

A DISCONTINUOUS GALERKIN METHOD FOR THE SHALLOW-WATER EQUATIONS WITH BATHYMETRIC TERMS AND DRY AREAS

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Abstract. *We develop a Runge-Kutta Discontinuous Galerkin method to approximate the two-dimensional Shallow-Water Equations with bathymetric terms and dry areas. We introduce a flux modification technique to ensure a well-balanced scheme (i.e. that preserves steady-states at rest) and a slope modification technique to deal with dry areas. Numerical results are presented to illustrate the performance of the proposed method in various test cases.*

1 INTRODUCTION

The study of a three-dimensional flow of water over a surface like in lakes, coasts or rivers is a subject of importance in the scientific community. Among the difficulties encountered in such situations, one of the most important is the computational cost of the study in a three-dimensional setting. A common useful approximation consists of treating the flow as *shallow*. More precisely, if the depth of the flow is small compared to its horizontal dimensions, one can average the original equations in the vertical direction. This approximation named *hydrostatic assumption* leads to a two-dimensional problem : The *Shallow-Water Equations*.

The Shallow-Water Equations have been the subject of extensive research and various methods have been designed for their approximation. In particular, recent work deals with a discretization of these equations by Discontinuous Galerkin methods^{9,10,4}. Indeed, in the more general framework of conservation laws, the development of Discontinuous Galerkin methods has been stimulated by several advantages such as their high order of accuracy and their sharp evaluation of *shocks*^{2,1}. The main difficulties encountered in the approximation of the Shallow-Water Equations are firstly to obtain a discretization of bathymetric terms which preserves *flows at rest* and secondly to treat appropriately dry areas. The goal of this work is to design and analyze a Discontinuous Galerkin

method applied to the two-dimensional Shallow-Water Equations with bathymetric terms and dry areas. The paper is organized as follows. In the second section, we present the two-dimensional Shallow-Water Equations with bathymetric terms and the original Discontinuous Galerkin method^{10,4}. In the third and fourth sections, we introduce a *flux modification* technique to treat adequately bathymetric terms and a *slope modification* technique to deal with dry areas. The last section illustrates the performance of the proposed scheme on various numerical tests.

2 SHALLOW-WATER EQUATIONS / DISCONTINUOUS GALERKIN METHOD

In this section, we introduce the Shallow-Water Equations and their discretization by the discontinuous Galerkin method.

2.1 Shallow-Water Equations

Let $T > 0$ and let Ω be an open polygonal subset of \mathbb{R}^2 . Denote by g the gravity acceleration and by $b : \Omega \rightarrow \mathbb{R}$ a smooth function which represents the bathymetry. The two-dimensional Shallow-Water Equations can be written as follows

$$\begin{cases} \frac{\partial W}{\partial t} + \frac{\partial \mathbb{F}(W)}{\partial x} + \frac{\partial \mathbb{G}(W)}{\partial y} = \mathbb{S}(W, b) \text{ in } \Omega \times]0, T[, \\ \text{Initial and Boundary conditions ,} \end{cases} \quad (1)$$

with

- $W := (\zeta, q_x, q_y)$, where ζ and $q = (q_x, q_y)$ are respectively the depth and the discharge of the flow,
- $\mathbb{F}(W) := (q_x, \frac{q_x^2}{\zeta} + \frac{g}{2} \zeta^2, \frac{q_x q_y}{\zeta})$,
- $\mathbb{G}(W) := (q_y, \frac{q_x q_y}{\zeta}, \frac{q_y^2}{\zeta} + \frac{g}{2} \zeta^2)$,
- $\mathbb{S}(W, b) := (0, -g \zeta \frac{\partial b}{\partial x}, -g \zeta \frac{\partial b}{\partial y})$.

