

Relaxational behavior of the infinite-range Ising spin-glass in a transverse field

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Abstract

We study the zero-temperature behavior of the infinite-ranged Ising spin glass in a transverse field. Using spin summation techniques and Monte Carlo methods we characterize the zero-temperature quantum transition. Our results are well compatible with a value $\nu = \frac{1}{4}$ for the correlation length exponent, $z = 4$ for the dynamical exponent and an algebraic decay t^{-1} for the imaginary-time correlation function. The zero-temperature relaxation of the energy in the presence of the transverse field shows that the system monotonically reaches the ground state energy due to tunneling processes and displays strong glassy effects.

64.70.Nr, 64.60.Cn

I. INTRODUCTION

The purpose of this work is to present some results concerning the zero-temperature critical and relaxational behavior of the Ising spin glass in the presence of a transverse magnetic field. While classical spin-glasses have been extensively studied during the recent years, the role of the quantum effects in the low-temperature regime are not so well understood. In particular, a large amount of work has been devoted to the study of the one-dimensional case [1] and the mean-field theory [14] [2]. These two limiting cases seem to capture one of the most relevant features associated to the quantum fluctuations, i.e. the presence of tunneling effects at zero-temperature. The effect of the transverse field is to allow the system to jump over the free energy barriers even at zero temperature. In this work we will focus our attention in the study of the zero-temperature critical behavior and on the nature of the relaxational dynamics. We have considered the infinite-range model where some analytical results can be obtained. The infinite-ranged model has been studied in several works. In particular, the phase diagram of the model has been obtained using spin summation techniques [2,17] while Miller and Huse [3] have obtained the imaginary-time correlation function at the zero temperature critical point using a theoretical analysis. On the other hand recent numerical work [4,5] reveals that the Monte Carlo method can yield good estimates of the critical exponents associated to the quantum transition using finite-size scaling techniques.

Our purpose is two fold. First we want to show how the Monte Carlo technique used in [4,5] is a powerful tool in order to determine the critical point and the critical exponents in the mean-field case. This will be done comparing the results obtained using finite-size scaling and numerical spin summation methods. Once the critical field is obtained we will obtain the main critical exponents z and ν and we will study the decay of the imaginary-time correlation function at the critical point. Unfortunately, our results are in disagreement with the theoretical prediction of Miller and Huse [3]. Second, we will consider the role of the quantum fluctuations on the zero-temperature relaxational behavior of the model. While these last results concern the quantum infinite-ranged model we expect that our main

conclusions are valid also in the short-ranged case.

II. THE MODEL

The model we are interested is defined by the Hamiltonian,

$$\mathcal{H} = - \sum_{i < j} J_{ij} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x \quad (1)$$

where the $\{\sigma_i; i = 1, N\}$ are the Pauli spin matrices and Γ is the transverse field. The J_{ij} are Gaussian distributed variables with zero mean and variance $\frac{1}{N}$. For $\Gamma = 0$ the model reduces to the classical Sherrington-Kirkpatrick spin-glass model [6]. It is well known [7] [8] that the ground state energy of the above Hamiltonian can be written as the free energy of a classical model with an extra imaginary-time dimension in the following way,

$$E_g(\Gamma) = - \lim_{\beta \rightarrow \infty} \lim_{M \rightarrow \infty} \frac{\log(Z_{eff})}{N\beta} \quad (2)$$

where

$$Z_{eff} = Tr_{\sigma_i} \exp(-\beta H_{eff}(\Gamma, M, \beta)) = \sum_{\sigma_i = \pm 1} \exp(A \sum_{i < j} \sum_{t=1}^M J_{ij} \sigma_i^t \sigma_j^t + B \sum_{i=1}^N \sum_{t=1}^M \sigma_i^t \sigma_i^{t+1} + C) \quad (3)$$

and the spins σ_i are classical variables which can take the values ± 1 . The parameters A and B are given by,

$$\begin{aligned} A &= \frac{\beta}{M} \\ B &= \frac{1}{2} \log(\coth(\frac{\beta\Gamma}{M})) \\ C &= \frac{MN}{2} \log(\frac{1}{2} \sinh(\frac{2\beta\Gamma}{M})) \end{aligned} \quad (4)$$

