Positivity-preserving rational bi-cubic spline interpolation for 3D positive data

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Abstract
This paper deals with the shape preserving interpolation problem for visualization of 3D positive data. A required display of 3D data looks smooth and pleasant. A rational bi-cubic function involving six shape parameters is presented for this objective which is an extension of piecewise rational function in the form of cubic/quadratic involving three shape parameters. Simple data dependent constraints for shape parameters are derived to conserve the inherited shape feature (positivity) of 3D data. Remaining shape parameters are left free for designer to modify the shape of positive surface as per industrial needs. The interpolant is not only local, $C^1$ but also it is a computationally economical in comparison with existing schemes. Several numerical examples are supplied to support the worth of proposed interpolant.

1. Introduction
Shape preserving interpolation problem for visualization of 3D positive data is one of the basic problem in computer graphics, computer aided geometric design, data visualization and engineering. It also arises frequently in many fields including military, education, art, medicine, advertising, transport, etc. Curve and surface design plays a significant role not only in these fields but also in manufacturing different products such as ship design, car modeling and airplane fuselages and wings.

In many interpolation problems, it is essential that the interpolant conserves some inherited shape features of data like positivity, monotonicity and convexity. The goal of this paper is to conserve the hereditary characteristic (positivity) of 3D data. Positivity-preserving problem occurs in visualizing a physical quantity that cannot be negative which may arise if the data is taken from some scientific, social or business environments. Depreciation of the price of computers in the market is an important example of positive data. Ordinary spline methods usually ignore these characteristics thus exhibiting undesirable inflections or oscillations in resulting curves and surfaces. Due to this reason, many investigations during the past years have been directed towards shape preserving interpolation schemes which are quoted as: A rational cubic, bi-cubic interpolants and rational bi-cubic partially blended function have a common feature in a way that no extra knots are used for shape preservation of positive 2D and 3D data. In contrast, the piecewise cubic Hermite interpolation and piecewise bi-cubic function [5] conserved the shape of data by inserting one or two extra knots in the interval where the interpolants do not conserve the desired shape characteristics of data. Shape preserving interpolants problem...
for visualization of positive data has been solved by using C\(^1\) piecewise rational bi-cubic functions [2,7–8], rational bi-quadratic splines [9] and rational bi-quadratic partially blended functions [10–13] with shape parameters. Simple data dependent conditions were derived for shape parameters to attain the desired shape feature of 3D data using rational splines [2,7–13].

In this paper, a C\(^1\) piecewise rational bi-cubic spline scheme with six shape parameters is developed to handle the problem of constructing a positivity-preserving surface through 3D positive data. This method is a contribution towards the advancement of such results that have been carried out by many authors. The method has many outstanding features like:

1. Abbas et al [2] extended the rational cubic function to rational bi-cubic function (cubic/cubic) in order to conserve a positive surface. This function was proved to be successful to interpolate the positive data for only non-zero partial derivatives in contrast the proposed scheme works for any value of partial derivatives.

2. No need of extra knots in the proposed interpolant. In contrast, the piecewise bi-cubic interpolant [5] achieves the required shape of surface by inserting of extra knots in the subinterval where the interpolant loses positivity of surface.

3. The schemes [7,10] do not allow the designer to refine the positive surface as per consumer’s demand. Whereas, this job is done by introducing free parameters which they are used in the description of rational bi-cubic function (5).

4. Hussain [8] constructed a rational bi-cubic function with eight shape parameters in cubic/cubic form. Data dependent constraints were derived for four shape parameters to conserve the positivity. Whereas, the proposed scheme is computationally economical as compared to scheme [8] because it has bunch of shape parameters in every rectangular patch.

5. The proposed positive interpolant has been demonstrated through several numerical examples and it is found not only local but also produces graphical pleasant results as compared to existing schemes [2,7–12] due to less number of constraints for shape parameters and flexibility bestowed to designer for refinement of surfaces as per consumer’s demand.

