tilde{f}"] \\ & A \end{tikzcd}$$

is commutative.

5.4. *Remarks.* Since we have defined the universal enveloping algebra of a Lie algebra by a universal property of morphisms, it is certainly unique if it exists. Existence is proved as usual in the following way: If L is a graded K-module let $T(L)$ denote the tensor algebra of L, i. e., as a graded K-module $T(L) = \bigoplus_p T_p(L)$ where $T_p(L)$ denotes the tensor product of L with itself p-times, e. g., $T_0(L) = K$, $T_1(L) = L$, and $T_2(L) = L \otimes L$. There is a natural isomorphism $T_p(L) \otimes T_q(L) \xrightarrow{\approx} T_{p+q}(L)$ which on passing to direct sums induces a multiplication $\varphi \colon T(L) \otimes T(L) \to T(L)$. Since $T_0(L) = K$, there are natural morphisms $\eta \colon K \to T(L)$ and $\varepsilon \colon T(L) \to K$ and $T(L)$ becomes an augmented algebra over K. The algebra $T(L)$ has the following property. If $f\colon L \to A$ is a morphism of graded K-modules and A is the underlying graded K-module of an algebra, then there is a unique morphism of algebras $f^{\#}\colon T(L) \to A$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[dr, "f"'] & T(L) \arrow[d, "f^{\#}"] \\ & A \end{tikzcd}$$

of graded K-modules is commutative, where $L = T_1(L)$ and i is the natural morphism.

Now if L is the underlying module of a Lie algebra, f a morphism of Lie algebras, and I the ideal in $T(L)$ generated by elements $xy - (-1)^{pq}yx - [x, y]$ for $x \in L_p$, $y \in L_q$, then $f^{\#}(I) = 0$, and thus, if we let $U(L) = T(L)/I$, there is a unique morphism of algebras $\tilde{f}\colon U(L) \to A$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "i_L"] \arrow[dr, "f"'] & U(L) \arrow[d, "\tilde{f}"] \\ & A \end{tikzcd}$$

is commutative, where i_L is the composition $L \xrightarrow{i} T(L) \to U(L)$.

Notice that the graded K-module 0 is a Lie algebra and that $U(0) = K$. Thus, since for any Lie algebra L, there is a unique morphism of Lie algebras $0_L\colon L \to 0$, we have that 0_L induces $\varepsilon\colon U(L) \to K$ and $U(L)$ is an augmented algebra. Now if $f\colon L \to L'$ is a morphism of Lie algebras, there is a unique morphism of algebras $U(f)\colon U(L) \to U(L')$ such that $U(f)i_L = i_{L'}f$, and thus $L \to U(L)$ is in fact a functor from the category of Lie algebras over K to the category of augmented algebras over K.

Since in our definition of Lie algebra L, we assumed that for some algebra A there was a morphism of Lie algebras $f\colon L \to A$ such that f was a monomorphism of graded K-modules, it follows that $i_L\colon L \to U(L)$ is a monomorphism of graded K-modules because $f = \tilde{f}i_L$.

5.5. Definition. If L, L' are Lie algebras over K, then $L \times L'$ is the Lie algebra over K such that

$$[(x, y), (x', y')] = \big([x, x'], [y, y']\big)$$

for

$$(x, y) \in (L \times L')_p = L_p \times L'_p\,, \qquad (x', y') \in (L \times L')_q = L_q \times L'_q\,.$$

The Lie algebra $L \times L'$ is called the *product* of the Lie algebras L and L'.

In fact the preceding definition is slightly incorrect since we have not shown that $L \times L'$ can be imbedded in the associated Lie algebra of an algebra. However, this omission is easily remedied by defining

$$i_{L\times L'}\colon L \times L' \longrightarrow U(L) \otimes U(L')$$

by $i(x, y) = x \otimes 1 + 1 \otimes y$ for $(x, y) \in (L \times L')_p$.

Recall that in an arbitary category $\mathcal{A}$ the product of objects A and A', if it exists, is an object $A \times A'$ together with morphisms $\pi\colon A \times A' \to A$ and $\pi'\colon A \times A' \to A'$ such that, if C is an object in $\mathcal{A}$ and $f\colon C \to A$, $f'\colon C \to A'$ are morphisms in $\mathcal{A}$, then there is a unique morphism $f \times f'\colon C \to A \times A'$ in $\mathcal{A}$ such that $\pi f \times f' = f$ and $\pi' f \times f' = f'$. It is exactly in this sence that $L \times L'$ is the product of the Lie algebras L and L' in the category $\mathfrak{L}_K$ of Lie algebras over K and morphisms of Lie algebras over K.

Suppose now we consider the category $(\mathcal{CH})_K$ of Hopf algebras over K with commutative comultiplication. Let $f\colon C \to A$, $f'\colon C \to A'$ be morphisms in this category. We have that $A \otimes A'$ is an object in the category and the composition

$$C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes f'} A \otimes A'$$

is a morphism in the category. Moreover $(A \otimes \varepsilon)(f \otimes f')\Delta_C = f$, and

$$\varepsilon \otimes A'(f \otimes f)\Delta_C = f' .$$

Thus $A \otimes A'$, together with the canonical morphisms $(A \otimes \varepsilon)\colon A \otimes A' \to A$ and $\varepsilon \otimes A'\colon A \otimes A' \to A'$, is easily seen to be the product of the objects A and A' in the category.

Observe that in the preceding, one used the fact that the comultiplication was commutative, for otherwise Δ_C would be only a morphism of algebras and not necessarily of Hopf algebras.

5.6. Proposition. *If L and L' are Lie algebras over K, then*

$$U(L \times L') = U(L) \otimes U(L') .$$

Proof. One verifies without trouble that the morphism of Lie algebras $i_{L\times L'}\colon L \times L' \to U(L) \otimes U(L')$ satisfies the necessary conditions for it to be the universal enveloping algebra of $L \times L'$.

5.7. Definition and comments. If L is a Lie algebra we have a natural morphism $\Delta\colon L \to L \times L$, where $\Delta(x) = (x, x)$ for $x \in L_n$. This induces $U(\Delta)\colon U(L) \to U(L) \otimes U(L)$ and $U(L)$ becomes a Hopf algebra. Henceforth we will consider that $U(L)$ is a Hopf algebra with this comultiplication. Notice that the comultiplication in question is commutative, thus $U(L)$ is an object in $(\mathcal{CH})_K$.

A Hopf algebra A is said to be *primitively generated* if the least subalgebra of A containing $P(A)$ is A itself.

5.8. PROPOSITION. *If A is a Hopf algebra, then $P(A)$ is a sub Lie algebra of the associated Lie algebra of A.*

The proof of the proposition is immediate from the definition. Now we may say that a Hopf algebra A is primitively generated if the natural morphism of Hopf algebras $U(P(A)) \to A$ is an epimorphism in the category of graded K-modules.

It is evident that every primitively generated Hopf algebra has commutative comultiplication. Thus the category $(\mathcal{PH})_K$ whose objects are the primitively generated Hopf algebras and morphisms are morphisms of Hopf algebras in a subcategory of the category $(\mathcal{CH})_K$ of Hopf algebras with commutative comultiplication. Recall that we have seen (4.18) that, if K is a field of characteristic zero, a connected Hopf algebra A is in $(\mathcal{PH})_K$ if and only if $P(A) \to Q(A)$ is an epimorphism, or equivalently the comultiplication of A is commutative.

We now have two functors, $U: \mathcal{L}_K \to (\mathcal{PH})_K$ and $P: \mathcal{H}_K \to \mathcal{L}_K$ where $\mathcal{H}_K$ is the category of all Hopf algebras over K.

5.9. PROPOSITION. *The functors $U: \mathcal{L}_K \to (\mathcal{PH})_K$ and $P: (\mathcal{CH})_K \to \mathcal{L}_K$ commute with products, and direct limits.*

This proposition is just a resumé of things stated earlier, together with a couple of observations. Note that we have not discussed the question of the existence of products in the category $\mathcal{H}_K$ but only in the subcategory $(\mathcal{CH})_K$.

In order to proceed further we need to recall some facts about bigraded K-modules. A bigraded K-module A is a family of K-modules $\{A_{p,q}\}$ where the indices p and q run through integers such that $p + q \geqq 0$. If A and B are bigraded K-modules, a morphism of bigraded K-modules $f: A \to B$ is a family of morphisms of K-modules $\{f_{p,q}\}$ such that $f_{p,q}: A_{p,q} \to B_{p,q}$.

If A and B are bigraded K-modules, then $(A \otimes B)$ is the bigraded K-module such that $(A \otimes B)_{p,q} = \bigoplus_{\substack{r+s=p \\ u+v=q}} A_{r,u} \otimes B_{s,v}$. The tensor product of morphisms of bigraded K-modules is defined similarly. There is also the dual of a bigraded K-module A. It is denoted by A^* and is the bigraded K-module such that $A^*_{p,q} = \mathrm{Hom}\,(A_{p,q}, K)$. If $f: A \to B$, then $f^*: B^* \to A^*$, and $f^*_{p,q} = \mathrm{Hom}\,(f_{p,q}, K)$.

If A and B are bigraded K-modules, the twisting morphism $T: A \otimes B \to B \otimes A$ is the morphism such that $T_{p,q}(a \otimes b) = (-1)^{(r+u)(s+v)} b \otimes a$ for $a \in A_{r,u}$,

$b \in B_{s,v}$, $r + s = p$, and $u + v = q$.

We are now in a position to carry out the work of the preceding chapters using bigraded K-modules instead of graded K-modules. The details of so doing are left to the reader. It is worth commenting that a bigraded K-module A is said to be *connected* if $A_{0,0} = K$. However connectedness no longer suffices for the theorems of the earlier paragraph in which the hypothesis was needed. We assume instead that we are dealing with algebras such that they are connected and either

(1) $A_{p,q} = 0$ for $p \leqq 0$, except for $(p, q) = (0, 0)$, or

(2) $A_{p,q} = 0$ for $p \geqq 0$, except for $(p, q) = (0, 0)$.