In the limit $M \rightarrow \infty$ the parameters A and B are highly anisotropic (the coefficient A goes to zero while B goes to infinity). This makes extremely difficult to perform Monte Carlo numerical tests of the quantum model. It has been recently shown [4,5] that it is better to work with a different Hamiltonian which nevertheless lies in the same universality class. To

this end, we have considered the family of models with parameters $A = \beta_{cl}$, $B = \beta_{cl}^n$, $C = 0$. Within this family of models the parameter β_{cl} has the role of the inverse of a classical temperature (not to be confused with the real temperature) which controls the intensity of the quantum fluctuations. In some sense, this effective classical temperature $\frac{1}{\beta_{cl}}$ plays the role of a transverse field in the *true* model (5). In this way, the new Hamiltonian is a more isotropic one. Also, in case universality holds, we expect the mean-field critical exponents to be independent of the particular model considered. We have concentrated our attention in the previous models with $n = 1$ (model (a)), and $n = 2$ (model (b)) and we have studied them using the Monte Carlo method and spin summation techniques. While our Monte Carlo numerical results are consistent with the universality hypothesis we have discovered that model (a) is still hampered by strong Monte Carlo sampling problems while model (b) gives more confident results.

III. SPIN SUMMATION RESULTS

In order to apply the spin summation techniques we have analitically solved the previous model eq.(3) using the replica trick with general coefficients A and B . The analytical solution of the infinite-range model has been already considered in the literature [14,17] and here we will only remind the results. Applying the replica trick and performing the usual technical steps in the theory of spin glasses i.e. (introducing the order parameters and decoupling the different sites) one gets the effective free energy,

$$F_{cl} = -\frac{\overline{\log(Z_{eff})}}{N\beta_{cl}} = \lim_{n \rightarrow 0} \frac{\overline{Z_{eff}^n}}{Nn\beta_{cl}} = \lim_{n \rightarrow 0} \frac{A[Q, R]}{n\beta_{cl}} \quad (5)$$

where $\overline{(\cdot)}$ stands for average over the disorder and n is an integer which denotes the number of replicas. The saddle-point free energy $A[Q, R]$ reads,

$$A[Q, R] = \frac{A^2}{4} \left(\sum_{\alpha \neq \beta} \sum_{t, t'} (Q_{\alpha\beta}^{tt'})^2 + \sum_{\alpha} \sum_{t \neq t'} (R_{\alpha}^{tt'})^2 \right) - \log F[Q, R] \quad (6)$$

with

$$F[Q, R] = \sum_{\sigma_\alpha^t} \exp[B \sum_{t, \alpha} \sigma_\alpha^t \sigma_\alpha^{t+1} + \frac{A^2}{2} (\sum_{\alpha \neq \beta} \sum_{t, t'} Q_{\alpha\beta}^{tt'} \sigma_\alpha^t \sigma_\beta^{t'} + \sum_{\alpha} \sum_{t \neq t'} R_\alpha^{tt'} \sigma_\alpha^t \sigma_\alpha^{t'})] \quad (7)$$

The indices $\alpha, \beta = 1, \dots, n$ stand for replica indices while the indices $t, t' = 1, \dots, M$ run over the imaginary-time direction with periodic boundary conditions (i.e. $\sigma_\alpha^{M+1} = \sigma_\alpha^1$). The saddle point equations yield the order parameters Q and R ,

$$Q_{\alpha\beta}^{tt'} = \langle \sigma_\alpha^t \sigma_\beta^{t'} \rangle \quad R_\alpha^{tt'} = \langle \sigma_\alpha^t \sigma_\alpha^{t'} \rangle \quad (8)$$

where the thermal averages $\langle \dots \rangle$ are done over the effective partition function defined in eq.(7). To solve the previous equation we impose the static condition (i.e. no dependence on the imaginary-time variables t, t') in the set of parameters Q while the R 's are assumed to be no static but traslationally-time invariant, i.e. depend only on the difference of times $t - t'$. In order to determine the critical value of β_{cl} it is enough to consider replica symmetry. In this case the order parameters assume the form $Q_{\alpha\beta}^{tt'} = q, R_\alpha^{tt'} = R(t - t')$ an the free energy reads,

$$\beta f = \frac{A^2}{4} \sum_{t \neq t'} (R^{tt'})^2 - \frac{A^2 M}{4} (1 - 2q) - \frac{A^2 M}{2} (1 - q) - \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{\frac{1}{2}}} e^{-\frac{x^2}{2}} \log \Theta(x) \quad (9)$$