6. In [9], the smoothness of bi-quadratic interpolant is C\(^0\) while in this paper it is C\(^1\).

7. In [10], the authors claimed that the rational bi-cubic partially blended functions (coon patches) generated a positive surface but unfortunately the visual models did not depict the positive surfaces due to the coon patches because they conserved the shape of data only on the boundaries of patch not inside the patch. In contrast, the proposed rational bi-cubic interpolant conserves the shape of data everywhere in the domain.

8. The proposed surface scheme is unique in its representation and it works well for both uniform and non-uniform space data. The proposed scheme is equally applicable for the data with derivative or without derivatives while the scheme developed by Casciola et al [13] work if partial derivatives at the knots are known.

This paper is organized as follows: A review of rational cubic spline function [1] with three shape parameters with shape control analysis is discussed in Section 2. The extension of rational cubic function to a rational bi-cubic function for the interpolation of regular 3D positive data is presented in Section 3. The arithmetic mean method for derivative approximation is discussed in the Section 4. Positivity-preserving interpolating rational bi-cubic scheme is constructed in Section 5. Several numerical examples are given in Section 6 to prove the worth of scheme. The concluding remarks are presented to end the paper.

2. Rational cubic spline function

Rational spline models have more authority than polynomial spline models as it can accommodate a much wider range of shapes, moderately simple form, better interpolatory properties, excellent extrapolatory powers, typically smoother, easy to handle computationally, less oscillatory and to model complicated structure with a fairly low degree in both the numerator and denominator. In this section, we rewrite the rational cubic spline function developed by Abbas et al. [1].

Let \( \{(x_i, f_i), i = 0, 1, 2, \ldots, n\} \) be the given set of data points such that \( x_i < x_{i+1}, i = 0, 1, 2, \ldots, n - 1 \). In each subinterval \( I = [x_i, x_{i+1}], i = 0, 1, 2, \ldots, n - 1 \), a piecewise rational cubic function is defined as:

\[
S(x) \equiv S_i(x) = \sum_{j=0}^{3} \alpha_j(1-t)^3+jx^j_i \frac{q_i(t)}{q_i(t)}.
\]

(1)

Let \( S'(x) \) denotes the first ordered derivative with respect to \( x \). The following conditions are imposed on rational cubic function (1) for the smoothness as:

\[
S(x_i) = f_i, \quad S(x_{i+1}) = f_{i+1},
\]

\[
S'(x_i) = d_i, \quad S'(x_{i+1}) = d_{i+1}.
\]

(2)

From (2), the values of unknown coefficients are

\[
\alpha_0 = \alpha_i f_i,
\]

\[
\alpha_1 = f_i(2\alpha_i + \beta_i + \gamma_i) + \alpha_i h_i d_i,
\]

\[
\alpha_2 = f_{i+1}(\alpha_i + 2\beta_i + \gamma_i) - \beta_i h_i d_{i+1},
\]

\[
\alpha_3 = \beta_i f_{i+1},
\]

\[
q_i(t) = \alpha_i (1-t)^2 + (\alpha_i + \beta_i + \gamma_i)t(1-t) + \beta_i t^2.
\]

(3)
where \( h_i = x_{i+1} - x_i \), \( t = \frac{x - x_i}{h_i} \) and \( \alpha, \beta, \gamma \) are the positive shape parameters that are used to control the desired shape of curve and provide the freedom to designer to modify the curve as desired. Let \( d_i \) denote the derivative at knots \( x_i \) to maintain the smoothness of interpolant. Using (3), the piecewise rational cubic function (1) is reformulated as:

\[
S_i(x) = \frac{p_i(t)}{q_i(t)},
\]

with

\[
p_i(t) = \alpha f_i(1-t)^3 + (f_i(2\alpha + \beta_i + \gamma_i) + \alpha h_i d_i) t(1-t)^2 + (f_{i+1}(\alpha_i + 2\beta_i + \gamma_i) - \beta_i h_i d_{i+1}) t^2 (1-t) + \beta_i f_{i+1} t^3
\]

\[
q_i(t) = \alpha_i (1-t)^2 + (\alpha_i + \beta_i + \gamma_i) t(1-t) + \beta_i t^2
\]

It is interesting to note that the piecewise rational cubic function (4) becomes a standard cubic Hermite spline with the values of shape parameters \( \alpha_i = 1, \beta_i = 1 \) and \( \gamma_i = 0 \).