In case (1) when dealing with a module B the condition of connectedness is replaced by assuming in addition that $B_{p,q} = 0$ for $p < 0$, and in case (2) we assume $B_{p,q} = 0$ for $p > 0$. We proceed in the same fashion for coalgebras and comodules over coalgebras. The bigraded case of a proposition in the preceding chapters such as 3.8 will be referred to as Proposition B 3.8. We leave the task of checking details to the reader.

Although we have dealt informally with filtrations in the preceding paragraphs, we now recall some further details. If A is a graded K-module, a filtration on A is a family of sub-graded modules of A $\{F_pA\}$ indexed on the integers such that $F_pA \subset F_{p+1}A$. If A and B are filtered modules, a morphism of filtered graded modules $f: A \to B$ is a morphism of graded modules such that $f(F_pA) \subset F_pB$. The associated bigraded module of the filtered graded module A is the bigraded module $E^0(A)$ such that $E^0(A)_{p,q} = (F_pA/F_{p-1}A)_{p+q}$. If $f: A \to B$ is a morphism of filtered graded modules there is induced a morphism $E^0(f): E^0(A) \to E^0(B)$ of associated bigraded modules.

The filtration $\{F_pA\}$ on the graded module A is *complete* [3] if

(1) $A = \lim_{p\to\infty} F_pA$, and

(2) $A = \lim_{-\infty\leftarrow p} A/F_pA$.

If A and B are complete filtered graded modules and $f: A \to B$ is a morphism of filtered modules, then if $E^0(f): E^0(A) \to E^0(B)$ is a monomorphism, epimorphism, or isomorphism, f is a monomorphism, epimorphism, or isomorphism as the case may be.

If A and B are filtered graded modules, a filtration is defined in $A \otimes B$ by letting $F_p(A \otimes B) = \operatorname{Im} (\bigoplus_{r+s=p} F_rA \otimes F_sB \to A \otimes B)$. There is a natural morphism $E^0(A) \otimes E^0(B) \to E^0(A \otimes B)$, which is always an epimorphism and is easily seen to be an isomorphism if either

(1) the sequences

$$0 \longrightarrow F_pA \longrightarrow A \longrightarrow A/F_pA \longrightarrow 0 \ ,$$
$$0 \longrightarrow F_qB \longrightarrow B \longrightarrow B/F_qB \longrightarrow 0$$

are split exact for all p, or

(2) the modules A/F_pA and B/F_qB are flat for all p, q.

In this paper we will assume that we are always dealing with a case such that

$$E^0(A) \otimes E^0(B) \xrightarrow{\approx} E^0(A \otimes B) \ .$$

Moreover, we will always deal with a case such that $F_pA = A$ for $p \geqq 0$, or $F_pA = 0$ for $p < 0$.

The ground ring K is always considered to be a filtered graded module with $F_pK = K$ for $p \geqq 0$, and $F_pK = 0$ for $p < 0$. We then have $E^0(K)$ is the bigraded module denoted by K.

A filtered algebra A over K is an algebra with a filtration on its underlying graded K-module such that $\varphi\colon A \otimes A \to A$ is a morphism of filtered graded modules. Thus if A is a filtered algebra over K, then $E^0(A)$ is a filtered bigraded algebra over K.

If A is a filtered algebra over K, a filtered left A-module M is a left A-module with a filtration on its underlying graded K-module such that $\varphi\colon A \otimes M \to M$ is a morphism of filtered graded modules. In this case $E^0(M)$ is a module over $E^0(A)$.

Similar considerations apply to coalgebras, comodules, and Hopf algebras.

5.10. Definition. If A is a primitively generated Hopf algebra over K, its *primitive filtration* is defined inductively as follows:

(1) $F_pA = 0$ for $p < 0$,

(2) $F_0A = K$, and

(3) $F_{p+1}A = \operatorname{Im} F_pA \oplus (P(A) \otimes F_pA) \to A$.

In other words F_pA is the sub graded K-module of A generated by products of $\leqq p$ primitive elements. This filtration could be defined on any Hopf algebra but it is complete if and only if A is primitively generated.

5.11. Proposition. *If A is a primitively generated Hopf algebra over the field K, then $E^0(A)$ is a bigraded Hopf algebra such that*

(1) $E^0(A)_{p,q} = 0$ *for* $p \leqq 0$ *except for* $(p, q) = (0, 0)$,

(2) $P(E^0(A)) \xrightarrow{\approx} Q(E^0(A))$,

(3) $P(E^0(A))_{p,q} = 0$ *for* $p \neq 1$, *and*

(4) $P(E^0(A))_{1,q} = E^0(A)_{1,q} = P(A)_{q+1}$.

Proof. Part 1 follows immediately from the way that the filtration on A was defined. Moreover it follows at once that $E^0(A)$ is primitively generated by $E^0_{1,q}(A)$ for all q. Thus $Q(E^0(A))_{p,q} = 0$ for $p \neq 1$, and $Q(E^0(A))_{1,q} = E^0(A)_{1,q}$. Since in A, if $x \in P(A)_m$, $y \in P(A)_n$, we have $[x, y] \in P(A)_{m+n}$, if follows that $E^0(A)$ is commutative, and the proposition follows from B 4.18 if the characteristic of K is zero. If the characteristic of K is finite, say d, then if $x \in P(A)_q$ and either $d = 2$ or q is even, we have $x^d \in P(A)_{dq}$. Thus in $E^0(A)$, we have $\xi(E^0_{p,q}(A)) = 0$ for $(p, q) \neq (0, 0)$, and the remainder of the proposition follows from B 4.23.

5.12. DEFINITION. If L is a Lie algebra, we define the Lie filtration on $U(L)$ inductively as follows:

(1) $F_pU(L) = 0$ for $p < 0$,

(2) $F_0U(L) = K$, and

(3) $F_{p+1}U(L) = \operatorname{Im}\big(F_pU(L)) \oplus (L \otimes F_pU(L)) \to U(L)\big)$.

Observe that the Lie filtration on $U(L)$ is complete, and if $\{F'_pU(L)\}$ is the primitive filtration on $U(L)$, then $F_pU(L) \subset F'_pU(L)$.

5.13. PROPOSITION. *If K is a field, L is a Lie algebra over K, and $U(L)$ is filtered by the Lie filtration, then*

(1) *$E^0(U(L))_{p,q} = 0$ for $p \leqq 0$ except for $(p, q) = (0, 0)$,*

(2) *$E^0(U(L))$ is a primitively generated, commutative bigraded Hopf algebra,*

(3) *$Q(E^0(U(L)))_{1,q} = E^0(U(L))_{1,q} = L_{q+1}$, and*

(4) *if the characteristic of K is zero, then*

$$P\big(E^0(U(L))\big) \xrightarrow{\approx} Q\big(E^0(U(L))\big) .$$

PROOF. Parts 1, 2, 3 follow from the definitions, while 4 follows from B 4.18.

5.14. NOTATION AND COMMENTS. For the time being we need a different notation for the underlying graded module of a Lie algebra L. It will be denoted by $L^\#$. The module $L^\#$ can also be considered as a Lie algebra by letting its Lie product be zero. We now denote $U(L^\#)$ by $A(L)$, i. e., $A(L) = T(L^\#)/I$ where I is the ideal in $T(L^\#)$ generated by elements $xy - (-1)^{pq}yx$ for $x \in L^\#_p$, $y \in L^\#_q$. The algebra $A(L^\#)$ has the universal property that if B is a commutative algebra and $f\colon L^\# \to B$ is a morphism of graded K-modules, then there is a unique morphism of algebras $\tilde{f}\colon A(L) \to B$ such that the diagram

$$\begin{tikzcd} L^\# \arrow[r] \arrow[dr, "f"'] & A(L) \arrow[d, "\tilde{f}"] \\ & B \end{tikzcd}$$

is commutative. It also has the universal property that if L' is a Lie algebra and if $f: L^{\#} \to L'$ is a morphism of graded K-modules, there is a unique morphism of bigraded Hopf algebras $E^0(A(L)) \to E^0(U(L'))$ induced by f.

5.15. THEOREM (Poincaré-Birkhoff-Witt). *If K is a field of characteristic zero, and L is a Lie algebra over K, then the natural morphism*

$$E^0(A(L)) \longrightarrow E^0(U(L))$$

is an isomorphism.

PROOF. By 5.13 we have that $Q(E^0(A(L))) \to Q(E^0(U(L)))$ and

$$P(E^0(A(L))) \longrightarrow P(E^0(U(L)))$$

are both isomorphisms. Hence applying B 3.8 and B 3.9, it follows that $E^0(A(L)) \to E^0(U(L))$ is an isomorphism, and the theorem is proved.

There is a more usual way of stating the preceding theorem. Recall that if A is a bigraded module we have associated with A a graded module $\bigoplus A$, where $(\bigoplus A)_n = \bigoplus_{p+q=n} A_{p,q}$. This functor from bigraded modules to graded modules takes algebras into algebras, coalgebras into coalgebras, and Hopf algebras into Hopf algebras. In particular if we have a graded K-module X, i. e., an abelian Lie algebra over the field K, then $\bigoplus E^0(A(X)) = A(X)$.

5.16. THEOREM. *If K is a field of characteristic zero, and L is a Lie algebra over K, then the natural morphism*

$$A(L) \longrightarrow \bigoplus E^0 U(L)$$

is an isomorphism.