2.2 Runge-Kutta Discontinuous Galerkin method

Let \mathcal{T}_h be a conforming triangulation of Ω . The mesh size is given by $h := \max_{K \in \mathcal{T}_h} h_K$, where h_K is the diameter of an element $K \in \mathcal{T}_h$. For $K \in \mathcal{T}_h$, we denote by $n_K = (n_{K,1}, n_{K,2})$ the outward unit normal vector to K and by $E_h(K)$ the set of edges of K . The space $\mathbb{P}^p(K), p \in \mathbb{N}, K \in \mathcal{T}_h$, is defined as the space of polynomial functions of two variables over K of total degree p at most. The discrete space is then given by

$\mathbb{P}_h^p := \{ v : \Omega \rightarrow \mathbb{R} : v|_K \in \mathbb{P}^p(K), \forall K \in \mathcal{T}_h \}$. Note that no matching condition is imposed on mesh cell interfaces.

Multiplying the first equation of (1) by $v_h \in [\mathbb{P}^p(K)]^3$, integrating over K , $K \in \mathcal{T}_h$, and applying Green's formula, we obtain the following approximation of (1) : Find $W_h := (\zeta_h, q_{h,x}, q_{h,y}) \in C^1([0, T], [\mathbb{P}_h^p]^3)$ such that, $\forall v_h \in [\mathbb{P}^p(K)]^3, \forall K \in \mathcal{T}_h, \forall t \in]0, T[$,

$$\begin{cases} \int_K v_h \frac{\partial W_h}{\partial t} + \int_{\partial K} v_h (\widehat{\mathbb{F}}(W_h), \widehat{\mathbb{G}}(W_h)) \cdot n_K \\ \quad - \int_K \frac{\partial v_h}{\partial x} \mathbb{F}(W_h) + \frac{\partial v_h}{\partial y} \mathbb{G}(W_h) = \int_K v_h \mathbb{S}_h(W_h, b), \\ \text{Initial condition,} \end{cases} \quad (2)$$

where $\mathbb{S}_h(\cdot, \cdot)$ is the discretization of the bathymetric term which will be introduced in the section 3 (see (8)) and $(\widehat{\mathbb{F}}(W_h), \widehat{\mathbb{G}}(W_h)) \cdot n_K$ is the numerical flux. In this work, we use the HLLC numerical flux^{9,10}. Moreover, boundary conditions are enforced weakly through the numerical flux and the use of characteristics⁵.

The discretization of the time derivative in (2) is performed in an explicit way. In order to ensure an equal order of accuracy in space and time, we use an explicit Runge-Kutta scheme of order $(p + 1)$. Writing the semi-discrete equation (2) as $\frac{\partial W_h}{\partial t} = \mathcal{H}_h(W_h)$ and defining the k -th iterate by W_h^k , $k \in \mathbb{N}$, the Runge-Kutta scheme of order $q \in \mathbb{N}$ can be written as follows

$$W_h^{k+1} = W_h^k + \sum_{i=1}^q c_i^q \mathcal{H}_h(W_h^{k,i}), \quad (3)$$

where $\{c_i^q\}_{i \in \{1, \dots, q\}} \in \mathbb{R}^q$ and $\{W_h^{k,i}\}_{i \in \{1, \dots, q\}}$ are sub-iterates³. Moreover, to ensure a stable method, the time step is determined adaptively. To compute the $(k + 1)$ -iterate, $k \in \mathbb{N}$, the time step $(\Delta t)_k$ is computed as

$$(\Delta t)_k := \min \left((\Delta t)_0, \frac{h}{(2p + 1) \max_{\substack{\partial K : K \in \mathcal{T}_h \\ \zeta_h^k > 0}} \left(\left| \frac{q_h^k}{\zeta_h^k} \cdot n_K \right| + \sqrt{g \zeta_h^k} \right)} \right), \quad (4)$$

where $W_h^k = (\zeta_h^k, q_h^k)$.