### 3. Shape control analysis

In this section, we illustrate the effect of the shape parameters \( \alpha, \beta, \gamma \) on the shape of a curve by mathematically and graphically. The free parameters \( \alpha, \beta, \gamma \) can be used properly to modify the shape of curve or surface according to the designer’s choice. To observe the demeanor of shape parameters on the rational cubic function (4), the function can be expressed as follows:

\[
S_i(x) = f_i(1-t) + f_{i+1} t + \frac{(d_i - \Delta_i)\alpha_i(1-t) + (\Delta_i - d_{i+1})\beta_i h_i t(1-t)}{\alpha_i(1-t)^2 + (\alpha_i + \beta_i + \gamma_i) t(1-t) + \beta_i t^2}.
\]

The following observation can be obtained from Eq. (4a),

\[
\lim_{\gamma_i \to \infty} S_i(x) = \lim_{\gamma_i \to \infty} \left\{ f_i(1-t) + f_{i+1} t + \frac{(d_i - \Delta_i)\alpha_i(1-t) + (\Delta_i - d_{i+1})\beta_i t h_i t(1-t)}{\alpha_i(1-t)^2 + (\alpha_i + \beta_i + \gamma_i) t(1-t) + \beta_i t^2} \right\} = f_i(1-t) + f_{i+1} t,
\]

\[
\lim_{\alpha_i \to \infty} S_i(x) = \lim_{\alpha_i \to \infty} \left\{ f_i(1-t) + f_{i+1} t + \frac{(d_i - \Delta_i)\alpha_i(1-t) + (\Delta_i - d_{i+1})\beta_i t h_i t(1-t)}{\alpha_i(1-t)^2 + (\alpha_i + \beta_i + \gamma_i) t(1-t) + \beta_i t^2} \right\} = f_i(1-t) + f_{i+1} t + (d_i - \Delta_i) h_i t(1-t),
\]

\[
\lim_{\beta_i \to \infty} S_i(x) = \lim_{\beta_i \to \infty} \left\{ f_i(1-t) + f_{i+1} t + \frac{(d_i - \Delta_i)\alpha_i(1-t) + (\Delta_i - d_{i+1})\beta_i t h_i t(1-t)}{\alpha_i(1-t)^2 + (\alpha_i + \beta_i + \gamma_i) t(1-t) + \beta_i t^2} \right\} = f_i(1-t) + f_{i+1} t + (\Delta_i - d_{i+1}) h_i t(1-t).
\]