5.17 PROPOSITION. *If K is a field of characteristic zero, and L is a Lie algebra over K, then the primitive and the Lie filtrations on $U(L)$ coincide.*

PROOF. First we consider the case where L is connected, i. e., $L_0 = 0$. In this case L is a direct limit of Lie algebras of finite type, and it suffices to prove the assertion when L is of finite type and connected. Now let $\{{}^{\mathrm{I}}F_p U(L)\}$ denote the Lie filtration on $U(L)$ and $\{{}^{\mathrm{II}}F_p U(L)\}$ the primitive filtration. We have that ${}^{\mathrm{I}}E^0(U(L)) \to {}^{\mathrm{II}}E^0 U(L)$ is a monomorphism in view of 5.11, 5.13, and the fact that $L \subset P(U(L))$. We are dealing with vector spaces of finite type, and for any given degree n the dimensions over K of $U(L)_n$, $\bigoplus_{p+q=n} {}^{\mathrm{I}}E^0_{p,q} U(L)$ and $\bigoplus_{p+q=n} {}^{\mathrm{II}}E^0_{p,q} U(L)$ are equal, which proves the result.

Suppose now X is a vector space over K, concentrated in degree 0. Let $L(X)$ be the Lie algebra $P(T(X))$ where $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in X_0$. We associate with $L(X)$ a graded Lie algebra $N(X)$. Recalling that

$$T(X) = \bigoplus_p T_p(X) ,$$

we let $N(X)_{2p} = L(X) \cap T_p(X)$, and $L(X)_{2p+1} = 0$; and using the induced Lie product, obtain a graded connected Lie algebra. Since the result is valid for this graded connected Lie algebra, it is also valid for $L(X)$ itself.

Now if L is any Lie algebra concentrated in degree 0, we have that there is a natural epimorphism $T(L^{\#}) \to U(L)$ and a resulting natural epimorphism $L(L^{\#}) \to U(L)$. Recalling that $T(L^{\#}) = U(L(L^{\#}))$, we have a commutative diagram

$$\begin{array}{ccc} {}^{\mathrm{I}}E^0U\big(L(L^{\#})\big) & \xrightarrow{\approx} & {}^{\mathrm{II}}E^0U\big(L(L^{\#})\big) \\ \downarrow & & \downarrow \\ {}^{\mathrm{I}}E^0U(L) & \longrightarrow & {}^{\mathrm{II}}E^0U(L) \end{array}$$

where the vertical morphisms are epimorphisms. It follows that

$${}^{\mathrm{I}}E^0U(L) \longrightarrow {}^{\mathrm{II}}E^0U(L)$$

is an epimorphism. We already know that this last epimorphism is a monomorphism since $L \subset PU(L)$, and thus it is isomorphism which proves the result if L is concentrated in degree 0. For any Lie algebra L there is a canonical exact sequence $0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0$ where L' is connected and L'' is concentrated in degree 0. It follows that $U(L')$, $U(L)$, and $U(L'')$ satisfy the hypothesis of Proposition 3.12, so we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & L'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & PU(L') & \longrightarrow & PU(L) & \longrightarrow & PU(L'') & & \end{array}$$

with exact rows. The proposition now follows from the 5-lemma.

5.18. THEOREM. *If K is a field of characteristic zero, and P: $(\mathcal{PH})_K \to \mathfrak{L}_K$ and U: $\mathfrak{L}_K \to (\mathcal{PH})_K$ are the natural functors, then*

(1) *the functor PU: $\mathfrak{L}_K \to \mathfrak{L}_K$ is the identity functor of $\mathfrak{L}_K$.*

(2) *the functor UP: $(\mathcal{PH})_K \to (\mathcal{PH})_K$ is the identity functor of $(\mathcal{PH})_K$.*

PROOF. The preceding proposition implies $PU(L) = L$, and hence 1. Since $U(P(A)) \to A$ has the property that $PU(P(A)) = P(A)$, it is a monomorphism. Certainly it is an epimorphism, hence an isomorphism, which implies 2, and hence the theorem.

6. Lie algebras and restricted Lie algebras

6.1. DEFINITIONS. Suppose that p is a prime, and that K is a commutative ring which is an algebra over the prime field with p-elements. If A is an algebra over K, define $\xi\colon A_n \to A_{pn}$ in case either n is even or $p = 2$ by $\xi(x) = x^p$ for $x \in A_n$. The graded K-module A together with its Lie product and the functions $\xi\colon A_n \to A_{pn}$ is called the *associated restricted Lie algebra of* A.

A *restricted Lie algebra* over K is a Lie algebra L together with functions $\xi\colon L_n \to L_{pn}$ for n even or $p = 2$ such that for some algebra A, there is a monomorphism of Lie algebras $f\colon L \to A$ such that the diagrams

$$\begin{array}{ccc} L_n & \xrightarrow{\xi} & L_{pn} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{pn}} \\ A_n & \xrightarrow{\xi} & A_{pn} \end{array}$$

are commutative.

If L and L' are restricted Lie algebras over K, a morphism $f\colon L \to L'$ of restricted Lie algebras is a morphism of Lie algebras such that the diagrams

$$\begin{array}{ccc} L_n & \xrightarrow{\xi} & L_{pn} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{pn}} \\ L'_n & \xrightarrow{\xi} & L'_{pn} \end{array}$$

are commutative

6.2. DEFINITION. If L is a restricted Lie algebra over the ring K of characteristic p, the *universal enveloping algebra of* L is an algebra $V(L)$ together with a morphism of restricted Lie algebras $i_L\colon L \to V(L)$ such that if A is an algebra and $f\colon L \to A$ is a morphism of restricted Lie algebras, there is a unique morphism of algebras $\tilde{f}\colon V(L) \to A$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "i_L"] \arrow[dr, "f"'] & V(L) \arrow[d, "\tilde{f}"] \\ & A \end{tikzcd}$$

is commutative.

6.3. REMARKS. The uniqueness of $V(L)$ is assured since it is defined by a universal property of morphisms. Existence is seen by considering L as a Lie algebra and letting $V(L) = U(L)/I$ where I is the ideal in $U(L)$ generated by elements of the form $x^p - \xi(x)$ for $x \in L_n$ where n is even or $p = 1$. Moreover since there is a monomorphism $f\colon L \to A$ for some associative algebra A, and $f = \tilde{f}\, i_L$, we have that $i_L\colon L \to V(L)$ is a monomorphism.

The graded K-module 0 is certainly a restricted Lie algebra and $0_L\colon L \to 0$ is a morphism of restricted Lie algebra. This induces $V(L) \to V(0) = K$, and thus $L \to V(L)$ is a functor from the category of restricted Lie algebras over

K to category of augmented algebras over K.

6.4. DEFINITION. If L, L' are restricted Lie algebras over the ring K of characteristic p, then $L \times L'$ is the restricted Lie algebra over K whose underlying Lie algebra is the product of the underlying Lie algebras of L and L' and such that

$$\xi(x, x') = \big(\xi(x), \xi(x')\big)$$

for $(x, x') \in (L \times L')_n$ where n is ever or $p = 1$.

6.5. PROPOSITION. *If L and L' are restricted Lie algebras over the ring K of characteristic p, then*

$$V(L \times L') = V(L) \otimes V(L') .$$

The proof is just as the proof of 5.6.

6.6. DEFINITIONS AND COMMENTS. If L is a restricted Lie algebra we have a natural morphism $\Delta: L \to L \times L$, where $\Delta(x) = (x, x)$ for $x \in L_n$. This induces $V(\Delta)$: $V(L) \to V(L) \otimes V(L)$ and $V(L)$ becomes a primitively generated Hopf algebra.

6.7. PROPOSITION. *If A is a Hopf algebra over the ring K of characteristic p, then $P(A)$ is a sub restricted Lie algebra of the associated restricted Lie algebra of A.*

The proof of the proposition is immediate from the definition.

We now have two functors V: $(\mathcal{RL})_K \to (\mathcal{PH})_K$ and P: $\mathcal{H}_K \to (\mathcal{RL})_K$ where $(\mathcal{RL})_K$ is the category of restricted Lie algebras.

6.8. PROPOSITION. *The functors V: $(\mathcal{RL})_K \to (\mathcal{PH}_K)$ and P: $(\mathcal{CH})_K \to (\mathcal{RL})_K$ commute with products and direct limits.*

The proof of the proposition is immediate from the definition.

6.9. DEFINITION. If L is restricted Lie algebra over the ring K of characteristic p, the *Lie filtration* on $V(L)$ is the filtration induced by $U(L) \to V(L)$, i.e., $F_q(V(L)) = \operatorname{Im}\big(F_q U(L) \to V(L)\big)$.

6.10. PROPOSITION. *If K is a field of characteristic $p \neq 0$, and L is a restricted Lie algebra over K, then the primitive and the Lie filtrations on $V(L)$ coincide.*

The proof is easy and follows the lines of the proof of 5.17.

6.11. THEOREM. *If K is a field of characteristic $p \neq 0$, and P: $(\mathcal{PH})_K \to (\mathcal{RL})_K$ and V: $(\mathcal{RL})_K \to (\mathcal{PH})_K$ are the natural functors, then*

(1) *the functor PV: $(\mathcal{RL})_K \to (\mathcal{RL})_K$ is the identity functor of $(\mathcal{RL})_K$, and*

(2) *the functor VP: $(\mathcal{PH})_K \to (\mathcal{PH})_K$ is the identity functor of $(\mathcal{PH})_K$.*

If L is a restricted Lie algebra, we may let $L^{\#}$ denote the restricted Lie algebra with Lie multiplication 0 and $\xi = 0$, we may then write $V(L^{\#}) = B(L)$.

