A Toeplitz algorithm for polynomial J-spectral factorization [1]

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Abstract

A block Toeplitz algorithm is proposed to perform the J-spectral factorization of a para-Hermitian polynomial matrix. The input matrix can be singular or indefinite, and it can have zeros along the imaginary axis. The key assumption is that the finite zeros of the input polynomial matrix are given as input data. The algorithm is based on numerically reliable operations only, namely computation of the null-spaces of related block Toeplitz matrices, polynomial matrix factor extraction and linear polynomial matrix equations solving.

Key words: Polynomial matrices, spectral factorization, numerical algorithms, computer-aided control system design.

1 Introduction

1.1 Polynomial J-spectral factorization

In this paper we are interested in solving the following *J-spectral factorization* (JSF) problem for polynomial matrices:

Let $A(s) = A_0 + A_1 s + \cdots + A_d s^d$ be an n-by-n polynomial matrix with real coefficients and degree d in the complex indeterminate s. Assume that $A(s)$ is para-Hermitian, i.e. $A(s) = A^T(-s)$ where T denotes the transpose. We want to find an n-by-n polynomial matrix $P(s)$ and a constant matrix J such that

$$A(s) = P^T(-s)JP(s), \quad J = \text{diag}\{I_{n_+}, -I_{n_-}, 0_{n_0}\}. \quad (1)$$

$P(s)$ is non-singular and its spectrum [3] lies within the left half-plane. $J = J^T$ is a signature matrix with $+1$, -1 and 0 along the diagonal, and such that $n_+ + n_- + n_0 = n$.

The J-spectral factorization of para-Hermitian polynomial matrices has important applications in control and systems theory, as described first in [Wiener, 1949]. See e.g. [Kwakernaak and Šebek, 1994] and [Grimble and Kučera, 1996] for comprehensive descriptions of applications in multivariable Wiener filtering, LQG control and H_∞ optimization.

1.1.1 Factorization of singular matrices

In its most general form, the JSF problem applies to singular matrices. In this paragraph, we show that the problem of factorizing a singular matrix can be converted into the problem of factorizing a non-singular matrix. Alternatively, we can also seek a non-square spectral factor $P(s)$.

If $A(s)$ has rank $r < n$, then $n_+ + n_- = r$ in the signature matrix J. The null-space of $A(s)$ is symmetric, namely if the columns of $Z(s)$ form a basis of the right null-space of $A(s)$, i.e. $A(s)Z(s) = 0$, then $Z^T(-s)A(s) = 0$. As pointed out in [Šebek, 1990], the factorization of $A(s)$ could start with the extraction of its null-space. There always exists an unimodular matrix

$$U^{-1}(s) = [B(s) \quad Z(s)]$$

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[3] The set of all the zeros (eigenvalues) of a polynomial matrix [Gohberg *et al.*, 1982b].

such that

$$U^{-T}(-s)A(s)U^{-1}(s) = \operatorname{diag}\{\bar{A}(s), 0\} \qquad (2)$$

where the $r\times r$ matrix $\bar{A}(s)$ is non-singular. Now, if $\bar{A}(s)$ is factorized as $\bar{A}(s) = \bar{P}^T(-s)\bar{J}\bar{P}(s)$, then the JSF of $A(s)$ is given by:

$$J = \operatorname{diag}\{\bar{J}, 0\}, \quad P(s) = \operatorname{diag}\{\bar{P}(s), I_{n-r}\}U(s).$$

Equivalently, if we accept $P(s)$ to be non-square, then we can eliminate the zero columns and rows in factorization (1) and obtain

$$A(s) = P^T(-s)JP(s), \quad J = \operatorname{diag}\{I_{n_+}, -I_{n_-}\} \qquad (3)$$

where $P(s)$ has size $r \times n$. The different properties of factorizations (1) and (3) are discussed later in section 3.4.

What is important to notice here is that any singular factorization can be reduced to a non-singular one. Therefore, the theory of non-singular factorizations can be naturally extended to the singular case.

1.2 Existence conditions

Suppose that the full rank n-by-n matrix $A(s)$ admits a factorization $A(s) = P^T(-s)JP(s)$ where constant matrix $J^T = J$ has dimension m. Let $\sigma[A(s)]$ be the spectrum of $A(s)$, then

$$m \geq m_0 = \max_{z\in i\mathbb{R}/\sigma[A(s)]} \mathcal{V}_+[A(z)] + \max_{z\in i\mathbb{R}/\sigma[A(s)]} \mathcal{V}_-[A(z)]$$

where $\mathcal{V}_+$ and $\mathcal{V}_-$ are, respectively, the number of positive and negative eigenvalues of $A(s)$ [Ran and Rodman, 1994]

If $A(s)$ has no constant signature on the imaginary axis, i.e. the difference $\mathcal{V}_+[A(z)] - \mathcal{V}_-[A(z)]$ is not constant for all $z \in i\mathbb{R}/\sigma[A(z)]$, then $n < m_0 \leq 2n$, see the proof of Theorem 3.1 in [Ran and Rodman, 1994].