**Fig. 1.** The piecewise rational cubic function (4) with \( \alpha_i = 200, \beta_i = 1 \) and \( \gamma_i = 0.5 \).
Hence, from (4b) it is clear that the increase in the shape parameter \( c_i \) reduces the piecewise rational cubic function (4) to the straight line \( f_i(1-t) + f_{i+1}t \) in the given interval \( I = [x_i, x_{i+1}], i = 0, 1, 2, \ldots, \frac{n}{2} - 1 \). The increase in the shape parameter \( \alpha_i \) or \( \beta_i \) reduces the rational cubic function (4) to quadratic polynomial and graphically, the shape of the curve be inclined on right and left side of the interval \( I = [x_i, x_{i+1}], i = 0, 1, 2, \ldots, \frac{n}{2} - 1 \) by increasing in the shape parameter \( \alpha_i \) and \( \beta_i \), respectively. These observations are demonstrated in Figs. 1–5 for a positive data set in Table 1 which was taken from [1]. It demonstrates the velocity of wind which is noted in different time interval. The velocity is inherently positive and we therefore, require a rational cubic function with shape parameters to conserve this shape characteristic. The \( x \)-values are time (min) and \( f \)-values are velocity of wind (km/min). For more details on positivity-preserving curve see the reference [1]. Figs. 1 and 2 illustrate the tension effect with increment in shape parameters \( \alpha_i \) and \( \beta_i \), respectively. Fig. 3 depicts the effect of increase in the shape parameter \( c_i \). A visually pleasing and smooth curve in Fig. 4 is obtained by simultaneously small increasing in both shape parameters \( \alpha_i \) and \( \beta_i \). Fig. 5 demonstrates the effect of the increase in the three shape parameters \( \alpha_i, \beta_i, c_i \), and this observation may or may not be positivity-preserving curve of positive data due to the manual input of shape parameters. The simultaneous increase in shape parameters \( (\alpha_i, \beta_i, c_i \to \infty) \) reduces the piecewise rational cubic function (4) to the straight line \( f_i(1-t) + f_{i+1}t \) in the given interval \( I = [x_i, x_{i+1}], i = 0, 1, 2, \ldots, \frac{n}{2} - 1 \). In this paper, for smooth and visually pleasing display of data, we only consider the simultaneously small increasing in both \( \alpha_i \) and \( \beta_i \). Fig. 6 depicts the positivity-preserving curve using shape preserving positive rational cubic curve interpolation developed in [1] (Theorem 1, page 210) with the values of free parameters \( \alpha_i = \beta_i = 2.5 \).
4. Rational bi-cubic spline function

The study of rational bi-cubic spline has upheld on bi-cubic polynomial splines interpolation due to more degrees of freedom in its presentation. This freedom has been used in a lot of purposes and goal to be achieved in many real life problems arising in several disciplines. In this section, we described the extension of rational cubic function (4) to a rational bi-cubic function $S(x, y)$ for the interpolation of regular data arranged over the rectangular Domain $\Omega = [a, b] \times [c, d]$. The partition of arbitrary intervals $[a, b]$ and $[c, d]$ can be defined as $\pi : a = x_0 < x_1 < x_2 < \cdots < x_n = b$, $\pi : c = y_0 < y_1 < y_2 < \cdots < y_m = d$, respectively. The rational bi-cubic function over each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0, 1, 2, \ldots, n - 1; j = 0, 1, 2, \ldots, m - 1$ defined as:

Table 1
A 2D positive data set.

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_i$</td>
<td>0</td>
<td>0.25</td>
<td>0.5</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$y_i$</td>
<td>2</td>
<td>0.6</td>
<td>0.10</td>
<td>0.13</td>
<td>1</td>
<td>0.5</td>
<td>1.1</td>
<td>0.25</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Fig. 4. The piecewise rational cubic function (4) with $x_i = 200$, $y_i = 200$ and $\gamma_i = 0.5$.

Fig. 5. The piecewise rational cubic function (4) with $x_i = 200$, $y_i = 200$ and $\gamma_i = 400$. 
Fig. 6. Positivity-preserving rational cubic curve [1] with $\alpha_i = 2.5, \beta_i = 2.5.$

\[ S(x,y) \equiv S_j(x,y) = b_0(s) \begin{pmatrix} f_{ij} \\ f_{i+1,j} \\ f_{ij} \\ f_{i+1,j} \end{pmatrix} + b_1(s) \begin{pmatrix} f_{ij+1} \\ f_{i+1,j+1} \\ f_{ij+1} \\ f_{i+1,j+1} \end{pmatrix} + b_2(s) \begin{pmatrix} f_{ij+2} \\ f_{i+1,j+2} \\ f_{ij+2} \\ f_{i+1,j+2} \end{pmatrix} + b_3(s) \begin{pmatrix} f_{ij+3} \\ f_{i+1,j+3} \\ f_{ij+3} \\ f_{i+1,j+3} \end{pmatrix}, \]

where $b_i(t), \hat{b}_i(s), i = 0(3)$ are the rational cubic blending functions defined as:

\[
\begin{align*}
    b_0(t) &= \frac{a_i(t-1)^3 + (\alpha_i + \beta_i + \gamma_i)t(1-t) + \beta_i t^2}{q_i(t)}, \\
    \hat{b}_1(s) &= \frac{a_i(s-1)^3 + (\alpha_i + \beta_i + \gamma_i)s(1-s) - \beta_i s^2}{q_i(s)}, \\
    \hat{b}_2(s) &= \frac{\alpha_i(s-1)^3 + (\alpha_i + \beta_i + \gamma_i)s(1-s) + \beta_i s^2}{q_i(s)}, \\
    \hat{b}_3(s) &= \frac{(\alpha_i(s-1)^3 + (\alpha_i + \beta_i + \gamma_i)s(1-s))}{q_i(s)}.
\end{align*}
\]

with

\[
q_i(t) = \alpha_i (1-t)^2 + (\alpha_i + \beta_i + \gamma_i)t(1-t) + \beta_i t^2,
\]

\[
q_i(s) = \alpha_i (1-s)^2 + (\alpha_i + \beta_i + \gamma_i)s(1-s) + \beta_i s^2,
\]

where $\lambda_i = 2 \alpha_i + \beta_i, \lambda_i = 2 \lambda_i + \beta_i$ and $\mu_i = \alpha_i + 2 \beta_i, \mu_i = \chi_i + 2 \beta_i$ and $t = (x-x_i)/h_i, s = (y-y_i)/\hat{h_i}$, with $h_i = x_{i+1} - x_i, \hat{h}_i = y_{i+1} - y_i.$

From (5) and (6), after carry out necessary algebra, we obtain

\[
S_j(x,y) = \frac{P_{ij}(1-t)^3 + Q_{ij}(1-t)^2 + R_{ij}t^2(1-t) + T_{ij}t^3}{\alpha_i (1-t)^2 + (\alpha_i + \beta_i + \gamma_i)t(1-t) + \beta_i t^2},
\]

where

\[
P_{ij} = f_{ij} \chi_i \zeta_{ij}(1-s)^3 + \chi_i f_{ij}(\gamma_i + \lambda_i) + f_{ij} \hat{h}_j \zeta_{ij}s(1-s)^2 + \chi_i (f_{ij+1}(\gamma_i + \lambda_i) - f_{ij+1} \hat{h}_j \zeta_{ij})s^2(1-s) + f_{ij+1} \chi_i \zeta_{ij} s^3/q_i(s),
\]

\[
Q_{ij} = \alpha_i (f_{ij}(\gamma_i + \lambda_i) + f_{ij} \hat{h}_j \zeta_{ij})(1-s)^3 + (\gamma_i + \lambda_i)(f_{ij}(\gamma_i + \lambda_i) + f_{ij} \hat{h}_j \zeta_{ij}) + h_i \alpha_i (f_{ij+1} \hat{h}_j \zeta_{ij} + f_{ij}(\gamma_i + \lambda_i)) s(1-s)^2
\]

\[
+ (f_{ij+1}(\gamma_i + \lambda_i) + f_{ij+1} \hat{h}_j \zeta_{ij} + f_{ij+1} \hat{h}_j \zeta_{ij} + f_{ij+1}(\gamma_i + \lambda_i)) s^2(1-s) + \hat{h}_j (f_{ij+1}(\gamma_i + \lambda_i) + f_{ij+1} \hat{h}_j \zeta_{ij}) s^3/q_i(s).
\]

\[
R_{ij} = \alpha_i (f_{ij+1}(\gamma_i + \lambda_i) - f_{ij+1} \hat{h}_j \zeta_{ij})(1-s)^3 + (\gamma_i + \lambda_i)(f_{ij+1}(\gamma_i + \lambda_i) + f_{ij+1} \hat{h}_j \zeta_{ij}) - h_i \alpha_i (f_{ij+1} \hat{h}_j \zeta_{ij} + f_{ij+1}(\gamma_i + \lambda_i)) s^2(1-s) + \hat{h}_j (f_{ij+1}(\gamma_i + \lambda_i) + f_{ij+1} \hat{h}_j \zeta_{ij}) s^3/q_i(s),
\]

\[
T_{ij} = f_{ij} \beta_i (1-s)^3 + \beta_i (f_{ij+1}(\gamma_i + \lambda_i) + f_{ij+1} \hat{h}_j \zeta_{ij}) s(1-s)^2 + \beta_i (f_{ij+1}(\gamma_i + \lambda_i) - f_{ij+1} \hat{h}_j \zeta_{ij}) s^2(1-s) + f_{ij+1} \beta_i \hat{h}_j s^3/q_i(s).
\]
5. Arithmetic mean method for derivative approximation