6.12. PROPOSITION. *If K is a field of characteristic $p \neq 0$, and L is a restricted Lie algebra over K, then*

$$B(L) = \bigoplus E^0 V(L) .$$

6.13. PROPOSITION. *If K is a field, f: $A \to B$ is a morphism of Hopf algebras which is a monomorphism of the underlying graded vector spaces, and B is primitively generated, then A is primitively generated.*

PROOF. We may suppose $A \subset B$. Let A' be the maximal primitively generated sub Hopf algebra of A, i.e., $A' = U(P(A))$ if the characteristic of K is zero, or $A' = V(P(A))$ if the characteristic of K is $p \neq 0$. Consider A' and B as filtered by their primitive filtrations. Define a filtration of A by letting $F_qA = A \cap F_qB$. Note that $F_1A = K + P(A)$. Now we have $A = \bigcup F_qA$ and $F_qA = 0$ for $q < 0$. Thus the filtration on A is complete, moreover the way we have defined it, the morphism $E^0(A) \to E^0(B)$ is a monomorphism.

Consider E^0A and E^0B as graded by the filtration degree. Then they are connected. Suppose the theorem is true for connected Hopf algebras, so that E^0A is primitively generated. Since $P(E^0(B))_{r,s} = 0$ for $r \neq 1$ it follows that $P(E^0(A))_{r,s} = 0$ for $r \neq 1$. It now follows that $E^0(A') \to E^0(A)$ is a morphism of primitively generated bigraded Hopf algebras, and that $P(E^0(A')) \to P(E^0(A))$ is an isomorphism.

Now a morphism of connected coalgebras is a monomorphism if and only if $P(f)$ is a monomorphism, and a morphism f of connected algebras is an epimorphism if and only if $Q(f)$ is an epimorphism. Thus $E^0(A') \to E^0(A)$ is an isomorphism, $A' \to A$ is an isomorphism, and the result follows.

It remains to prove the theorem for connected A and B. Using Proposition 4.13, it suffices to prove it for A of finite type. Now it follows by an argument similar to the proof of Proposition 4.13 that B is the direct limit of primitively generated sub Hopf algebras of finite type. If A is of finite type, then it is contained in one of these primitively generated Hopf algebras of finite type $B' \subset B$. Hence it suffices to prove the result when A and B are both of finite type and connected. In this case, Proposition 4.20 (applied to the dual Hopf algebra) implies that A (or B) is primitively generated if and only if A^* (or B^*) has an associative commutative multiplication, and $\xi(A^n) = 0$ for $n > 0$ (or $\xi(B^n) = 0$ for $n > 0$), where $\xi(x) = x^p$, $p =$ characteristic of the field K. If f: $A \to B$ is a monomorphism, it follows that f^*: $B^* \to A^*$ is an epimorphism, and thus these properties carry over from B^* to A^*, and the theorem is proved.

6.14. DEFINITIONS. If L is a Lie algebra over K, then a subalgebra L' of L

is a *normal subalgebra* if there exists a Lie algebra L'', and a morphism of Lie algebras $f: L \to L''$ such that L' is the kernel of f.

6.15. PROPOSITION. *If K is a field, B is a primitively generated Hopf algebra over K, and A is a sub Hopf algebra of B, then the following conditions are equivalent*:

(1) *$P(A)$ is a normal sub Lie algebra of $P(B)$,*

(2) *the algebra of A is a normal sub algebra of the algebra of B, and*

(3) *the algebra of A is a left normal sub algebra of the algebra of B.*

PROOF. Suppose $x \in P(A)_m$, $y \in P(B)_n$, and (1) is satisfied, then $xy = [x, y] + (-1)^{pq}xy$ and $[x, y] \in P(A)_{m+n}$. Thus $I(A)B = BI(A)$ since B is primitively generated and (1) implies (2). Certainly (2) implies (3). Suppose (3), and let $C = K \otimes_A B$. Now C is a primitively generated Hopf algebra, and we have that the sequence

$$0 \to P(A) \to P(B) \to P(C)$$

is an exact sequence of Lie algebras which implies (1).

Note that the preceding proposition shows that $K \otimes_A B = C = B \otimes_A K$, and as indicated in § 3 we often write $B//A$ for C.

6.16. PROPOSITION. *If K is a field, B is a primitively generated Hopf algebra, and A is a normal sub Hopf algebra of B, then the sequence of Lie algebras*

$$0 \to P(A) \to P(B) \to P(B//A) \to 0$$

is exact.

PROOF. We have that the sequence $0 \to P(A) \to P(B) \to P(B//A)$ is exact. Let L be the cokernel of $P(A) \to P(B)$, and let $C = U(L)$ if the characteristic of K is zero, or $C = V(L)$ if the characteristic of K is p. Now there is a natural morphism $C \to B//A$ and $0 \to P(C) \to P(B//A)$ is exact and $C \to B//A$ is a monomorphism. Moreover the composition $B \to C \to B//A$ is just the natural morphism $B \to B//A$. Hence $C \to B//A$ is an epimorphism. This implies $C \to B//A$ is an isomorphism, and the desired result follows.

We now proceed to use an old trick to obtain the Poincaré-Birkhoff-Witt theorem for Lie algebras over a field K of characteristic $p \neq 0$.

6.17. DEFINITIONS, RECOLLECTIONS, AND NOTATION. Let $\mathbf{Z}$ denote the ring of rational integers, and let $\mathbf{Z}[[t]]$ denote the ring of formal power series in one variable t over $\mathbf{Z}$. If $f \in \mathbf{Z}[[t]]$, let $f(n)$ denote the coefficient of t^n in f. Now f is a unit in the ring $\mathbf{Z}[[t]]$ if and only if $f(0) = \pm 1$. Indeed let $f^{-1}(0) = f(0)$. Then the formula $f^{-1}(n) = -f(0)^{-1}\left(\sum_{i=1}^{n} f(i)f^{-1}(n-i)\right)$ defines $f^{-1}(n)$ inductively for $n > 0$.

Let I be the ideal in $\mathbf{Z}[[t]]$ consisting of those functions with no constant term. Let $\beta\colon I \to \mathbf{Z}[[t]]$ be defined by $\beta(f)(k)$ is the coefficient of t^k in the power series corresponding to the rational function

$$\prod_{1 \leqq j \leqq k+1} (1 + (-1)^{j+1} t^j)^{(-1)^{j+1} f(j)} .$$

Note that

$$\beta(f + g) = \beta(f)\beta(g),\ \beta(f)(0) = 1,\ \beta(t^{2n}) = (1 - t^{2n})^{-1} ,$$

and

$$\beta(t^{2n+1}) = (1 + t^{2n+1}) .$$

We have that β is a morphism of I into the group G of those units of $\mathbf{Z}[[t]]$ with leading term 1. Suppose now $f \in G$. Define f_n inductively as follows:

$$f_0 \quad = f ,$$

and

$$f_{n+1} = f_n (1 + (-1)^n t^{n+1})^{-f_n(+n1)} .$$

Let $g(0) = 0$, and $g(n) = (-1)^{n-1} f_{n-1}(n)$. Now $\beta(g) = f$, and it follows that $\beta\colon I \to G$ is an isomorphism.

Suppose now that K is a commutative ring and X is a free graded K-module of finite type. The *Euler-Poincaré series of* X is the element $\alpha(X) \in \mathbf{Z}[[t]]$ such that $\alpha(X)(n)$ is the dimension of X_n over K. We have $\alpha(X \oplus Y) = \alpha(X) + \alpha(Y)$, and $\alpha(X \otimes Y) = \alpha(X)\alpha(Y)$.

If $f, g \in \mathbf{Z}[[t]]$, we say that $f \leqq g$ is $f(n) \leqq g(n)$ for all n. Suppose $h\colon X \to Y$ is an epimorphism, then $\alpha(Y) \leqq \alpha(X)$ and, for h to be an isomorphism, it is necessary and sufficient that $\alpha(Y) = \alpha(X)$.

If K is a field, and X is such that $X_0 = 0$, then $\alpha(X) \in I$, and so $\beta\alpha(X)$ is defined. If $p \neq 2$, or $X_{2q+1} = 0$ for all q, then $\beta\alpha(X) = \alpha(A(X))$. Indeed this is the reason for defining β in the way in which we have. The fact that $\beta\alpha(X) = \alpha(A(X))$ follows easily from the identities $\beta(t^{2n}) = (1 - t^{2n})^{-1}$, and $\beta(t^{2n+1}) = (1 + t^{2n+1})$.

We now want to deal with the bigraded case of the preceding. Thus let $\mathbf{Z}[[s, t]]$ be the power series ring over $\mathbf{Z}$ in two variables s and t, and I the ideal of series with 0 constant term. Let G be the group of units with leading term 1, and for a power series f, let $f(m, n)$ be the coefficient of $s^m t^n$. Define $\beta\colon I \to G$ by letting $\beta(f)(m, n)$ be the coefficient of $s^m t^n$ in the power series corresponding to the rational function

$$\prod_{1 \leqq i \leqq m+1} \prod_{1 \leqq j \leqq n+1} (1 + (-1)^{i+j+1} s^i t^j)^{(-1)^{i+j+1} f(i, j)} .$$

The function β has the following properties

(1) $\beta(f+g) = \beta(f)\beta(g)$,

(2) β is an isomorphism,

(3) $\beta(s^m t^n) = (1 + (-1)^{m+n+1} s^m t^n)^{(-1)^{m+n+1}}$.

For bigraded modules over K we consider only two special cases. In case 1 we deal with free bigraded modules X of finite type such that $X_{p,q} = 0$ for $p < 0$. We let $\alpha(X)(m, n)$ be the dimension over K of $X_{m,m+n}$. In case 2 we deal with free bigraded modules X of finite type such that $X_{p,q} = 0$ for $p > 0$, and we let $\alpha(X)(m, n)$ be the dimension over K of $X_{-m,n-m}$.