2.3 Slope limiting

It is well-known that in the context of conservation laws a shock can appear in finite time even if the initial data is smooth. Moreover high-order methods can yield spurious oscillations in the vicinity of a shock. To avoid this situation, slope limiting is necessary. Slope limiting consists of replacing the right hand side of (3) by

$$W_h^k + \sum_{i=1}^q c_i^q \Lambda \Pi(\mathcal{H}_h(W_h^{k,i})), \quad \forall k \in \mathbb{N}, \quad (5)$$

where $\Lambda\Pi(\cdot)$ is an operator which firstly *detects* shocks and then reconstructs the slope of the approximation if necessary. In this work, we use the classical slope limiting process designed for Discontinuous Galerkin methods² with a slightly modified slope limiting function to be less diffusive⁶. Moreover, the slope limiting is applied to the water height (i.e. $\zeta + b$) rather than to the water depth⁴. Let us give details about the criterion used to detect shock¹¹. Let $(\zeta_h, q_h) \in [\mathbb{P}_h^p]^3$ be the approximation. For $K \in \mathcal{T}_h$, we define the subset $E_h^-(K)$ of $E_h(K)$ as the *inflow* edges of K , more precisely

$$E_h^-(K) := \left\{ \sigma \in E_h(K) : \int_{\sigma} q_h \cdot n_{K,\sigma} \leq 0 \right\},$$

where $n_{K,\sigma}$ is the outward unit normal vector to K on σ . Moreover, we introduce

$$\mathcal{I}_{K,\sigma} := \frac{\left| \int_{\sigma} (\zeta_h|_K - \zeta_h|_{K_\sigma}) \right|}{h_K^{(p+1)/2} |\sigma| \|\zeta_h\|_K}, \quad \forall K \in \mathcal{T}_h,$$

where K_σ is the element of \mathcal{T}_h sharing the edge σ with K and $|\sigma|$ denotes the measure of σ . Then, setting

$$\mathcal{I}_K := \sum_{\sigma \in E_h^-(K)} \mathcal{I}_{K,\sigma}, \quad (6)$$

we apply slope limiting on K if $\mathcal{I}_K \geq 1$.

3 TREATMENT OF BATHYMETRIC TERMS : FLUX MODIFICATION TECHNIQUE

A desirable property of a discretization of the Shallow-Water Equations with bathymetric terms is to ensure the preservation of *equilibrium states* and especially of *steady-states at rest*. These states are given by $\zeta + b \equiv C$, where C is a constant, and $q \equiv 0$ over the domain. Since at the discrete level, $\zeta_h \in \mathbb{P}_h^p$, $p \in \mathbb{N}$, it is not possible to have $\zeta_h + b \equiv C$ and $b \notin \mathbb{P}_h^p$. Therefore, the discrete version of the property of preserving equilibrium states is written in terms of a function $b_h \in \mathbb{P}_h^p$ which is an approximation of the bathymetric function b . In this work, we consider b_h to be the L^2 -orthogonal projection of b onto \mathbb{P}_h^p , which verifies

$$\int_K b v_h = \int_K b_h v_h, \quad \forall v_h \in \mathbb{P}_h^p, \quad \forall K \in \mathcal{T}_h.$$

In the case of a steady-state at rest, the scheme defined by (2) can not ensure the preservation of quiescent states leading to the so-called *numerical waves* (the numerical test of subsection 5.1 illustrates the importance to eliminate these numerical waves). Indeed, the preservation of flows at rest requires a compatibility between the numerical flux and the source term. In the framework of Finite Volumes, several methods have been

proposed to satisfy this property leading to the so-called *well-balanced schemes*. In this work, we present a *flux modification* technique inspired from the *hydrostatic reconstruction* developed for a kinetic scheme⁸.