where the function $\Theta(x)$ is given by,

$$\Theta(x) = \sum_{\sigma^t} \exp(\Xi(x, \sigma)) = \sum_{\sigma^t} \exp\left(B \sum_t \sigma^t \sigma^{t+1} + (A^2 q)^{\frac{1}{2}} x \sum_t \sigma^t + \sum_{t \neq t'} (R_{tt'} - q) \sigma^t \sigma^{t'}\right) \quad (10)$$

and the order parameters q and $R(t - t')$ can be obtained solving the equations,

$$q = \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{\frac{1}{2}}} e^{-\frac{x^2}{2}} \left(\frac{\sum_{\sigma} \sigma^t \exp(\Xi(x, \sigma))}{\Theta(x)} \right)^2 \quad (11)$$

$$R(t - t') = \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{\frac{1}{2}}} e^{-\frac{x^2}{2}} \left(\frac{\sum_{\sigma} \sigma^t \sigma^{t'} \exp(\Xi(x, \sigma))}{\Theta(x)} \right) \quad (12)$$

We have numerically solved the previous non linear equations for the models (a) and (b) at different values of M ranging from 2 to 15. Similarly as done in [2] we have extrapolated the different parameters q and $R(t - t')$ to the $M \rightarrow \infty$ limit. We have found that a second degree polynomial in $\frac{1}{M}$ yields very stable and good results. In case of model (a) we found

a phase transition at $T_c^{(a)} = 2.81 \pm .01$ while for model (b) we obtain $T_c^{(b)} = 2.11 \pm 0.01$. The spin summation method yields the thermodynamic quantities with good precision but is inadequate to obtain the critical exponents at the transition.

IV. MONTE CARLO RESULTS

In order to characterize the quantum critical point we have done Monte Carlo (MC) numerical simulations of class of models (a) and (b). While model (a) displays strong Monte Carlo sampling problems (and needs a lot of computational time) the model (b) yields the critical behavior with modest computational effort. Note that model (a) corresponds the case considered in references [4,5]. In what follows, and otherwise stated, we will present numerical results for model (b). In order to simulate the system described by eq.(3) we consider M coupled systems along the time direction with the same realization of disorder. To increase the speed of the computations we have considered the case of discrete couplings $J_{ij} = \pm \frac{1}{\sqrt{(N)}}$ which yields the same behavior in the large N limit as in the case of a Gaussian distribution of couplings. We have simulated two different replicas $\{\sigma_i^t, \tau_i^t; i = 1, \dots, N; t = 1, \dots, M\}$ of the system eq.(6) with the same disorder realization. The main quantity we are interested in is the spin-spin overlap

$$Q = \frac{1}{NM} \sum_{i=1}^N \sum_{t=1}^M \sigma_i^t \tau_i^t \quad (13)$$

which yields the spin-glass susceptibility,

$$\chi_{SG} = N(\overline{\langle q^2 \rangle} - \overline{\langle q \rangle}^2) \quad (14)$$

Following [4,5] we consider the Binder parameter for different values of N and M . This adimensional parameter measures the Gaussianity of the statistical fluctuations and is defined by,

$$g = \frac{1}{2} \left[3 - \frac{\overline{\langle q^4 \rangle}}{\overline{\langle q^2 \rangle}^2} \right] \quad (15)$$

In the vicinity of the critical point the spin-glass susceptibility eq.(14) and the Binder parameter eq.(15) are expected to scale with the size of the system N and the temporal dimension M in the following way,

$$\chi_{SG} = N^p \hat{\chi}(N(T - T_c)^q, N/M^r) \quad (16)$$

$$g = \hat{g}(N(T - T_c)^q, N/M^r) \quad (17)$$

where $\hat{\chi}, \hat{g}$ are scaling functions and p, q, r are mean-field exponents related to the exponent ν and the dynamical exponent z .¹