On the other hand, if $A(s)$ has constant signature, then $m = m_0 = n$, and matrix J can be chosen as the unique square matrix given in (3). In [Ran and Rodman, 1994] it is shown that any para-Hermitian matrix $A(s)$ with constant signature admits a JSF. So, constant signature of $A(s)$ is the basic existence condition for JSF that we will assume in this work.

In [Ran and Zizler, 1997] the authors give necessary and sufficient conditions for a self-adjoint polynomial matrix to have constant signature, see also [Gohberg *et al.*, 1982a]. These results can be extended to para-Hermitian polynomial matrices but this is out of the scope of this paper. Some other works giving necessary and sufficient conditions for the existence of the JSF are [Meinsma, 1995] and [Ran, 2003]. There, the conditions are related to the existence of an stabilizing solution of an associated Riccati equation. A deeper discussion of all the related results on the literature would be very extensive and also out of our objectives.

1.2.1 Canonical factorizations

It is easy to see that if $A(s)$ is para-Hermitian, then the degrees δ_i for $i = 1, 2, \ldots, n$ of the n diagonal entries of $A(s)$ are even numbers. We define the diagonal leading matrix of $A(s)$ as

$$\mathcal{A} = \lim_{|s|\to\infty} = D^{-T}(-s)A(s)D^{-1}(s) \qquad (4)$$

where $D(s) = \operatorname{diag}\{s^{\frac{\delta_1}{2}}, \ldots, s^{\frac{\delta_n}{2}}\}$. We say that $A(s)$ is diagonally reduced if $\mathcal{A}$ exists and is non-singular. The JSF (3) can be defined for both diagonally or non-diagonally reduced matrices. From [Kwakernaak and Šebek, 1994] we say that the JSF of a diagonally reduced matrix $A(s)$ is canonical if $P(s)$ is column reduced [4] with column degrees equal to half the diagonal degrees of $A(s)$, see also [Gohberg and Kaashoek, 1986].

In this paper we extend the concept of canonical factorization to matrices with no assumption on its diagonally reducedness (see section 3.3) or even singular matrices (see section 3.4).

1.3 Current algorithms and contributions

The interest in numerical algorithms for polynomial JSF has increased in the last years. Several different algorithms are now available. As for many problems related to polynomial matrices, these algorithms can be classified in two major approaches: the state-space approach and the polynomial approach.

The state-space methods usually relate the problem of JSF to the stabilizing solution of an algebraic Riccati equation. One of the first contributions was [Tuel, 1968], where an algorithm for the standard sign-definite case ($J = I$) is presented. The evolution of this kind of methods is resumed in [Stefanovski, 2003] and references inside. This paper, based on the results of [Trentelman and Rapisarda, 1999], describes and algorithm that can handle indefinite para-hermitian matrices (with constant signature) with zeros along the imaginary axis. The case of singular matrices is tackled in [Stefanovski, 2004] only for the discrete case.

[4] Let q be the degree of the determinant of polynomial matrix $P(s)$ and k_i the maximum degree of the entries of the ith column of $P(s)$. Then $P(s)$ is column reduced if $q = \sum_{i=1}^{n} k_i$.

The popularity of the state-space methods is related with its good numerical properties. There exist several numerical reliable algorithms to solve the Riccati equation, see for example [Bittanti et al. (Eds.), 1991]. On the negative side, sometimes the *reformulation* of the polynomial problem in terms of the state-space requires elaborated preliminary steps [Kwakernaak and Šebek, 1994] and some concepts and particularities of the polynomial problem are difficult to recover from the new representation. On the other hand, the advantage of polynomial methods is their conceptual simplicity and straightforward application to the polynomial matrix, resulting, in general, in faster algorithms. On the negative side, the polynomial methods are often related with elementary operations over the ring of polynomials, and it is well known that these operations are numerically unstable.

In this paper we follow the polynomial approach to develop a new numerically reliable algorithm for the most general case of JSF. Our algorithm follows the idea of symmetric factor extraction used first by Davis for the standard spectral factorization [Davis, 1963]. Davis' algorithm was improved in [Callier, 1985] and [Kwakernaak and Šebek, 1994] under the assumption that the polynomial matrix is not singular and has no zeros on the imaginary axis. Moreover, the computations are still based on unreliable elementary polynomial operations. Our contributions are twofold:

(1) Generality: Our algorithm can handle the most general case of a possibly singular, indefinite para-Hermitian matrix, with no assumptions on its diagonally reducedness or the locations of its zeros along the imaginary axis. As far as we know, the only polynomial method which can deal with this general case is the diagonalization algorithm of [Kwakernaak and Šebek, 1994]. The latter algorithm is based on iterative elementary polynomial operations and thus, it can be quite sensitive to numerical round-off errors.
(2) Stability: No elementary polynomial operations are needed. Our algorithm is based on numerically reliable operations only, namely computation of the null-spaces of constant block Toeplitz matrices along the lines sketched in [Zúñiga and Henrion, 2004a], factor extraction as described in [Henrion and Šebek, 2000], as well as solving linear polynomial matrix equations.