Let us consider $K \in \mathcal{T}_h$ and $\sigma \in E_h(K)$. We define $K_\sigma \in \mathcal{T}_h$ as the element sharing the edge σ with K . In practice, we evaluate the numerical flux on σ in (2) by a quadrature. At each quadrature point, we define the new numerical flux function as follows (the reference to integration points is omitted for shortness) :

$$(\widehat{\mathbb{F}}^*(W_h), \widehat{\mathbb{G}}^*(W_h)) \cdot n_K = (\widehat{\mathbb{F}}(W_h^*), \widehat{\mathbb{G}}(W_h^*)) \cdot n_K + \delta_K(W_h, b_h, n_K), \quad (7)$$

where

- $W_h^* := (\zeta_h^*, q_h^*),$
- $q_h^* := \zeta_h^* q_h / \zeta_h,$
- $\zeta_h^*|_K := \max(\zeta_h|_K - \max(b_h|_{K_\sigma} - b_h|_K, 0), 0),$
- $\zeta_h^*|_{K_\sigma} := \max(\zeta_h|_{K_\sigma} - \max(-b_h|_{K_\sigma} + b_h|_K, 0), 0),$
- $\delta_K(W_h, b_h, n_K) := \left(0, \frac{g}{2} (\zeta_h|_K^2 - \zeta_h^*|_K^2) n_{K,1}, \frac{g}{2} (\zeta_h|_K^2 - \zeta_h^*|_K^2) n_{K,2} \right).$

Furthermore, we define the discrete bathymetric term $\mathbb{S}_h(\cdot, \cdot)$ in (2) as

$$\mathbb{S}_h(W_h, b) = \mathbb{S}(W_h, b_h). \quad (8)$$

The following proposition holds

Proposition 3.1 *The scheme defined by (2) and (8) associated with the flux (7) preserves steady-states at rest, i.e., for $k \in \mathbb{N}$, defining $W_h^k = (\zeta_h^k, q_h^k)$ as the k -th iterate and $W_h^{k+1} = (\zeta_h^{k+1}, q_h^{k+1})$ as the $(k+1)$ -th iterate, we have*

$$\left(\zeta_h^k + b_h \equiv C \text{ and } q_h^k \equiv 0 \right) \Rightarrow \left(\zeta_h^{k+1} + b_h \equiv C \text{ and } q_h^{k+1} \equiv 0 \right).$$

Numerical tests of section 5 show that this flux modification technique preserves the order of accuracy of the Discontinuous Galerkin method.

4 TREATMENT OF DRY AREAS : SLOPE MODIFICATION TECHNIQUE

Many applications of Shallow-Water Equations like dam-break problems or nearshore flows involve dry states. One major problem to treat numerically these dry areas is the following : let (ζ_h, q_h) be the approximate solution (we omit time dependence for shortness). If ζ_h has “small” values, one can obtain a negative water depth during the computation. Besides the non-physical meaning of such values, this poses difficulties in the numerical flux computation since the quantity $\sqrt{g \zeta_h}$ appears in the wave speed. To solve this problem, we propose to modify the slope of ζ_h on the elements of the mesh where ζ_h admits negative values. For the degree $p = 1$, the process splits up as follows :

1. If the average of ζ_h is negative on the element, we set $\zeta_h = q_h := 0$ on the element.
2. If the average of ζ_h is positive, this implies that ζ_h is negative at one or two vertices of the element. We keep the average of ζ_h but modify its slope in such a way that ζ_h vanishes at the vertices in question. Then, the discharge q_h is modified by only setting its value at those vertices to zero. This process preserves the mass on the element but not the average of the discharge.

The numerical tests presented below indicate that this slope modification procedure is stable and accurate. For $p = 0$, the process reduces to setting to zero those states with negative water depth. In the case $p \geq 2$, we restrict the order of the approximation to $p = 1$ in the vicinity of areas where negative values appear. In consequence, at each time step, we have to find the minima of ζ_h over each mesh cell I_n , $n \in \{1, \dots, N\}$, which is easy for $p \leq 2$. For $p > 2$, this can be more expensive but this situation is not considered henceforth.

Remark 4.1 *An interesting property of the HLLC flux is that it keeps ζ_h positive in mean provided a sufficiently small time step. This eliminates the step 1 of the post-process described above.*

Remark 4.2 *In practice, we use fixed positive threshold ε to detect dry areas as in Finite Volume methods.*

Remark 4.3 *Slope limiting is not applied in the vicinity of an element where dry areas occur because the slope modification procedure can activate artificially the shock detector.*

5 NUMERICAL TESTS

For all the tests, we set $g = 9.81ms^{-2}$.