Now we face the problem that the finite-size scaling depends on two variables N, M . As noted in [4] the phase transition is signalled by the behavior of the parameter g as a function of N and M . For large values of M the system behaves as a one-dimensional system and for small values of M the system behaves as the classical SK model. Then the Binder parameter (15) is expected to go to zero for large and small values of M . At intermediate values of M a maximum for g is expected. Above the critical temperature the system becomes disordered and the value of g associated to that maximum decreases with N . Below T_c it increases with N since the system tends to order. At the critical point $T = T_c$ the maximum value of g is constant with N . According to eq.(16) the scaling with N of the value of M corresponding to the position of maximum determines the mean-field exponent r . The previous criterium yields the critical temperature with very good precision. We find $T = 2.11 \pm .01$ in agreement with the results that we obtained in the last section. Our results for the spin-glass susceptibility χ and the Binder parameter g are shown in figures 1 and 2 at $T = 2.11$. The values of N we studied cover the range $N = 32 - 160$ with 5000 samples in each case. We have observed that small values of N (in fact, less than $N \simeq 50$) are affected by strong subdominant corrections to the critical behavior. The reason is easy to

¹This exponent z should not to be confused with the dynamical exponent asociated with the critical-time dynamics in classical systems.

understand since in the model we are studying the maximum of g is located at quite small values of M (for instance, at $N = 32$ the position value of M where the g has its maximum is located at a value less than 2 which is certainly very small).

Larger values of N (we only show data for N larger than 64) allow to extract the values of the critical exponents. The exponents p, q, r can be derived as a function of ν and the dynamical exponent z . These are given by, $\nu = \frac{pq}{2}$, $z\nu = \frac{q}{r}$ which yield $\gamma = 2\nu$. The numerical results for g show that the exponent $r = \frac{1}{2}$ fits very well the scaling of the function g at the critical point. The fit of the spin glass susceptibility as a function of the temperature in the region of scaling $M = 0.42N^{\frac{1}{2}}$ is shown in the inset of figure 2 and is quite consistent with $q = \frac{3}{2}, p = \frac{2}{3}$ which yields $\nu = \frac{1}{4}$ and $\gamma = \frac{1}{2}$ as predicted within the Gaussian approximation [13]. Unfortunately it is difficult for us to conclude, from the numerical data, on the exact value of the exponent z . Our best fit reveals $r = \frac{1}{2}, z = 3$ which yields $\beta = \frac{7}{8}$. But it is very plausible that these exponents are an artifact of the aforementioned subdominant finite M corrections. On the light of these considerations we expect that the *canonical* exponents $r = \frac{2}{3}, z = 4$ (which would also yield $\beta = 1$) are the correct ones. These are the values of the exponents used to scale data in figures 1 and 2. To definitely conclude on this point we should explore larger sizes. But this is a very difficult task due to the long-ranged nature of the model we are studying which makes simulations very much time consuming. It is interesting to note that the critical value of g ($g_c = \text{Max}(g(N, M, T_c))$) is close to 0.056 and smaller than the values obtained in two and three dimensions [4,5] as expected. As previously said we have also performed numerical simulations of model (a) which shows a critical value of g of order 0.07 slightly higher than that of model (b). But in this case we have not been able to make the data for g to collapse in a single universal curve. We are suspicious that strong Monte Carlo sampling problems are the reason for such bad results. This is presumably related to the value of B in the critical point which is higher in model (a) than in model (b). This implies stronger anisotropic interactions in the first case.

Recently Miller and Huse have obtained the imaginary-time correlation function at the critical point using a theoretical analysis [3]. Our mean-field exponents are in disagreement

with their results. At the critical point they obtain,

$$C(t) = \langle \sigma_i^0 \sigma_i^t \rangle \sim t^{-\alpha} \quad (18)$$

with the value $\alpha = 2$. Figure 3 is a check of the theoretical prediction by Miller and Huse for the imaginary-time correlation function at the critical temperature $T = 2.11$. Simulations have been done for a large system $N = 2272$, $M = 20$ such that it is in the scaling region where we expect the $g(N, M, T_c)$ takes its maximum value. We have carefully checked that the system is in thermal equilibrium and data has been averaged over 8 samples. The results for the decay of the correlation function eq.(7) yields an exponent $\alpha \simeq 1.2$ consistent with the exponent $\alpha = \frac{\beta}{\nu z}$ which ranges from 1 to 7/6 depending if $z = 3$ or $z = 4$. Note that the decay of the imaginary-time correlation function eq.(18) is quite sensitive to how much close we are to the critical region. Obviously, if we are not precisely in the critical region we expect the system to be slightly more disordered and the correlation function to decay faster. In any case the fitted value 1.2 is an upper limit to the true exponent α which we find natural to be 1 and then $z = 4$. It is not clear to us how the predicted exponent $\alpha = 2$ can fit the numerical data.