2 Eigenstructure and factor extraction

In this section we review some theory on polynomial matrices that we use in our algorithm. Formal mathematics of this theory can be found in the literature [Gohberg *et al.*, 1982b]. Here we adopt a practical approach and we present the results in a form convenient for the sequel.

2.1 Eigenstructure of polynomial matrices

The eigenstructure of a polynomial matrix $A(s)$ contains the finite structure, the infinite structure and the null-space structure.

2.1.1 Finite structure

A finite zero of a non-singular polynomial matrix $A(s)$ is a complex number z such that there exists a non-zero complex vector v satisfying $A(z)v = 0$. Vector v is called characteristic vector or eigenvector associated to z.

If z is a finite zero of $A(s)$ with algebraic multiplicity m_A and geometric multiplicity m_G, then there exists a series of integers $k_i > 0$ for $i = 1, 2, \ldots, m_G$ such that $m_A = k_1 + k_2 + \cdots + k_{m_G}$ and a series of eigenvectors $v_{i1}, v_{i2}, \ldots, v_{ik_i}$ for $i = 1, 2, \ldots, m_G$ associated to z such that

$$\begin{bmatrix} \bar{A}_0 & & 0 \\ \bar{A}_1 & \bar{A}_0 & \\ \vdots & & \ddots \\ \bar{A}_{k_i-1} & \cdots & \bar{A}_1 \;\; \bar{A}_0 \end{bmatrix} \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{ik_i} \end{bmatrix} = T_Z[A(s), k_i]V = 0 \quad (5)$$

with $v_{11}, v_{21}, \ldots, v_{m_G1}$ linearly independent and where

$$\bar{A}_j = \frac{1}{j!}\left[\frac{\mathrm{d}^j A(s)}{\mathrm{d}s^j}\right]_{s=z}.$$

Integer k_i is the length of the ith chain of eigenvectors associated to z.

2.1.2 Infinite structure

From [Gohberg *et al.*, 1982b] we can associate the infinite structure of polynomial matrix $A(s)$ with the finite structure at $s = 0$ of the dual matrix

$$A_{\mathrm{dual}}(s) = A_d + A_{d-1}s + \cdots + A_0 s^d.$$

So, if $s = 0$ in $A_{\mathrm{dual}}(s)$ has algebraic multiplicity m_∞ and geometric multiplicity m_G, then there exists a series of integers $k_i > 0$ for $i = 1, 2, \ldots, m_G$ such that $m_\infty = k_1 + k_2 + \cdots + k_{m_G}$ and a series of vectors $v_{i1}, v_{i2}, \ldots, v_{ik_i}$ for $i = 1, 2, \ldots, m_G$ such that

$$\begin{bmatrix} A_d & & 0 \\ A_{d-1} & A_d & \\ \vdots & & \ddots \\ A_{d-k_i+1} & \cdots & A_{d-1} \;\; A_d \end{bmatrix} \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{ik_i} \end{bmatrix} = 0 \quad (6)$$

with $v_{11}, v_{21}, \ldots, v_{m_G1}$ linearly independent.

For our purposes in this paper, we define vectors $v_{i1}, v_{i2}, \ldots, v_{ik_i}$ for $i = 1, 2, \ldots, m_G$ as the eigenvectors at infinity of $A(s)$. Integer k_i is the length of the ith chain of eigenvectors at infinity. We also say that matrix $A(s)$ has m_∞ zeros at infinity. Notice, however, that from these m_∞ zeros only those corresponding to the chains that have more than d eigenvectors will appear as zeros at infinity in the Smith–MacMillan form at infinity of $A(s)$, see e.g. section 6.5.3 in [Kailath, 1980]. To avoid confusions, we use the terminology *infinite Smith zeros* to refer to this subset of zeros.

2.1.3 Null-space structure

A basis of the right null-space of $A(s)$ contains the $n-r$ non-zero polynomial vectors $v(s) = v_0 + v_1 s + v_2 s^2 + \cdots + v_\delta s^\delta$ such that $A(s)v(s) = 0$, or equivalently

$$\begin{bmatrix} A_0 & & & 0 \\ \vdots & A_0 & & \\ A_d & \vdots & \ddots & \\ & A_d & & A_0 \\ & & \ddots & \vdots \\ 0 & & & A_d \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_\delta \end{bmatrix} = T_N[A(s), \delta+1]V = 0. \qquad (7)$$

Similarly, a basis of the left null-space of $A(s)$ contains the $n-r$ vectors $u(s)$ such that $A^T(s)u(s) = 0$.

Let δ_i be the degree of each vector in the basis of the null-space. If the sum of all the degrees δ_i is minimal then we have a minimal basis.

2.2 Polynomial matrix factor extraction

Factor extraction on a polynomial matrix $A(s)$ consists in finding a right factor $R(s)$ containing a desired part of its eigenstructure, for instance a set of finite zeros with their respective chains of eigenvectors, and such that $A(s) = L(s)R(s)$. Left factor $L(s)$ contains the remainder of the eigenstructure of $A(s)$. Nevertheless, it is not always possible to extract in $R(s)$ any arbitrary part of the eigenstructure of $A(s)$. Theoretical conditions are presented in section 7.7 of [Gohberg *et al.*, 1982b]. Here, in order to analyze this problem, and for the effects of our algorithm, we use the following result extracted from section 3.6 in [Vardulakis, 1991].