Usually, the derivatives \( f_{ij}^x \) and \( f_{ij}^y \) are not known, so must be calculated either from the given data or by some other sources. Let us denote \( f_{ij}^{xh} \) and \( f_{ij}^{yh} \) as the first order derivatives with respect to \( x \) and \( y \), respectively, at the data point \( f_{ij} \). Similarly, let the mixed derivatives be denoted by \( f_{ij}^{xy} \). Arithmetic mean method, as proposed in [7], is the three-point difference approximation based on arithmetic calculation for the positive curve manipulation. This method can be oriented and extended for the 3D data visualization as follows:

\[
\begin{align*}
f_{ij}^{xh} &= \Delta_{ij}^x + \frac{(\Delta_{ij} - \Delta_{i-1,j})h_0}{(h_0 + h_1)}, \quad f_{ij}^{yh} = \Delta_{ij}^y + \frac{(\Delta_{i,j-1} - \Delta_{i,j-2})h_{n-1}}{(h_{n-1} + h_{n-2})}, \\
f_{ij}^x &= \frac{(\Delta_{ij} + \Delta_{i-1,j})}{2}, \quad i = 1, 2, 3, \ldots, n - 1, \quad j = 0, 1, 2, \ldots, m, \\
f_{ij}^y &= \frac{(\Delta_{i,j-1} + \Delta_{i,j})}{2}, \quad i = 0, 1, 2, \ldots, n \quad j = 1, 2, 3, \ldots, m - 1, \\
f_{ij}^{xy} &= \frac{1}{2} \left\{ \frac{f_{i+1,j}^x - f_{i-1,j}^x}{h_{n-1} + h_i} + \frac{f_{i-1,j+1}^x - f_{i-1,j-1}^x}{h_j + h_i} \right\}, \quad i = 1, 2, \ldots, n - 1; \quad j = 1, 2, \ldots, m - 1,
\end{align*}
\]

where \( \Delta_{ij} = \frac{f_{i+1,j}^{xy} - f_{ij}^{xy}}{h_i} \), \( \hat{\Delta}_{ij} = \frac{f_{ij}^{xy} - f_{i-1,j}^{xy}}{h_j} \).

6. The proposed algorithm

In this section the positivity-preserving surface scheme is constructed for regular surface data using the rational bi-cubic function (5).

**Theorem 5.1**. The piecewise rational bi-cubic function \( S(x,y) \) defined over the rectangular mesh \([x_i, x_{i+1}] \times [y_j, y_{j+1}], \quad i = 0, 1, 2, \ldots, n - 1; \quad j = 0, 1, 2, \ldots, m - 1 \) in (5) conserves the positivity of surface through regular 3D positive data if the following constraints are satisfied:

\[ \alpha_i > 0, \quad \beta_i > 0, \quad \gamma_i > 0, \quad \delta_i > 0, \]

\[
\hat{\gamma}_{ij} > \operatorname{Max} \left( 0, -\frac{f_{ij}^{xh}h_{ij}^2}{f_{ij}} + \frac{f_{i+1,j}^{xh}h_{ij}^2}{f_{i+1,j}}, -\frac{f_{ij}^{yh}h_{ij}^2}{f_{ij}} + \frac{f_{i+1,j}^{yh}h_{ij}^2}{f_{i+1,j}} \right).
\]