5.1 Steady-state at rest

The aim of this test is to verify that a flow at rest is preserved by the flux modification technique and to underline the necessity to preserve steady-states at rest. We set $T = 1s$ and Ω is a square with a length $L = 1m$. For shortness, the bathymetric function b is not detailed herein. The initial condition is defined as $\zeta_0 + b = 1m$ and $q_0 = 0ms^{-2}$. The degree p of the approximation is equal to 2. Figure 1 clearly shows numerical waves which can occur with the original Discontinuous Galerkin scheme and the preservation of steady-states obtained with the flux modification technique.

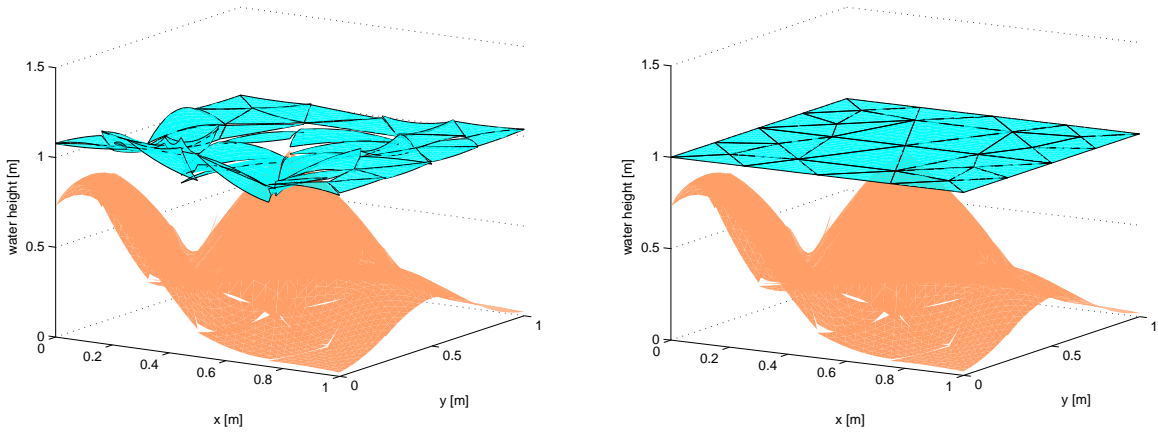


Figure 1: Approximate water height at final time without (left) and with (right) flux modification technique

5.2 Subcritical flow

The goal of this test is to assess the accuracy obtained with the flux modification technique. We set $T = 140s$. The domain Ω is a rectangular channel with length $L = 25m$ and width $l = 5m$. The bathymetric function is defined as $b(x, y) = \max(0.2 - 0.05(x - 10)^2, 0)$. The initial condition are given $\zeta_0 + b = 2m$, $q_{0,x} = 4.42ms^{-2}$ and $q_{0,y} = 0ms^{-2}$. Figure 2 shows the initial and final approximate water heights for $p = 2$. For $p = 0, 1$ and 2 , the convergence rates of the error on the water depth in the L^2 -norm are presented in figure 3. These results confirm that the flux modification technique preserves the original order of accuracy of the method see¹⁰.

5.3 Rarefaction

This test illustrates the performance of the slope modification technique designed to treat dry areas. The final time is $T = 10s$, the domain Ω is a square with length $L = 40m$ and the bathymetry is flat. The initial water depth is defined as $\zeta_0 = 0.102m$ if $x \leq 20m$