Lemma 1 *The number of poles of a polynomial matrix $A(s)$ (it has only poles at infinity) is equal to the number k of finite zeros (including multiplicities) plus the number of infinite Smith zeros plus the sum d_r of the degrees of the vectors in a minimal basis of the right null-space of $A(s)$ plus the sum d_l of the degrees of the vectors in a minimal basis of the left null-space of $A(s)$.*

Corollary 2 *Let $A(s)$ be a polynomial matrix of degree d and rank r. Then*

$$rd = k + m_\infty + d_r + d_l$$

PROOF. Proof is direct from Lemma 1 and our definition of zeros at infinity in Section 2.1.2.

Consider a square full-rank polynomial matrix $A(s)$ of dimension n and degree d with a set $\{z_1, z_2, \ldots, z_k\}$ of finite zeros and with m_∞ zeros at infinity. From Corollary 2 it follows that $nd = k + m_\infty$. Suppose that we want to extract an $n \times n$ factor $R(s)$ of degree d_R containing a subset of $\bar{k}$ finite zeros of $A(s)$:

- If $\bar{k} = nd_R$ and $R(s)$ contains only the $\bar{k}$ finite zeros of $A(s)$ then we say that the *factorization is exact.*
- If $\bar{k}$ is not an integer multiple of n, then the exact factorization of a subset of $\bar{k}$ finite zeros of $A(s)$ is not possible. In this case we can see that $R(s)$ contains the $\bar{k}$ finite zeros but also some zeros at infinity.

The condition that the degree $d_R = \bar{k}/n$ of factor $R(s)$ should be an integer is only a necessary condition to have an exact factorization. We also require that equation $L(s)R(s) = A(s)$ can be solved for a polynomial factor $L(s)$. Solvability of the above polynomial matrix equation is related to the fact that $A(s)$ and the extracted factor $R(s)$ have the same Jordan pairs associated to the extracted zeros as explained in Chapter 7 of [Gohberg *et al.*, 1982b].

Example 3 *Consider the matrix*

$$A(s) = \begin{bmatrix} s & -s^2 \\ 1 & 0 \end{bmatrix}$$

which has two finite zeros at $s = 0$ and two zeros at infinity. Suppose that we want to extract exactly a factor $R(s)$ containing the two finite zeros. We can see that the degree of $R(s)$ should be $d_R = 2/2 = 1$. For instance we can propose

$$R(s) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}.$$

Nevertheless there is no solution $L(s)$ for the polynomial equation $L(s)R(s) = A(s)$. As a result, $R(s)$ is not a valid right factor of $A(s)$. One possible factorization is given by

$$R(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^2 \end{bmatrix}, \quad L(s) = \begin{bmatrix} s & -1 \\ 1 & 0 \end{bmatrix}.$$

Notice that $R(s)$ has the two finite zeros at $s = 0$ but also two zeros at infinity, so the factorization is not exact.

By analogy, we say that $A(s)$ can have an exact factorization of its infinite zeros whenever n_∞ is an integer multiple of n. By duality, notice that zeros at the origin are introduced when exact factorization is impossible, see Example 5.

When $A(s)$ has rank $r < n$, from Corollary 2 we can see that $rd = k + m_\infty + d_l + d_r$ where d_l and d_r. So, for $A(s)$ to have an exact factorization of its right null-space, $R(s)$ should have full row-rank equal to r and its degree d_R should verify $rd_R = d_r$. When the factorization of the null-space is not exact, $R(s)$ has also some zeros at infinity, see Examples 4 and 6.

3 The algorithm

As a basic assumption we consider that the finite zeros of polynomial matrix $A(s)$ are given as input data. Finding the finite zeros of a general polynomial matrix in a numerical sound way is a difficult problem in numerical linear algebra. The most widely accepted method consists in applying the QZ algorithm over a related pencil or linearization of $A(s)$ [Moler and Stewart, 1973]. The QZ algorithm is backward stable, nevertheless it can be shown that a small backward error in the coefficients of the pencil can sometimes yield large errors in the coefficients of the polynomial matrix [Tisseur and Meerbergen, 2001]. In [Lemonnier and Van Dooren, 2004] an optimal scaling of the pencil is proposed and it is shown that the zeros of $A(s)$ can be computed with a small backward error in its coefficients.

Once the zeros of $A(s)$ are given, we divide the problem of JSF into two major parts: first the computation of the eigenstructure of $A(s)$, second the extraction of factor $P(s)$.

3.1 Computing the eigenstructure

In [Zúñiga and Henrion, 2004a] we outlined some block Toeplitz algorithms to obtain the eigenstructure of a polynomial matrix $A(s)$. We showed how the infinite structure and the null-space structure of $A(s)$ can be obtained by solving iteratively linear systems (6) and (7) of increasing size. We also showed that if we know the finite zeros of $A(s)$, then solving iteratively systems (5) of increasing size allows to obtain the finite structure.