If $f, g \in \mathbf{Z}[[s, t]]$ we say that $f \leqq g$ if $f(m, n) \leqq g(m, n)$ for all m, n. If $h: X \to Y$ is an epimorphism in either of the cases under consideration then $\alpha(Y) \leqq \alpha(X)$ and, for h to be an isomorphism, it is necessary and sufficient that $\alpha(Y) = \alpha(X)$.

If K is a field, and X is such that $X_{0,0} = 0$, then $\alpha(X) \in I$, $\beta\alpha(X)$ is defined, and $\beta\alpha(X) = \alpha A(X)$ if $p \neq 2$ or $X_{m,n} = 0$ for all $m + n$ odd.

6.18. DEFINITION. If X is a graded module over K, let $L(X)$ denote the sub Lie algebra of $T(X)$ generated by X. The Lie algebra $L(X)$ is called the *free Lie algebra generated by* X.

Note that if K is a field of characteristic 0, then $PT(X) = L(X)$, and we have already used this notation in 5.17. Observe further that it is clear, from the universal properties of $UL(X)$ and $T(X)$, that $UL(X) = T(X)$.

6.19. PROPOSITION. *If K is a field, X is a graded vector space over K, and $U(L(X))$ is filtered by its Lie filtration, then the natural morphism*

$$A\big(L(X)\big) \longrightarrow E^0 U\big(L(X)\big)$$

is an isomorphism.

PROOF. Because of the usual argument concerning direct limits, we may assume that X is of finite type. Suppose the characteristic of K is 2. There is a functor from the category of graded K-modules into itself which doubles dimension; namely, to a graded K-module B, it assigns the graded K-modules $D(B)$ such that $D(B)_{2q} = B_q$, and $D(B)_{2q+1} = 0$. To a morphism $f: B \to B'$, it assigns the morphism $D(f): D(B) \to D(B')$ such that $D(f)_{2q} = f_q$. The functor D has the property that $D(B \otimes B') = D(B) \otimes D(B')$. Moreover $A(D(B)) = DA(B)$ for any graded K-module B, $L(D(B)) = D(L(B))$, and $U(L(D(B))) = DU(L(B))$. Consequently in order to prove the proposition it suffices to assume that if the characteristic of K is 2, then $X_{2q+1} = 0$ for all q.

We now want to make a further reduction. This is done by introducing a special filtration on $L(X)$, and $UL(X) = T(X)$. This filtration is defined by setting $F_q L(X) = 0$ for $q \leqq 0$, and $F_q L(X) = L(X) \cap \bigoplus_{r \leqq q} T_r(X)$ for $q > 0$. The filtration on $T(X)$ is such that $F_q T(X) = \bigoplus_{r \leqq q} T_r(X)$. A filtration is induced on X itself. We let Y be the bigraded vector space $E^0(X)$, and note that $Y_{r,s} = 0$ for $r \neq 1$, and $Y_{1,s} = X_{s+1}$. In particular $Y_{0,0} = 0$. We have $E^0T(X) = T(Y) = U(L(Y))$. Since this filtration is complete, it suffices to prove, in order to prove the proposition, the bigraded version of the proposition dealing with a bigraded vector space Y where $Y_{0,0} = 0$.

We may assume without loss that $Y = K \otimes_{\mathbf{Z}} W$ where W is a free bigraded $\mathbf{Z}$ module. Now if we set $f = \alpha(W)$, we have $f = \alpha(K \otimes_{\mathbf{Z}} W)$ for any field K. Moreover $T(Y) = K \otimes_{\mathbf{Z}} T(W)$, and thus $\alpha(T(Y)) = \alpha(T(W)) = (1 - f)^{-1}$ for any field K. Suppose the characteristic of K is zero. In this case we have $AL(Y) \to \bigoplus E^0UL(Y)$ is an isomorphism by § 5.16. Consequently $\beta(\alpha(L(Y)) = (1 - f)^{-1}$. Suppose we let $g = \alpha(L(W))$, and $h = \alpha(L(Y))$. Since there is a natural epimorphism $K \otimes_{\mathbf{Z}} L(W) \to L(Y)$, it follows that $h \leqq g$. Since this last epimorphism is an isomorphism if the characteristic of K is 0, it follows that $\beta(g) = (1 - f)^{-1}$. Since β is order preserving and $h \leqq g$, it follows that $\beta(h) \leqq (1 - f)^{-1}$. However $\beta(h) = \alpha A(L(Y))$ and the natural morphism $AL(Y) \to \bigoplus E^0UL(Y)$ is an epimorphism. Therefore $\beta(h) \geqq (1 - f)^{-1}$. It follows that $\beta(h) = (1 - f)^{-1}$, $AL(Y) \to \bigoplus E^0UL(Y)$ is an isomorphism, and since β is an isomorphism, that $h = g$, i.e., $K \otimes_{\mathbf{Z}} L(W) \to L(Y)$ is an isomorphism. Thus the proposition is proved.

It is worth remarking that neither the assertion that $K \otimes_{\mathbf{Z}} L(W) \to L(Y)$ is an isomorphism or the assertion that $\alpha(AL(Y)) = \beta\alpha L(Y)$ is true if the characteristic of K is 2, and $W_{1,2q} \neq 0$ for some q.

6.20. THEOREM (Poincaré-Birkhoff-Witt). *If K is a field, L is a Lie algebra over K, and $U(L)$ is filtered by its Lie filtration, then the natural morphism*

$$A(L) \longrightarrow \bigoplus E^0U(L)$$

is an isomorphism.

PROOF. For any Lie algebra L, temporarily let $C(L) = \bigoplus E^0U(L)$. Now choose a graded vector space X so that there is an exact sequence of Lie algebras $0 \to L' \to L(X) \to L \to 0$. We have a commutative diagram

$$\begin{array}{ccc} A(L') & \longrightarrow & C(L') \\ \downarrow & & \downarrow \\ A(L(X)) & \xrightarrow{\approx} & C(L(X)) \end{array}.$$

Since $A(L') \to A(L(X))$ is a monomorphism, it follows that $A(L') \to C(L')$ is a monomorphism. Since the natural morphism $A(L) \to C(L)$ is an epimorphism for any Lie algebra L, it follows that $A(L') \to C(L')$ is an isomorphism. The fact that $U(L) = U(L(X))//U(L')$ implies that $C(L) = C(L(X))//C(L')$. Moreover $A(L) = A(L(X))//A(L')$. Thus $A(L) \to C(L)$ is an isomorphism, and the theorem is proved.

7. Some classical theorems

Convention. In this section quasi Hopf algebra will refer only to those quasi Hopf algebras whose multiplication is associative.

7.1. DEFINITION. Let A be an augmented algebra over K. Define $F_pA = A$ for $p \geqq 0$, and $F_pA = I(A)^{-p}$ for $p < 0$. The filtration $\{F_pA\}$ is called the augmentation filtration of A.

7.2. PROPOSITION. *If A is an augmented algebra over K filtered by its augmentation filtration, then*

(1) *$E^0(A)$ is a bigraded connected algebra over K,*

(2) *$Q(E^0(A))_{p,q} = 0$ for $p \neq -1$,*

(3) *$Q(E^0(A))_{-1,q} = Q(A)_{q-1}$.*

7.3. PROPOSITION. *If A and B are augmented algebras over the field K filtered by their augmentation filtrations, then $E^0(A \otimes B) = E^0(A) \otimes E^0(B)$.*

The proof of the preceding propositions is immediate.

7.4. PROPOSITION. *If A is a quasi Hopf algebra over the field K, and A is filtered by its augmentation filtration, then $E^0(A)$ is a primitively generated bigraded Hopf algebra over K with comultiplication $E^0(\Delta)$ induced by the comultiplication Δ of A.*

Once more the proof is immediate from the definitions.

Using the augmentation filtration, we have associated with any quasi Hopf algebra A over the field K a primitively generated Hopf algebra $\bigoplus E^0(A)$ which is closely related to A. That it is not in general isomorphic with A as an algebra, can be seen by letting A be the Steenrod algebra or the group algebra of an appropriately chosen finite group, e.g., the Sylow p-subgroup of the symmetric group on p^2 letters for p an odd prime. We will see, however, that if A is connected, and has commutative multiplication, then A and $\bigoplus E^0(A)$ are isomorphic with A as algebras if the field K is perfect. This can be viewed as a major part of the content of the theorems of Hopf, Leray, and Borel if one chooses to do so. These theorems will be proved in this section.

7.5. THEOREM (Leray). *If A is a connected commutative quasi Hopf algebra over the field K of characteristic zero and $X = Q(A)$, then if $f: X \to I(A)$ is a morphism of graded vector spaces such that the composition $X \xrightarrow{f} I(A) \xrightarrow{\pi} X$ is the identity morphism of X, where π is the natural morphism, then there is an isomorphism of algebras $A(X) \to A$ induced by f.*

PROOF. Choose f satisfying the required conditions. Let $f\colon A(X) \to A$ denote the morphism of augmented algebras induced by f. Now f induces $E^0(f)\colon E^0(A(X)) \to E^0(A)$, and $E^0(f)$ is a morphism of bigraded connected Hopf algebras. Applying 7.2 and the manner in which f was chosen, we have that $Q(E^0(f))\colon Q(E^0(A(X))) \to Q(E^0(A))$ is an isomorphism. It is essentially the identity morphism of X. Applying 4.17 we have a commutative diagram

$$\begin{array}{ccc} P(E^0(A(X))) & \longrightarrow & P(E^0(A)) \\ \downarrow & & \downarrow \\ Q(E^0(A(X))) & \longrightarrow & Q(E^0(A)) \end{array}$$

such that the morphisms represented by the vertical arrows are monomorphisms. Thus $P(E^0(f))$ is a monomorphism. It results that $E^0(f)$ is an isomorphism. Hence f is an isomorphism and the theorem is proved.