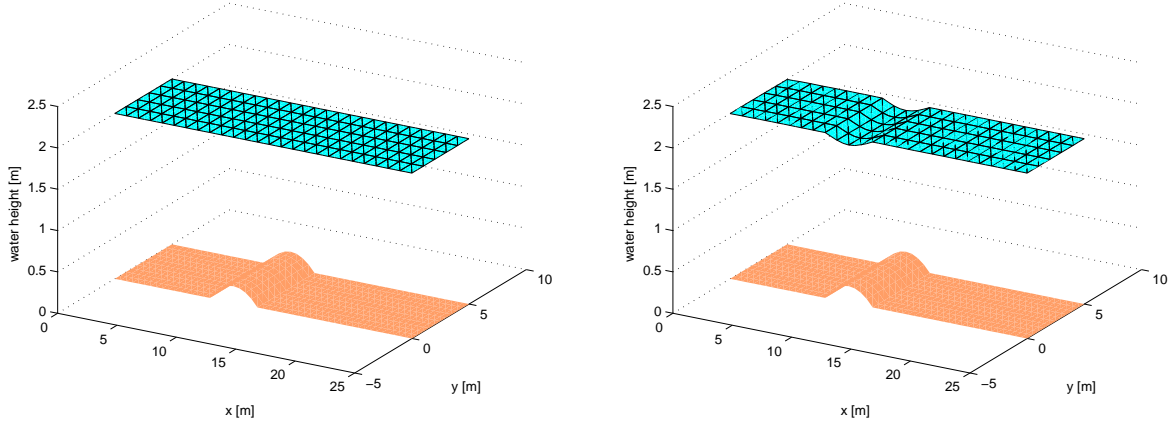
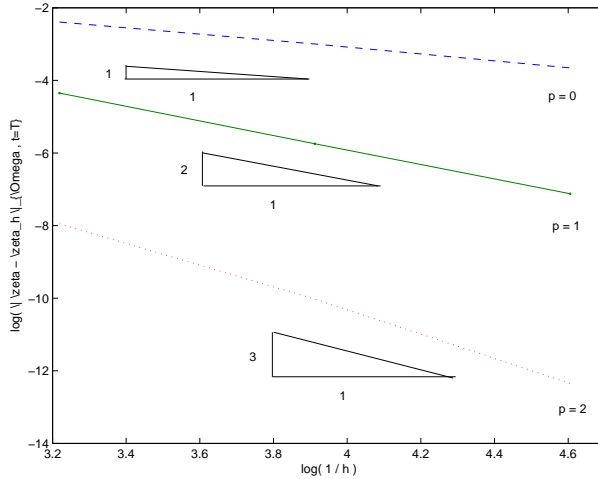


Figure 2: Initial (left) and final (right) approximate water height


 Figure 3: $\log(\|\zeta - \zeta_h\|_{0,\Omega,t=T})$ versus $\log(1/h)$ for $p = 0, 1$ and 2

and $0m$ otherwise. The initial discharge is set to zero. The order of the approximation is $p = 1$. Figure 4 shows the approximate water depth at different times and figure 5 illustrates the convergence rate of the error on the water depth in the L^2 -norm. Owing to the regularity of the solution, the observed order of convergence is 1.

5.4 Flow over a bump

For this test, we set $T = 50s$, the domain Ω is a square with length $L = 100m$ and $b(x, y) = 1.5 \exp^{-2.5e-4((x-50)^2+(y-50)^2)}$. The initial condition are given by $\zeta_0 = \max(1 - b, 0)m$ and $q_{0,x} = q_{0,y} = 0ms^{-2}$. From time $t = 0s$ to time $t = 8s$, we impose an inlet wave at the left of the domain defined by $\zeta_{in}(t) = \zeta_0 + 0.5 \sin(0.125\pi t)$. Figure