In order to solve systems (5), (6) or (7) we use reliable numerical linear algebra methods such as the Singular Value Decomposition (SVD), the Column Echelon Form (CEF) or the LQ (dual to QR) factorization. All these methods are numerically stable, see for example [Golub and Van Loan, 1996]. In [Zúñiga and Henrion , 2004b] we also presented a *fast version* of the LQ factorization and the CEF. The fast algorithms are based on the displacement structure theory [Kailath and Sayed, 1999] and can be a good option when the dimensions of systems (5), (6) or (7) are very large. Numerical stability of fast algorithms is more difficult to ensure however.

Preliminary results on the backward error analysis of the algorithms summarized in [Zúñiga and Henrion, 2004a] are presented in [Zúñiga and Henrion, 2005]. There we determine a bound for the backward error produced by the LQ factorization in the coefficients of matrix $A(s)$ when solving (5), (6) or (7). We summarize here these results.

Consider that $A(s)$ has a vector $v(s)$ of degree δ in the basis of its null-space. The computed vector $\hat{v}(s)$, obtained from (7) via the LQ factorization, is the exact null-space vector of the slightly perturbed matrix $A(s) + \Delta(s)$, where matrix coefficients of perturbation polynomial matrix $\Delta(s) = \Delta_0 + \Delta_1 s + \cdots + \Delta_d s^d$ satisfy

$$\|\Delta_i\|_2 \leq O(\epsilon)\|T_N[A(s), \delta + 1]\|_2.$$

where $O(\epsilon)$ is a constant of the order of the machine precision ϵ.

Now consider that $A(s)$ has a zero z and a chain of k associated eigenvectors $\{v_1, v_2, \ldots, v_k\}$. The computed vectors $\{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_k\}$ associated to the computed zero $\hat{z}$, obtained from (5) via the LQ factorization, are the exact vectors associated to the exact zero $\hat{z}$ of the slightly perturbed matrix $A(s) + \Delta(s)$ with

$$\|\Delta_j\|_2 \leq O(\epsilon)\|T_Z[a(|\hat{z}|), k]\|_2 ,$$

and where $a(s) = \|A_d\|s^d + \cdots + \|A_0\|$.

3.2 Extracting a polynomial factor

Consider a square non-singular n-by-n polynomial matrix $A(s)$ with a set of finite zeros $z = \{z_1, z_2, \ldots, z_q\}$. The finite zero z_j for $j = 1, 2, \ldots, q$ has a number m_G of chains of associated eigenvectors, or geometric multiplicity. The ith chain of eigenvectors satisfies (5). For index $i = 1, 2, \ldots, m_G$ define the $(d + 1)n$-by-k_i block Toeplitz matrix

$$V_i = \begin{bmatrix} v_{i1} & v_{i2} & \cdots & v_{i(k_i-1)} & v_{ik_i} \\ v_{i0} & v_{i1} & \cdots & v_{i(k_i-2)} & v_{i(k_i-1)} \\ v_{i(-1)} & v_{i0} & \cdots & v_{i(k_i-3)} & v_{i(k_i-2)} \\ & & \vdots & & \\ v_{i(-d+1)} & v_{i(-d+2)} & \cdots & v_{i(-d+k_i-1)} & v_{i(-d+k_i)} \end{bmatrix}$$

with $v_{it} = 0$ for $t \leq 0$, and build up the matrix $W_j = [V_1 \cdots V_{m_G}]$ called the characteristic matrix of z_j. Now suppose we want to extract a factor $R(s)$ containing the set of zeros z and such that $A(s) = L(s)R(s)$. Since $R(s) = R_0 + R_1 s + \cdots + R_d s^d$ contains the zero z_j and shares the same eigenvectors, equation (5) is also true for $T_Z[R(s), k_i]$. Notice that this equation can be rewritten as

$$\begin{bmatrix} \bar{R}_0 & \bar{R}_1 & \cdots & \bar{R}_d \end{bmatrix} V_i = 0$$

and finally that

$$\begin{bmatrix} \bar{R}_0 & \bar{R}_1 & \cdots & \bar{R}_d \end{bmatrix} = RS =$$

$$\begin{bmatrix} R_0 & \cdots & R_d \end{bmatrix} \begin{bmatrix} I & & & \\ z_j I & I & & \\ \vdots & \vdots & \ddots & \\ z_j^d I & d z_j^{d-1} I & \cdots & I \end{bmatrix}$$

So, since square matrix S is not singular, we can see that the coefficients of $R(s)$ can be obtained from a basis of the left null-space of a constant matrix, namely,

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_d \end{bmatrix} \begin{bmatrix} W_1 & W_2 & \cdots & W_q \end{bmatrix} = RW = 0.$$

To solve this last equation we can use some of the numerical linear algebra methods mentioned above. Notice that matrix W has a left null-space of dimension larger than n, in general. In other words, we have several options for the rows of $R(s)$. We choose the rows in order to have a minimum degree and $R(s)$ column reduced, for more details see [Henrion and Šebek, 2000]. Column reducedness of factor $R(s)$ controls the introduction of infinite Smith zeros. As pointed out in [Callier, 1985], this control is important for the spectral symmetric factor extraction. Nevertheless, notice that, even if $R(s)$ has no infinite Smith zeros, when the factorization is not exact it has some zeros at infinity.