Note that, in the preceding proof, connectedness of A was used to guarantee that the augmentation filtration of A was complete, allowing us to conclude that f was an isomorphism from the fact that $E^0(f)$ was an isomorphism.

The theorem of Hopf is that special case of the preceding where, for some integer n, we have $A_q = 0$ for $q > n$. This guarantees $Q(A)_q = 0$ for q even, and that, as an algebra, A is an exterior algebra generated by elements of odd degree.

7.6. DEFINITIONS AND REMARKS. If A is an algebra over K, a submodule Y of A is a *generating submodule* if the natural morphism $T(Y) \to A$ induced by $Y \to A$ is an epimorphism. A set of generators for A is a graded set X which is a set of generators for a generating submodule. If A is connected, we may by 3.8 always choose a set of generators to be contained in $I(A)$. If K is a field, then we will have that the set of generators has no superfluous elements if its image in $Q(A)$ is a basis for $Q(A)$.

7.7. DEFINITION. If A is an algebra over K and $x \in A_n$ for some n, the *height of* x is the least integer q such that $x^q = 0$; or, if no such integer exists, the height of x is infinity.

7.8. PROPOSITION. *Let A be a connected quasi Hopf algebra over the field K which as an algebra has one generator x of degree n, then A is a Hopf algebra, and*

(1) *if the characteristic of K is zero, the height of x is two for n odd and infinity for n even;*

(2) *if the characteristic of K is p odd, then the height of x is two for n odd and either infinity or a power of p for n even; and*

(3) *if the characteristic of K is two, then the height of x is either in-*

finity or a power of two.

PROOF. We must have $\Delta(x) = x \otimes 1 + 1 \otimes x$. Thus x is primitive, Δ is associative, and A is a Hopf algebra. If for n odd the characteristic of K is zero odd, then $x^2 = 0$ since A is commutative. In the remaining cases consider the formula

$$\Delta(x^q) = \sum_{i+j=q} (i, j) x^i \otimes x^j ,$$

where (i, j) denotes the appropriate binomial coefficient and $x^0 = 1$. If the characteristic of K is zero, we have by induction that $x^q \neq 0$ for all q. If the characteristic of K is p, we must have that if x is of finite height q, then all of the binomial coefficients $(i, q - i)$ such that $0 < i < q$ are zero, and thus q is a power of p.

7.9. PROPOSITION. *If A is a connected quasi Hopf algebra over the field K of characteristic $p \neq 0$, and the multiplication of A is commutative, then if*

(1) *$Q(A)_r = 0$ for $r > n$, and*

(2) *$\xi^f(I(A)) = 0$ for some integer f,*

it follows that $P(A)_r = 0$ for $r > p^{f-1}n$.

PROOF. Suppose first that A has one generator x in degree k. If both p and k are odd, then $x^i = 0$, x is a basis for $I(A)$ and the proposition is trivially true. Otherwise, since $x, x^i, \cdots, x^{p^f-1}$ generate $I(A)$ and $\Delta(x^q) = \sum_{i+j=q}(i, j)x^i \otimes x^j$ where (i, j) denotes the appropriate binomial coefficient, we have that $P(A)$ is generated by $x, \xi(x), \cdots, \xi^{f-1}(x)$, and the proposition is again true.

Suppose now that the assertion is proved for quasi Hopf algebras with q or less generators as an algebra, and that A has $q + 1$ generators. Choosing a set of $q + 1$ generators, let X be one of highest degree and let A' be the sub quasi Hopf algebra generated by the remaining generators. Let $A'' = A//A'$, and note that A'' is a quasi Hopf algebra with one generator satisfying (1) and (2). Modifying 3.12 slightly, since we have not assumed that A has an associative comultiplication, there results an exact sequence $0 \to P(A') \to P(A) \to P(A'')$. By induction the proposition is true for A'. It is true for A'' since A'' has one generator. Thus the proposition is true for finitely generated quasi Hopf algebras. The usual direct limit argument now shows that it is true in general.

7.10. PROPOSITION. *If A is a connected quasi Hopf algebra over the perfect field K, the multiplication of A is commutative, A' is a sub quasi Hopf algebra of A, $A'' = A//A'$, and*

(1) *$0 \to Q(A') \to Q(A) \to Q(A'') \to 0$ is exact,*

(2) *$Q(A)_i = 0$ for $r > n$, and*

(3) *A'' has one generator x in degree n, then A is isomorphic with $A' \otimes A''$ as an algebra.*

PROOF. Let π: $A \to A''$ be the natural morphism. If f: $A'' \to A$ is a morphism of graded vector spaces such that $\pi f = A$, then by 1.7, the composition

$$A' \otimes A'' \xrightarrow{i \otimes f} A \otimes A \xrightarrow{\varphi} A$$

is an isomorphism of left A'-modules. Since the multiplication of A is commutative φ is a morphism of algebras. Thus for the composition to be an isomorphism of algebras, it is necessary and sufficient that f be a morphism of algebras.

In other words, in order to prove the proposition, it suffices to show that there exists $y \in A_n$ such that $\pi(y) = x$ and the height of y is the same as the height of x. Clearly any y such that $\pi(y) = x$ will suffice if n is odd and the characteristic of K is zero or odd; or, if the height of x is infinity. Thus we may suppose without loss that we are in one of the remaining cases, the characteristic of K is p, and the height of x is p^f. Let $B' = \xi^f(A')$ and observe that, since the field K is perfect, B' is the sub Hopf algebra $K\xi^f(A')$ of A'. Let $C' = A'//B'$, $C = A//B'$, $C'' = C//C'$. Now 4.11 (extended to the case of quasi Hopf algebras) implies that the natural morphism $A'' \to C''$ is an isomorphism. We have also that C' is a sub quasi Hopf algebra of C, and $Q(A') = Q(C')$, $Q(A) = Q(C)$. Let $\pi' \colon C \to C''$ be the natural morphism and choose $z \in C_n$ such that $\pi'(z) = x \in C''_n = A''_n$. We have $\Delta(z) = z \otimes 1 + 1 \otimes z + u$ where $u \in I(C') \otimes I(C')$. Thus $\xi^f(u) = 0$ and $\Delta(\xi^f z) = \xi^f z \otimes 1 + 1 \otimes \xi^f z$. This means $\xi^f(z) \in P(C)_{p^f n}$. However, $P(C)_{p^f n} = 0$ since $0 \to P(C') \to P(C) \to P(C'')$ is exact, and both C' and C'' satisfy the hypotheses of 7.9. Thus $\xi^f(z) = 0$.

Let $\alpha \colon A \to C$ be the natural morphism. Choose $w \in A_n$ such that $\alpha(w) = z$. Now $\Delta(w) = w \otimes 1 + 1 \otimes w + v$ where $v \in I(A') \otimes I(A')$. Thus $(A \otimes \alpha)\Delta(\xi^f w) = \xi^f w \otimes 1$ using that $\xi^f z = 0$, and both that $(A \otimes \alpha)v \in I(A') \otimes I(C')$ and that ξ^f is zero in $I(A') \otimes I(C')$. Consequently applying 4.9 (extended to the case of quasi Hopf algebras) we have that $\xi^f w \in B'$. Thus there exists $w_0 \in A'_n$ such that $\xi^f w_0 = \xi^f w$. Letting $y = w - w_0$, we have $\pi(y) = x$ and the height of y is p^f. Hence the proposition is proved.

7.11. THEOREM (Borel). *If A is a connected quasi Hopf algebra over the perfect field K, the multiplication in A is commutative, and the underlying graded vector space of A is of finite type, then as an algebra, A is isomorphic with a tensor product $\bigotimes_{i \in I} A_i$ of Hopf algebras A_i, where A_i is a Hopf algebra with a single generator x_i.*

PROOF. Note that, under the hypotheses of the theorem, $A(n)$ is finitely generated as an algebra. Thus the theorem follows by induction from the preceding proposition for $A(n)$. A simple direct limit argument completes the proof.

7.12. EXAMPLE. Let K be a field of characteristic p which is not perfect. Let B be the Hopf algebra which, as an algebra, is the free commutative algebra generated by two generators x and y of degree n even. Let $k \in K$ be an element which does not have a $p^{r\text{th}}$ root in K, and let A be the sub Hopf algebra generated by $\xi(x) - k\xi(y)$. Finally let $C = B//A$. We have that C is a primitively generated Hopf algebra with two generators x and y having commutative multiplication. However, the algebra of C cannot be decomposed into a tensor product of two algebras with one generator.

7.13. THEOREM. *If B is a connected quasi Hopf algebra with commutative multiplication over the perfect field K, and B is filtered by its augmentation filtration, then B is isomorphic with $\bigoplus E^0(B)$ as an algebra.*

PROOF. Note that for any augmented algebra A, we have $\bigoplus E^0(A)$ is an augmented algebra and $E^0(\bigoplus E^0(A)) = E^0(A)$. Let $\mathfrak{A}$ be the set of pairs (A, f) such that

(1) A is a sub Hopf algebra of B such that the sequence $0 \to Q(A) \to Q(B)$ is exact,

(2) $f\colon \bigoplus E^0(A) \to A$ is a morphism of algebras such that $E^0(f)\colon E^0(A) \to E^0(A)$ is the identity morphism, and

(3) if $Q(f)_n\colon Q(\bigoplus E^0(A))_n \to Q(B)_n$ is not an isomorphism, then $Q(\bigoplus E^0(A))_q = 0$ for $q > n$.

Observe that condition 2 implies that f is an isomorphism of algebras. Further since $K \in \mathfrak{A}$, we have $\mathfrak{A}$ is non-empty.

Order the pairs (A, f) by saying that $(A, f) < (C, g)$ if

(1) $A \subset C$,

(2) if $Q(A)_n \to Q(C)_n$ is not an isomorphism, then $Q(A)_q = 0$ for $q > n$,

(3) the diagram

$$\begin{array}{ccc} \bigoplus E^0(A) & \longrightarrow & \bigoplus E^0(C) \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ A & \longrightarrow & C \end{array}$$

is commutative.