V. ZERO-TEMPERATURE RELAXATIONAL DYNAMICS

Once we have characterized the zero temperature quantum transition we want to present some results concerning the real-time dynamical behavior of the quantum model at zero temperature. We face the problem of defining a reasonably real-time dynamics for a quantum system. We have considered the simple possibility that real time Monte Carlo dynamics is an appropriate tool to explore the slow dynamic process in the presence of tunneling effects. In the classical case (zero transverse field) we already know that the relaxation at zero temperature of the system stops whenever it finds a metastable state [16]. Because the dynamics is non ergodic in the classical case (the system cannot jump over energy barriers) then the system cannot reach the ground state energy. When a transverse field is applied

the system can jump over energy barriers allowing for a new type of relaxation. In order to investigate this point we have considered the relaxational dynamics of the *true* quantum model of eq.(3) with the coefficients A, B, C given in eq.(5) at very low temperatures as a function of the transverse field. Concretely we are interested in the behavior of the model for large β in the limit $M \rightarrow \infty$ with $\frac{\beta}{M}$ as much small as possible ².

In this limit the Hamiltonian eq.(3) is strongly anisotropic, the coefficient A goes like $\frac{1}{M}$ while B is much larger and goes like $\log(M)$. The total energy in eq.(3) can be decomposed in two parts plus a configuration independent constant C : $E = AE_J + BE_F + C$ where E_J is the sum of all interaction energies in the different imaginary-time slices and E_F is a nearest-neighbour ferromagnetic interaction between spins in the different imaginary-time slices. Our main quantity of interest is the relaxational behavior of the interaction part E_J as a function of time. We will show that the dynamical evolution of the system is the same if the Monte Carlo time is rescaled by the factor $(\frac{\beta}{M})^2$. This is a natural result since the parameters A and B of the effective Hamiltonian of eq.(7) are only a function of that ratio. Note that in the limit $M \rightarrow \infty$ the relaxation of the energy E_J is extremely slow with time (because the main contribution to the full energy in the Hamiltonian eq.(3) is due to the ferromagnetic term E_F). This clarifies the appropriate regime of parameters β and M in which the zero temperature relaxation of the model is defined. Moreover, depending on the values of β and M one is considering, it unambiguously determines a different region of the real dynamical time which is explored.

We performed two kind of experiments. We have studied zero temperature dynamical relaxations at a fixed transverse field. We have considered the model at different low temperatures and different values of M such that yield nearly the same thermodynamic properties. In figure 4 we show the relaxation of the energy E_J as a function of Monte Carlo time for different values of M and β such that the ratio $\frac{\beta}{M}$ is small. The simu-

²Note that in eq.(2) the limit $\beta \rightarrow \infty$ is performed after the limit $M \rightarrow \infty$

lations were performed for two different sizes $N = 320, 640$ finding the same qualitative results. We studied several different ratios $\frac{\beta}{M}$ ranging from 0.2 to 0.001. The explored temperatures were $T = 0.1, 0.02, 0.01$, deep in the zero-temperature region, and the values of $M = 50, 100, 500, 1000$. Relaxations were studied with a small transverse field $\Gamma = 0.1$ (the critical values of the transverse field is close to 1.6 [2]). In order to make the relaxation curves collapse in a single curve we have rescaled the time by the factor $(\frac{\beta}{M})^2$. It is interesting to observe that the energy E_J decreases to a value close to -0.76 which is the expected value in the classical SK model at zero temperature at first order of replica symmetry breaking [15]. Note also that the energy E_J decreases with time but it can fluctuate and increase due to the tunneling effects in the presence of the transverse field. Indeed we have clearly appreciated this effect especially in the large-time regime.

Another interesting aspect of the quantum model we are considering concerns its glassy properties due to tunneling effects. The transverse field controls the intensity of quantum fluctuations and we expect strong hysteresis effects as the transverse field is varied. This is shown in figure 6 where we plot the relaxation of the energy E_J at three different *cooling-heating* rates as a function of the transverse field Γ ³. The cooling rate is defined by the number of Monte Carlo steps per temperature step ($\Delta T=0.05$ in figure 6). Hysteresis curves for different values of M and β collapse in the same curve once the cooling rate is appropriately scaled by the time factor $(\frac{\beta}{M})^2$. The area enclosed in the hysteresis curves decreases as the cooling-heating rate decreases very similar to what happens in real glasses.