Example 4 *Consider the matrix of Example 2. We want to extract a factor $R(s)$ containing the two finite zeros at $s = 0$. With the algorithms of* [Zúñiga and Henrion, 2004a] *we obtain the two characteristics vectors associated to $s = 0$*

$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then we can construct matrix

$$W = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]$$

such that a basis of its left null-space allows to find factor $R(s)$:

$$\begin{bmatrix} R_0 & R_1 & R_2 \end{bmatrix} W = \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] W = 0.$$

Notice that no pair of rows in the basis of the left null-space of W gives a factor

$$R(s) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}.$$

Choosing rows 1 and 4 we obtain a column reduced factor $R(s)$ which is the same as in Example 2.

Now we naturally extend these results to extract a factor sharing the same null-space as a square singular $n \times n$ polynomial matrix $A(s)$ of rank r and degree d. Suppose that a minimal basis of the null-space of $A(s)$ contains the vectors $v_i(s) = v_{i0} + v_{i1}s + \cdots + v_{id_i}s^{d_i}$ for $i = 1, 2, \ldots, n - r$ such that $A(s)v_i(s) = 0$. Define the $(d + 1)n$-by-$(d_i + d + 1)$ block Toeplitz matrix

$$W_i = \begin{bmatrix} v_{i0} & \cdots & v_{id_i} & & 0 \\ & \ddots & & \ddots & \\ 0 & & v_{i0} & \cdots & v_{id_i} \end{bmatrix},$$

and build up the matrix $W = [W_1 \cdots W_{n-r}]$. Then we can see that a factor $R(s) = R_0 + R_1 s + \cdots + R_d s^d$ sharing the same null-space as $A(s)$ can be obtained by solving the system

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_d \end{bmatrix} W = 0.$$

Example 5 *Consider the matrix*

$$A(s) = \begin{bmatrix} s & 0 & 1 \\ s^2 & 0 & s \\ 2s - s^2 & 0 & 2 - s \end{bmatrix}.$$

We want to extract a factor $R(s)$ sharing the same right null-space as $A(s)$. With the algorithms of [Zúñiga and Henrion, 2004a] *we obtain the vectors*

$$v_1(s) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2(s) = \begin{bmatrix} -0.71 \\ 0 \\ 0.71s \end{bmatrix}$$

generating a basis for the right null-space of $A(s)$. Then we can construct matrix

$$W = \begin{bmatrix} 0 & 0 & 0 & -0.71 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.71 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.71 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.71 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.71 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.71 \end{bmatrix}$$

such that a basis of its left null-space allows to find factor $R(s)$

$$\begin{bmatrix} R_0 & R_1 & R_2 \end{bmatrix} W = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] W = 0.$$

So, the minimum degree and rank r factor is

$$R(s) = \begin{bmatrix} s & 0 & 1 \end{bmatrix}.$$

Notice that $R(s)$ has no zeros at infinity, so the factorization is exact.

Finally we sketch our algorithm for the JSF.

Algorithm `spect`
For an $n \times n$ para-Hermitian polynomial matrix $A(s)$ with constant signature, rank r and degree d, this algorithm computes the JSF $A(s) = P^T(-s)JP(s)$. We consider that the set z of finite zeros of $A(s)$ is given.

(1) If $r < n$, extract an $r \times n$ polynomial factor $R_n(s)$ sharing the same right null-space as $A(s)$. Solve the polynomial equation

$$A(s) = R_n^T(-s)X_1(s)R_n(s)$$

where $X_1(s)$ is a full-rank $r \times r$ polynomial matrix containing only the finite zeros of $A(s)$ and some zeros at infinity. If $r = n$ let $A(s) = X_1(s)$.

(2) Extract a polynomial factor $R_f(s)$ containing the finite left half plane zeros of $A(s)$ and half of its finite zeros on the imaginary axis. Solve the polynomial equation

$$X_1(s) = R_f^T(-s)X_2(s)R_f(s)$$

where $X_2(s)$ is a full-rank unimodular matrix.

(3) Extract from $X_2(s)$ a factor $R_\infty(s)$ containing half of its zeros at infinity. Solve the polynomial equation

$$X_2(s) = R_\infty^T(-s)CR_\infty(s)$$

where C is a constant matrix such that $C^T = C$.

(4) Finally factorize $C = U^T JU$. At the end we obtain the searched spectral factor

$$P(s) = UR_\infty(s)R_f(s)R_n(s).$$

Reliable algorithms to solve polynomial equations of the type $A(s)X(s)B(s) = C(s)$ are based on iteratively solving linear systems of equations, see documentation of functions `axb`, `xab` and `axbc` of the Polynomial Toolbox for Matlab [PolyX Ltd., 1998]. Factorization of matrix C in step 4 is assured by the Schur algorithm [Golub and Van Loan, 1996].