A simple direct limit argument with a linearly ordered subset of $\mathfrak{A}$ shows that $\mathfrak{A}$ has maximal elements. Let (A, f) be such a maximal element, and n the least integer such that $Q(A)_n \neq Q(B)_n$ if such an integer exists. Let $y \in B_n$ be an element whose image in $Q(B)_n$ is not in $Q(A)_n$ and let C be the sub Hopf algebra of B generated by A and y. Applying 7.10 we have that C is isomorphic with $A \otimes C//A$. Now $E^0(C) = E^0(A) \otimes E^0(C//A)$. Moreover $\bigoplus E^0(C//A)$ is isomorphic with $C//A$ since $C//A$ has a single generator as an algebra. Therefore we may choose $g\colon \bigoplus E^0(C) \to C$ so that $(C, g) \in \mathfrak{A}$ and $(A, f) < (C, g)$. This contradicts the maximality of (A, f) and shows that we must have had $A = B$ to start out with. Hence the theorem is proved.

7.14. LEMMA. *If A is a connected primitively generated Hopf algebra with commutative multiplication over the field K of characteristic $p \neq 0$, then*

$$P(K\xi(A)) = K\xi(P(A)) .$$

PROOF. By 6.11 we have that $A = V(P(A))$. Now $K\xi(A) = V(K\xi(P(A)))$, and the result follows.

7.15. PROPOSITION. *If A is a connected primitively generated Hopf algebra over the perfect field K, the multiplication in A is commutative, A' is a sub Hopf algebra of A, $A'' = A//A'$, and*

(1) $0 \to Q(A') \to Q(A) \to Q(A'') \to 0$,

(2) $Q(A)_r = 0$ *for* $r > n$, *and*

(3) A'' *has one generator x in degree n, then A is isomorphic with $A' \otimes A''$ as a Hopf algebra.*

PROOF. In the notation of 7.10 it suffices to prove there exists $y \in A_n$ such that $\pi(y) = x$, height $y =$ height x, and y is primitive. We now follow through the proof of 7.10 observing that we may choose z, w, and w_0 primitive, this last being true since we obtain $\xi^f w \in P(\xi^f A') = \xi^f P(A')$ by the preceding lemma and using the fact that K is perfect.

7.16. THEOREM. *If A is a connected primitively generated Hopf algebra over the perfect field K, the multiplication in A is commutative, and the underlying graded vector space of A is of finite type, then there is an isomorphism of Hopf algebras of A with $\bigotimes_{i \in I} A_i$ where each A_i is a Hopf algebra with a single generator x_i.*

PROOF. One applies 7.15 inductively, and passes to the direct limit.

Note that in 7.11 we have an isomorphism of algebras and that, under the stronger hypotheses of 7.16, we have an isomorphism of Hopf algebras.

7.17. DEFINITION. The algebra A is *strictly commutative* if it is commutative and if, in addition, $x^2 = 0$ for every X of odd degree in A.

7.18. REMARKS. If A is an algebra over a field of characteristic not two, then A is commutative if and only if A is strictly commutative. Over any ring K, the free strictly commutative algebra generated by a graded module X is the quotient of $A(X)$ by the ideal I generated by the elements x^2 where x is an element of odd degree of $A(X)$. If X is concentrated in odd degrees then the free strictly commutative algebra generated by X is called the *Grassmann* or *exterior algebra of* X. It is usually denoted by $E(X)$, and has the property that

$E(X \oplus Y) = E(X) \otimes E(Y)$. The morphism $X \to X \oplus X$ where $x \to (x, x)$ induces a morphism which makes $E(X)$ into a Hopf algebra in a canonical way. If X is projective of finite type, then $E(X)^*$ is easily seen to be the same thing as $E(X^*)$.

If X is concentrated in even degrees, the algebra $A(X)$ is sometimes called the symmetric algebra generated by X. If X is free and generators $\{X_i\}_{i \in I}$ for X are given, it is called also the polynomial algebra generated by $\{X_i\}_{i \in I}$.

7.19. EXAMPLE. Let K be a field of characteristic p which is not perfect. Let B be the Hopf algebra which, as an algebra, is the free commutative algebra generated by an element x of degree 2 and an element y of degree $2p$. Let k be an element of K which does not have a p' th root, and let A be the sub Hopf algebra of B generated by $\xi^2(x) - k\xi(y)$. Letting $C = B//A$ we have that C is a primitively generated Hopf algebra with two generators x and y. However, neither 7.11 or 7.13 is true for C. This example differs from that of 7.12 in that 7.13 is true for the example of 7.12.

7.20. THEOREM (Samelson-Leray). *Let A be a connected Hopf algebra over the field K having strictly commutative multiplication, and such that $Q(A)_n = 0$ for n even, then the natural morphism $E(P(A)) \to A$ induced by $P(A) \to A$ is an isomorphism.*

PROOF. It suffices to prove the theorem assuming that A is of finite type. Now $P(A) \to Q(A)$ is a monomorphism by 4.21 or 4.20 depending on the characteristic of K. Note that in the case of finite characteristic $\xi(I(A)) = 0$ since A is generated by elements of odd degree and is strictly commutative. We thus have that A^* is primitively generated, i.e., $P(A^*) \to Q(A^*)$ is an epimorphism. However $P(A^*)_n = 0$ for n even thus $[x, y] = 0$ for x, y primitive elements of A^*. Since A^* is primitively generated, this means that the multiplication in A^* is commutative. If the characteristic of K is two and x is a primitive element of odd degree, then x^2 is primitive of even degree so $x^2 = 0$. Consequently A^* is strictly commutative and $Q(A^*)_n = 0$ for n even. Applying 4.21 or 4.20 again we have for any field K that $P(A^*) \to Q(A^*)$ is a monomorphism. It results that $P(A) \to Q(A)$ is an isomorphism and the theorem follows.

Note that in the preceding it was important that A be a Hopf algebra and not just a quasi Hopf algebra. The result of the theorem is not velid if associativity of the comultiplication is not assumed.

7.21. PROPOSITION. *If B is a connected Hopf algebra over the field K of characteristic different from two, and both the multiplication and the comultiplication of B are commutative, then B is isomorphic with $A \otimes C$ where A is a Grassman algebra with generators of odd degree, and C is a Hopf algebra which is zero in odd degrees.*

PROOF. Let $L_q = P(B)_q$ for q odd and $L_q = 0$ for q even. Let A be the sub Hopf algebra of B generated by $L(A = E(L))$. Let $C = B//A$. Assuming B is of finite type, we have L^* is the odd degree part of $P(B^*)$. There results a morphism of Hopf algebras $j^*: A^* \to B^*$ whose dual $j: B \to A$ has the property that the composition $A \to B \to A$ is the identity morphism of A. Using j, we form $A\backslash\backslash B$, and observe that the composition $A\backslash\backslash B \to B \to C$ is an isomorphism. Thus $B = A \otimes C$, and the proposition is proved.

8. Morphisms of connected coalgebras into connected algebras

8.1. DEFINITIONS. If A is a connected coalgebra and B is a connected algebra, let $G(A, B)$ denote the set of morphism of modules $f: A \to B$ such that f_0 is the identity morphism of K. If $f, g \in G(A, B)$, let $f*g$ be the composition

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} B \otimes B \xrightarrow{\varphi} B .$$

8.2. PROPOSITION. *If A is a connected coalgebra and B is a connected algebra, then $G(A, B)$ is a group under the operation $*$ with identity*

$$A \xrightarrow{\varepsilon} K \xrightarrow{\eta} B.$$

PROOF. The fact that $G(A, B)$ is a monoid is clear from the definitions. It remains to show that, if $f \in G(A, B)$, there exists $f^{-1} \in G(A, B)$. Suppose f^{-1} is defined in degrees less than n, $x \in A_n$, and $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x'_i \otimes x''_i$. Note we may assume $n > 0$, and that $0 <$ degree $x''_i < n$ for all i. Let $f^{-1}(x) = -x - \sum x'_i f^{-1}(x''_i)$.

8.3. PROPOSITION. *If $f: A' \to A$ is a morphism of connected coalgebras and $g: B \to B'$ is a morphism of connected algebras, there is induced a morphism of groups $G(f, g): G(A, B) \to G(A', B)$ by $G(f, g)\, h = ghf$.*

8.4. DEFINITION. If A is a connected Hopf algebra, the *conjugation* of A is the inverse in $G(A, A)$ of the identity morphism of A. It is denoted by C_A or C.

8.5. PROPOSITION. *If A and B are connected Hopf algebras, then $C_A \otimes C_B: A \otimes B \to A \otimes B$ is the conjugation of $A \otimes B$.*

The proof is straightforward.

8.6. PROPOSITION. *If A is a connected Hopf algebra, then the diagram*

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dd, "C"] & A \otimes A \arrow[d, "T"] \\ & A \otimes A \arrow[d, "C \otimes C"] \\ A \arrow[r, "\Delta"] & A \otimes A \end{tikzcd}$$

is commutative.