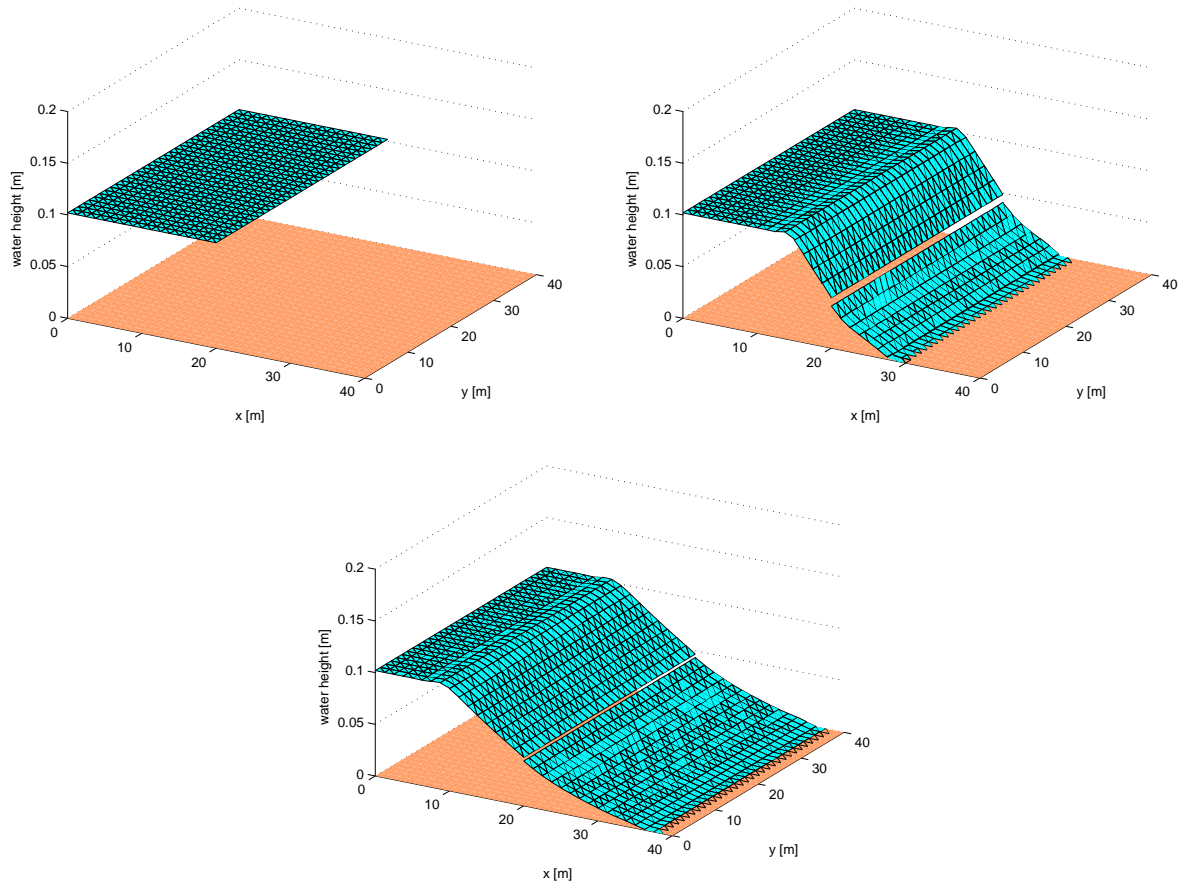


Figure 4: Evolution of the approximate water height

6 represents the approximate water height at different times showing the flooding of the initial dry area. To prevent non physical reflections at inflow and outflow boundaries, we use non-reflecting boundary conditions⁷.