3.3 Canonicity

When $A(s)$ is positive definite, factorization in step 3 is always exact and factor $P(s)$ has always half of the zeros at infinity of $A(s)$, see [Callier, 1985]. If $A(s)$ is indefinite, there are cases where an exact factorization in step 3 is not possible. For instance consider that $X_2(s)$ results in the unimodular matrix of Example 3.3 in [Kwakernaak and Šebek, 1994]. In that case notice that $X_2(s)$ can have a factorization with a non-minimal degree spectral factor [5]. When $A(s)$ is diagonally reduced, i.e. without Smith infinite zeros [Callier, 1985], this non-minimality in the factorization of $X_2(s)$ implies that $P(s)$ is not canonical in the sense of [Kwakernaak and Šebek, 1994]. Nevertheless notice that, even if we cannot conclude about the diagonal reducedness of $A(s)$, the non-minimality in the factorization of $X_2(s)$ can be detected. So, in general we can say that a JSF is not canonical if $P(s)$ has more than half of the zeros at infinity of $A(s)$.

Example 6 *Consider the matrix*

$$A(s) = \begin{bmatrix} 0 & 12 - 10s - 4s^2 + 2s^3 \\ 12 + 10s - 4s^2 - 2s^3 & -16s^2 + 4s^4 \end{bmatrix}$$

[5] In the sense that the degree of $X_2(s)$ is not the double of the degree of $R_\infty(s)$.

which has finite zeros $\{-1, 1, -2, 2, -3, 3\}$ *and two zeros at infinity. First we extract the factor*

$$R_f(s) = \begin{bmatrix} 3+4s+s^2 & 0 \\ 0 & 2+s \end{bmatrix}$$

containing the negative finite zeros of $A(s)$*, and we derive the factorization*

$$A(s) = R_f^T(-s)X_2(s)R_f(s) = R_f^T(-s)\begin{bmatrix} 0 & 2 \\ 2 & -4s^2 \end{bmatrix}R_f(s).$$

Notice that $X_2(s)$ *is unimodular and has 4 zeros at infinity. The only column reduced factor containing 2 zeros at infinity that we can extract from* $X_2(s)$ *is given by* $R_\infty(s) = \mathrm{diag}\{1, s^2\}$*. Notice that* $R_\infty(s)$ *has also two finite zeros at 0, so the factorization is not exact. With Algorithm 3.2 of* [Kwakernaak and Šebek, 1994] [6] *we can factorize* $X_2(s)$ *as* $X_2(s) = R_\infty^T(-s)\mathrm{diag}\{1,-1\}R_\infty(s)$ *with*

$$R_\infty(s) = \begin{bmatrix} 1 & 1-s^2 \\ 1 & -1-s^2 \end{bmatrix}.$$

Finally, the spectral factor $P(s)$ *is given by*

$$P(s) = \begin{bmatrix} 3+4s+s^2 & 2+s-2s^2-s^3 \\ 3+4s+s^2 & -2-s-2s^2-s^3 \end{bmatrix}.$$

Notice that $\mathcal{A}$ *given by (4) does not exist, we cannot conclude about the diagonal reducedness of* $A(s)$*, but we can say that the factorization is not canonical according to our definition. Spectral factor* $P(s)$ *has finite zeros* $\{-1, -2, -3\}$ *but also 3 zeros at infinity, namely, more than half of the zeros at infinity of* $A(s)$*.*

3.4 Singular factorizations

When $A(s)$ is rank deficient, we extract a non-square factor $R_n(s)$ at step 1. Equivalently we can extract a square factor as in (2) with the reliable methods presented in [Henrion and Šebek, 1999]. Differences between both approaches are showed in the following example.

Example 7 *Consider the singular matrix*

$$A(s) = \begin{bmatrix} s^2+s^8 & s+s^7 & s^4 \\ -s-s^7 & -1-s^6 & -s^3 \\ s^4 & s^3 & 1 \end{bmatrix}.$$

[6] Numerical reliability of this algorithm is not guaranteed, however.

Matrix $A(s)$ *has rank 2 and 14 zeros at infinity. A right null-space basis is given by* $v(s) = [-0.71\ 0.71s\ 0]^T$*. A factor* $R_n(s)$ *sharing the same right null-space as* $A(s)$ *is given by*

$$R_n(s) = \begin{bmatrix} 0 & 0 & 1 \\ s & 1 & 0 \end{bmatrix},$$

so we have the factorization $R_n^T(-s)X_2(s)R_n(s) = A(s)$ *with*

$$X_2(s) = \begin{bmatrix} 1 & s^3 \\ -s^3 & -1-s^6 \end{bmatrix}.$$

Matrix $X_2(s)$ *has only zeros at infinity and it can be exactly factorized as* $X_2(s) = R_\infty^T(-s)\mathrm{diag}\{1,-1\}R_\infty(s)$ *with*

$$R_\infty(s) = \begin{bmatrix} 1 & s^3 \\ 0 & 1 \end{bmatrix}.$$

Therefore the non-square spectral factor $P(s)$ *is given by*

$$P(s) = R_\infty(s)R_n(s) = \begin{bmatrix} s^4 & s^3 & 1 \\ s & 1 & 0 \end{bmatrix}.$$