PROOF. Since Δ is a morphism of algebras, we have by 8.3 that ΔC is the inverse of Δ. It remains to show that $(C \otimes C)\, T\Delta$ is also an inverse of Δ. Since T is an automorphism of the Hopf algebra $A \otimes A$ we have $T(C \otimes C) = (C \otimes C)T$. Now

$$\begin{aligned}
&\Delta^* \big(T(C \otimes C)\Delta\big) \\
&\quad = (\varphi \otimes \varphi)(A \otimes T \otimes A)A \otimes A \otimes T\big(\Delta \otimes (C \otimes C)\Delta\big)\Delta \\
&\quad = (\varphi \otimes A)(A \otimes T)(A \otimes \varphi \otimes A)(A \otimes A \otimes C \otimes C)(A \otimes \Delta \otimes A)(A \otimes \Delta)\Delta \\
&\quad = (\varphi \otimes A)(A \otimes T)\big(\big(A \otimes \varphi(A \otimes C)\Delta\big) \otimes C\big)(A \otimes \Delta)\Delta \\
&\quad = (\varphi \otimes A)(A \otimes T)(A \otimes \eta\varepsilon \otimes C)(A \otimes \Delta)\Delta\ .
\end{aligned}$$

Observing that $T(\eta\varepsilon \otimes C)\Delta = (C \otimes \eta\varepsilon)\Delta$, the last line above becomes

$$(\varphi \otimes A)(A \otimes C \otimes \eta\varepsilon)(\Delta \otimes A)\Delta = \big(\big(\varphi(A \otimes C)\Delta\big) \otimes \eta\varepsilon\big)\Delta = (\eta\varepsilon \otimes \eta\varepsilon)\Delta = \eta\varepsilon\ ,$$

and the proposition is proved.

Note that the associativity of the comultiplication was used in the preceding.

8.7. PROPOSITION. *If A is a connected Hopf algebra, then the diagram*

$$\begin{tikzcd}
A \otimes A \arrow[r, "\varphi"] \arrow[d, "C \otimes C"] & A \arrow[dd, dashed, "C"] \\
A \otimes A \arrow[d, "T"] & \\
A \otimes A \arrow[r, "\varphi"] & A
\end{tikzcd}$$

is commutative.

The proof of this proposition is just the dual of the proof of 8.6.

8.8. PROPOSITION. *If A is a Hopf algebra with either commutative multiplication or commutative comultiplication, then $C \circ C\colon A \to A$ is the identity morphism of A.*

PROOF. It suffices to show that $C^2 = C \circ C$ is the inverse of C in $G(A, A)$. Now $C^2 * C = \varphi(C \otimes C)(C \otimes A)\Delta = C\varphi T(C \otimes A)\Delta$ by 8.7. If either φ or Δ is commutative, this becomes $C\varphi(C \otimes A)\Delta = C\,\eta\varepsilon = \eta\varepsilon$, and the proposition is proved.

8.9. COMMENTS. If we consider the category $\mathfrak{A}$ of augmented coalgebras with commutative comultiplication over K, it is easily seen to be a category with products. The product of A and B is just $A \otimes B$. This is not the case if we do not restrict ourselves to commutative comultiplication. Moreover K is a point in this category, i.e., given any object A in $\mathfrak{A}$ there is a unique morphism

η: $K \longrightarrow A$ and a unique morphism ε: $A \longrightarrow K$. This means that we can define the notion of monoid in the category. Letting $A \times B$ denote the product in the category of A and B, we have that a monoid in the category is an object A together with a morphism φ: $A \times A \longrightarrow A$ such that the diagrams

$$\begin{tikzcd}
A \times A \times A \arrow[r, "\varphi \times A"] \arrow[d, "A \times \varphi"'] & A \times A \arrow[d, "\varphi"] \\
A \times A \arrow[r, "\varphi"] & A ,
\end{tikzcd}$$

$$\begin{tikzcd}
A = K \times A \arrow[r, "\eta \times A"] \arrow[dr, dashed, "A"'] & A \times A \arrow[d, "\varphi"] \\
 & A ,
\end{tikzcd}$$

$$\begin{tikzcd}
A = A \times K \arrow[r, "A \times \eta"] \arrow[dr, dashed, "A"'] & A \times A \arrow[d, "\varphi"] \\
 & A
\end{tikzcd}$$

are commutative. In other words a monoid in the category $\mathfrak{A}$ is just a Hopf algebra with commutative comultiplication. A group in the category is a monoid in the category together with C: $A \longrightarrow A$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[dr, dashed, "\varepsilon"'] & A \times A \arrow[r, "A \times C"] & A \times A \arrow[r, "\varphi"] & A \\
 & K \arrow[urr, dashed, "\eta"'] & &
\end{tikzcd}$$

is commutative. Thus we have just seen that the connected Hopf algebras with commutative comultiplication are groups in the category $\mathfrak{C}$.

If we let $\mathcal{T}$ denote the category of topological spaces with base point, then if K is a field, there is a natural functor $H_*(\quad, K)\colon \mathcal{T} \to \mathfrak{C}$ which to every space X assigns its singular homology with coefficients in K. The comultiplication $H_*(X, K) \to H_*(X, K) \otimes H_*(X, K)$ is the morphism induced by the diagonal $\Delta\colon X \to X \times X$. Note that if P is a point, $H_*(P, K) = K$. Thus $P \to X$ induces $K \to H_*(X; K)$ and $X \to P$ induces $H_*(X; K) \to K$. The functor $H_*(\quad, K)\colon \mathcal{T} \to \mathfrak{C}$ preserves products, sends monoids into monoids, and groups into groups. Indeed it sends connected H-spaces with homotopy associative multiplication into groups. If G is a group in $\mathcal{T}$ and G operates on X on the left, then $G \times X \to X$ induces $H_*(G; K) \otimes H_*(X, K) \to H_*(X, K)$, and $H_*(X, K)$ becomes a module coalgebra over the Hopf algebra $H_*(G, K)$. We assume the usual associativity conditions etc. for the operation of G on X.

We will conclude this section by showing one more analogy between connected Hopf algebras with commutative comultiplication and groups; namely, that in this case, there is just one notion of normal subalgebra (3.3).

8.10. Proposition. *If B is a connected Hopf algebra with commutative comultiplication, and A is a sub Hopf algebra of B, then the following are equivalent*:

(1) *A is a left normal subalgebra of B,*

(2) *A is a right normal subalgebra of B, and*

(3) *A is a normal subalgebra of B.*

Proof. It suffices to show that, if $I(A)B \subset BI(A)$, then $I(A)B = BI(A)$. Applying the conjugation operation, we have $C(I(A)B) \subset C(BI(A))$. However, $C(I(A)B) = C(B)C(I(A)) = BI(A)$, and $C(BI(A)) = C(I(A))C(B) = I(A)B$ by 8.7 and 8.8, which proves the proposition.

Appendix. The characteristic zero homology of H-spaces

By Corollary 4.18 we have that, if A is a connected Hopf algebra over a field K of characteristic zero, then it is primitively generated if and only if the comultiplication in A is commutative. Applying 5.18, we have that such a Hopf algebra is the universal enveloping algebra of the Lie algebra of its own primitive elements, i.e., $A = U(P(A))$.

Now if G is an H-space with unit and homotopy associative multiplication, then $H_*(G; K)$ is a Hopf algebra. Moreover G is connected if and only if $H_*(G; K)$ is connected. Assuming G connected, we have that $H_*(G; K)$ is primitively generated, since the fact that the diagonal $\Delta\colon G \to G \times G$ is commutative, implies that the comultiplication in $H_*(G; K)$ is commutative. Thus

$H_*(G, K) = U(P(H_*(G; K)))$. In this situation it is possible to give $P(H_*(G; K))$ a more geometric interpretation. We now proceed to indicate how this is done.

If G satisfies the above conditions, and $\pi_q(G)$ stands for the q' th homotopy group of G based at the identity element of G, there is defined a pairing

$$[\ ,\ ]: \pi_p(G) \otimes \pi_q(G) \longrightarrow \pi_{p+q}(G)$$

having the following properties:

(1) if $x \in \pi_p(G)$, and $y \in \pi_q(G)$, then $[x, y] = (-1)^{pq+1}[y, x]$,

(2) if $x \in \pi_p(G)$, $y \in \pi_q(G)$, $z \in \pi_r(G)$, then

$$(-1)^{pr}[x, [y, z]] + (-1)^{pq}[y, [z, x]] + (-1)^{qr}[z, [x, y]] = 0 ,$$

and

(3) if $x \in \pi_p(G)$, $y \in \pi_q(G)$ and $\lambda_n: \pi_n(G) \longrightarrow H_n(G)$ is the Hurewicz morphism, then

$$\lambda_{p+q}[x, y] = \lambda_p(x)\lambda_q(y) - (-1)^{pq}\lambda_q(y)\lambda_p(x) .$$

Consequently K being of characteristic zero, the graded vector space $\pi(G; K)$ such that $\pi_n(G; K) = \pi_n(G) \otimes K$ becomes a Lie algebra over K, and the induced morphism $\lambda: \pi_*(G; K) \longrightarrow H_*(G, K)$ is a morphism of Lie algebras. Details concerning the preceding may be found in several places, e.g., [8] or [9]. The Lie product in $\pi_*(G; K)$ is frequently called the Samelson product.

Now one has that the image of λ is contained in $P(H_*(G; K))$. It is in fact exactly $P(H_*(G; K))$ as has been proved by Cartan and Serre. The proof is most easily carried out by looking at the natural Postnikov system of G [7], and observing that at each stage the fibre is totally non-homologous to zero. This last fact is obtained by observing that each stage of this Postnikov system is a principal fibration with fibre, base, and total space all H-spaces of the type under consideration. It results that the characteristic zero k invariant of the fibration is zero due to its being a primitive element of a degree such that the space of primitive elements of this degree is zero by inductive hypothesis. Combining these facts there results the following theorem.

THEOREM. *If G is a pathwise connected homotopy associative H-space with unit, and $\lambda: \pi(G; K) \longrightarrow H_*(G; K)$ is the Hurewicz morphism of Lie algebras, then the induced morphism $\tilde{\lambda}$: $U(\pi(G; K)) \longrightarrow H_*(G; K)$ is an isomorphism of Hopf algebras.*

Note that if G is a pathwise connected topological group, then the conditions of the theorem obtain.

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