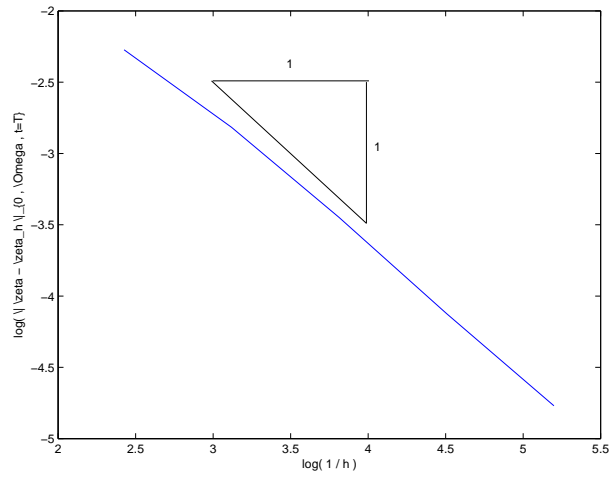


Figure 5: $\log(\|\zeta - \zeta_h\|_{0,\Omega,t=T})$ versus $\log(1/h)$

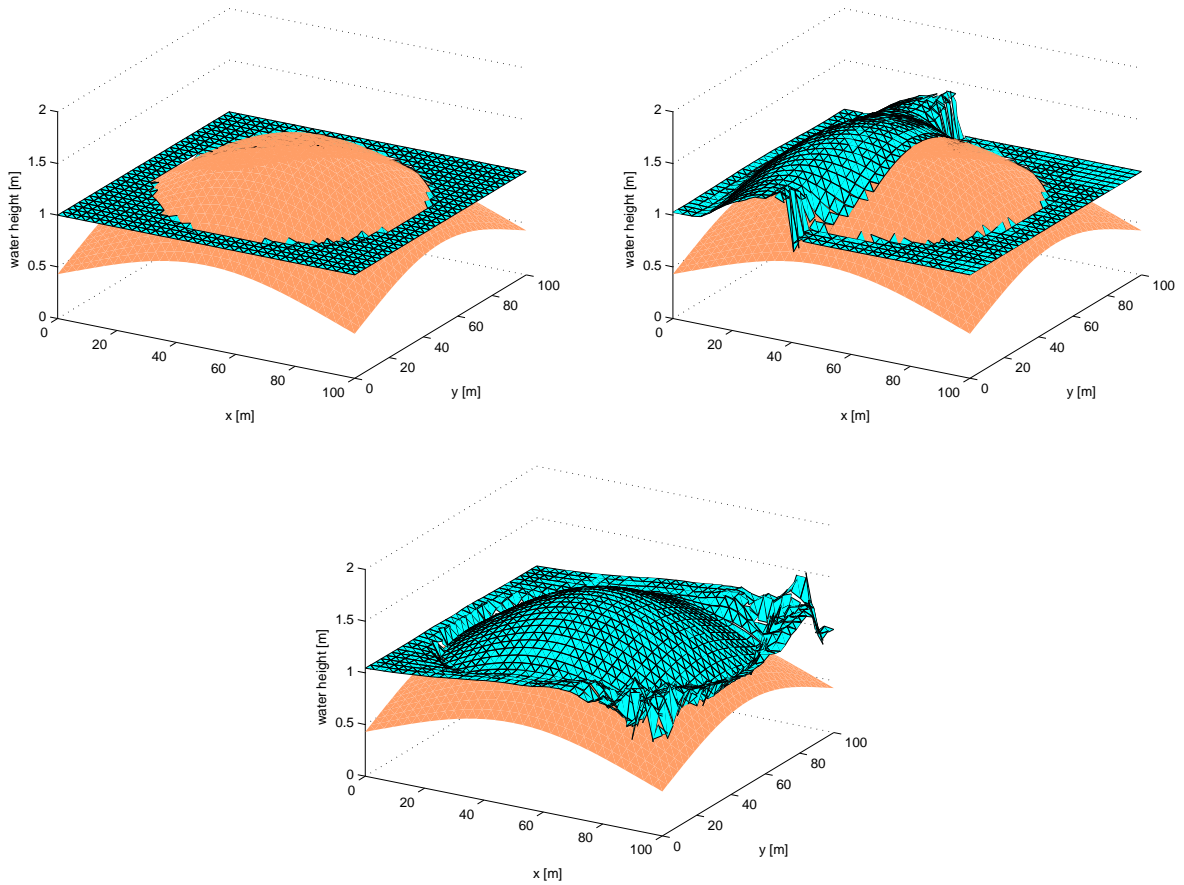


Figure 6: Evolution of the approximate water height

6 CONCLUSIONS

In this work, we have designed a well-balanced Discontinuous Galerkin method dealing with dry areas. To do so, we have introduced a flux modification technique inspired from kinetic solvers and a slope modification technique based on a threshold strategy. Numerous numerical tests have confirmed the accuracy of the scheme and ability to treat dry areas. Further work aims at including friction terms and Coriolis forces into the method.

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