Notice that $\mathcal{A}$ *given by (4) has rank 1, we cannot conclude about the diagonal reducedness of* $A(s)$*, but we can say that the factorization is canonical according to our definition. Spectral factor* $P(s)$ *has 7 zeros at infinity, namely, half of the zeros at infinity of* $A(s)$*. On the other hand, if at step 1 of the algorithm we take a square factor* $R_n(s)$ *such that* $R_n^T(-s)X_2(s)R_n(s) = A(s)$ *with*

$$X_2(s) = \begin{bmatrix} -1 & s^3 & 0 \\ -s^3 & 1+s^6 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$R_n(s) = \begin{bmatrix} s+s^7 & 1+s^6 & s^3 \\ s^4 & s^3 & 1 \\ 1.4+0.71s^6 & 0.71s^5 & 0.71s^2 \end{bmatrix},$$

then square non-singular factor $P(s)$ *is given by*

$$P(s) = \begin{bmatrix} s^4 & s^3 & 1 \\ s & 1 & 0 \\ 1.4+0.71s^6 & 0.71s^5 & 0.71s^2 \end{bmatrix}.$$

In this case notice that the factorization is not canonical according to our definition. Factor $P(s)$ *now has full* *rank, but this implies the introduction of zeros at infinity. $P(s)$ has 18 zeros at infinity, namely, more than half of the zeros at infinity of $A(s)$.*

4 Conclusions

A numerical algorithm for the polynomial matrix J-spectral factorization was presented in this paper. In contrast with almost all the existing algorithms in the polynomial approach [Callier, 1985, Kwakernaak and Šebek, 1994] our algorithm can deal with a possibly singular, indefinite polynomial para-Hermitian matrix with zeros along the imaginary axis. Moreover, no elementary operations over polynomials are needed, and the algorithm is only based on numerically reliable operations:

- computation of the eigenstructure with the block Toeplitz methods described in [Zúñiga and Henrion, 2004a];
- successive factor extractions along the lines described in [Henrion and Šebek, 2000];
- linear polynomial matrix equations solving.

Another approach to solving the J-spectral factorization problem is the state-space approach, where the problem is related with the solution of an algebraic Riccati equation. Maybe the most efficient algorithms with this approach are presented in [Stefanovski, 2003] and [Stefanovski, 2004] for the discrete case. Our algorithm is as general as the algorithm in [Stefanovski, 2003] but we can also handle singular matrices using the null-space factor extraction presented here or the reliable triangularization methods of [Henrion and Šebek, 1999].

Direct comparisons between our algorithm and the algorithm in [Stefanovski, 2003] are difficult to achieve and it is out of the scope of this paper. We claim that both algorithms improve classical methods in their own style. On the one hand, algorithm in [Stefanovski, 2003] avoids some preparatory steps usually necessary in the state-space approach via a minimal state-space dimension. So, it brings improvements in terms of computational effort. On the other hand, our algorithm avoids elementary operations over polynomials which are the basis of standard polynomial methods. So, it brings improvements in terms of numerical stability.

Notice, however, that global backward stability of algorithm `spect` cannot be guaranteed. Consider for example the null-space factor extraction: the null-space computed by algorithms in [Zúñiga and Henrion, 2004a] is used as input of the factor extraction algorithm described here in section 3.2. So, it is the sensitivity of the null-space structure problem that determines how the backward error of the factor extraction algorithm is projected into the coefficients of the analyzed polynomial matrix, and unfortunately this null-space problem is generally ill-posed.

Even if there exist satisfactory results on computing the finite zeros of a polynomial matrix [Lemonnier and Van Dooren, 2004], in practice, a real advantage of the state-space methods over our algorithm is that they do not need to compute these finite zeros beforehand. Nevertheless notice that, by exploiting the information on the eigenstructure of the polynomial matrix, further insight is lend into several cases of J-spectral factorization. Consider for example the non-canonical factorization: we have seen that it is related with a non exact extraction of the infinite structure of the polynomial matrix.

Another advantage of using the polynomial eigenstructure is that our algorithm can be applied almost verbatim to the discrete time case [7]. In fact, a discrete time polynomial matrix can be considered as a continuous time rational matrix. The structure at infinity of the discrete polynomial matrix is mapped into the infinite structure and the structure at $s = 0$ of the associated rational matrix. This fact explains the presence of the spurious zeros [Stefanovski, 2004] but also allows us to handle them naturally. For the discrete time case, factorization in step 3 of the algorithm `spect` presented here, can be achieved in two different ways: (1) by extracting a factor containing the finite zeros at $s = 0$ (also zeros at infinity of the discrete time polynomial matrix) and (2) by extracting a factor containing only zeros at infinity. In the case (1) we recover the result of Algorithm 1 in [Stefanovski, 2004] with the presence of spurious zeros, and in the case (2) we recover the result of Algorithm 2 in [Stefanovski, 2004].

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