VI. CONCLUSIONS

In this work we have studied the zero temperature behavior of the infinite-range quantum Ising spin glass in a transverse field. In particular we have studied the critical properties at

³In our case the parameter which is varied is the transverse field and not the temperature as in real glasses

the quantum transition point and the relaxational behavior as a function of the transverse field.

Concerning the static properties we have studied an effective model (the so called model (b)) which is expected to be in the same universality class as the original quantum model eq.(3,5). Also this effective model does not present strong Monte Carlo sampling problems and gives enough confident results. Even though our results for model (b) show strong finite M corrections for small sizes, our data is in agreement with the mean-field quantum exponents $\nu = 1/4, \beta = 1, \gamma = 1/2, z = 4$ [13]. Unfortunately we have not been able to corroborate the prediction of Miller and Huse for the imaginary time autocorrelation function eq.(18) where $\alpha = 2$. This is the result expected for a dynamical quantum exponent $z = 2$ which we definitely rule out from the analysis of the data shown in figures 1 and 3. In particular, numerical data shown in figure 3 reveals an exponent of $\alpha \simeq 1.2$ which should be a little bit lower if we are not precisely within the scaling region. The value $\alpha = 1$ seems us the natural exponent compatible with our numerical results. This is an interesting point which deserves further investigation. Unfortunately it is very difficult to go to larger sizes since we would need much more computing time.

Concerning the dynamical properties of the model we have investigated the zero temperature relaxational behavior of the model. We have found that the quantum model eq.(3,5) in the zero temperature limit $\beta \rightarrow \infty$, with $\frac{\beta}{M} \rightarrow 0$ shows a universal relaxational behavior when the Monte Carlo time is rescaled by the factor $(\frac{\beta}{M})^2$. For a low transverse field we have observed that the universal curve for the interaction energy E_J monotonically decreases approximately to the static value predicted in the classical SK model at first order of replica symmetry breaking (obviously there are small corrections due to the finite value of the transverse field). Because the effective model (3,5) mainly depends on the ratio $\frac{\beta}{M}$ we expect that similar conclusions about the dynamical behavior of the infinite-ranged model are also valid in the short-ranged case. We have also observed the glassy features of the model by studying the hysteresis effects as a function of the *cooling-heating* rate variation of the transverse field. The results shown in figure 5 indicate a dynamical behavior of the

model reminiscent of that observed in real glasses. In the presence of a transverse field the system can jump over energy barriers due to tunneling effects. Then, at zero temperature, the system is not constrained to remain forever in a metastable state. It can be instructive to speculate if this jumping of the system over the energy barriers corresponds to some kind of activated processes in classical glassy models. This interesting point deserves further investigation.

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Figure Captions

Fig. 1 Binder parameter $g(N, M)$ in model (b) at $T = T_c = 2.11$ for different sizes $N = 64, 96, 128, 160$ as a function of $M/N^{\frac{2}{3}}$.

Fig. 2 Spin-glass susceptibility $\chi(N, M)/N^{\frac{1}{3}}$ in model (b) at $T = T_c = 2.11$ for different sizes $N = 64, 96, 128, 160$ as a function of $M/N^{\frac{2}{3}}$. The inset shows the $\chi(N, M)/N^{\frac{1}{3}}$ as a function of the temperature for $N = 32, 96, 160$ for values of N, M where the g takes its maximum value.

Fig. 3 imaginary-time correlation function in model (b) for $N = 2272, M = 20$ in the scaling region averaged over 8 samples. The fit is of the form $C(t) = A/t^\alpha + A/(20 - t)^\alpha$ with the best fit parameters $\alpha = 1.2, A = 0.3$.

Fig. 4 Relaxation of the energy E_J with $\Gamma = 0.1$ for different ratios β/M as a function of the rescaled Monte Carlo time $t^* = t(\frac{\beta}{M})^2$.

Fig. 5 Hysteresis cycles of the energy E_J at three different cooling rates r (dotted line $r = 100$, dashed line $r = 10$, continuous line $r = 1$).