\[
\gamma_{ij} > \operatorname{Max} \left( 0, -\frac{f_{ij}^{xh}h_{ij}^2}{f_{ij}} + \frac{f_{i+1,j}^{xh}h_{ij}^2}{f_{i+1,j}}, -\frac{h_{ij}^2}{f_{ij}} \left( f_{i+1,j}^{xh}h_{ij}^2 - f_{i,j+1}^{xh}h_{ij}^2 \right) \right),
\]

\[
-\frac{h_{ij}^2}{f_{ij}} \left( f_{i,j} h_{ij}^2 - f_{i+1,j}^{xh}h_{ij}^2 \right),
\]

\[
-\frac{h_{ij}^2}{f_{ij}} \left( f_{i,j} h_{ij}^2 - f_{i,j+1}^{xh}h_{ij}^2 \right),
\]

\[
\hat{\delta}_{ij} = \alpha_{ij} + \operatorname{Max} \left( 0, -\frac{f_{ij}^{xh}h_{ij}^2}{f_{ij}} + \frac{f_{i+1,j}^{xh}h_{ij}^2}{f_{i+1,j}}, -\frac{f_{ij}^{yh}h_{ij}^2}{f_{ij}} + \frac{f_{i+1,j}^{yh}h_{ij}^2}{f_{i+1,j}} \right), \quad \alpha_{ij} > 0
\]
\[
\gamma_{ij} = b_{ij} + \max \left( 0, -\frac{f_{ij}^x h_{ij}}{\delta_i}, -\frac{f_{ij}^y h_{ij}}{\delta_j}, -\frac{h_{ij} \left( f_{ij}^x h_{ij}^2 + f_{ij}^y h_{ij}^2 \right)}{\delta_i^2} \right), \quad b_{ij} > 0.
\]

**Proof.** Let \((\mathbf{x}, \mathbf{y}, f_{ij}), i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, m\) be the given set of 3D positive regular data points which is arranged over the rectangular region \(I = [x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0, 1, 2, \ldots, n - 1; j = 0, 1, 2, \ldots, m - 1\). Since the data is positive so \(f_{ij} > 0, \forall i, j\). The piecewise rational bi-cubic function \(S(x, y)\) is positive on rectangular grid \(\Omega = [a, b] \times [c, d]\) such that \(S(x_i, y) = f_{ij}, i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, m\) if

\[
S(x, y) > 0, \forall (x, y) \in \Omega.
\]

From (7), the piecewise rational bi-cubic function conserves the positivity if both

\[
\begin{align*}
& \left\{ \begin{array}{l}
P_{ij}(1-t)^3 + Q_{ij}t (1-t)^2 + R_{ij}t^2(1-t) + T_{ij}t^3 > 0 \\
\alpha_{ij}(1-t)^2 + \beta_{ij}(1-t) + \gamma_{ij}t^2 > 0.
\end{array} \right.
\end{align*}
\]

The following constraints are assumed to be positive on shape parameters throughout the paper as:

\[
\begin{align*}
& \alpha_{ij} > 0, \gamma_{ij} > 0, \beta_{ij} > 0 \\
& \hat{\alpha}_{ij} > 0, \hat{\gamma}_{ij} > 0, \hat{\beta}_{ij} > 0.
\end{align*}
\]

Thus, Eq. (13) is satisfied if the following constraints hold.

\[
P_{ij} > 0, Q_{ij} > 0, R_{ij} > 0, T_{ij} > 0.
\]

Let \(P_{ij} > 0\), such that

\[
P_{ij} = f_{ij} \alpha_{ij} \hat{\alpha}_{ij} (1-s)^3 + \alpha_{ij} (f_{ij} (\gamma_{ij} + \hat{\gamma}_{ij}) + f_{ij}^x h_{ij} \hat{h}_{ij}) s(1-s)^2 + \alpha_{ij} (f_{ij}^x (\gamma_{ij} + \hat{\gamma}_{ij}) - f_{ij}^x h_{ij} \hat{h}_{ij}) s^2(1-s) + f_{ij}^x \alpha_{ij} \hat{\alpha}_{ij} s^3/q_j(s) > 0.
\]

We have \(q_j(s) > 0\) if Eq. (14) is satisfied. \(\Box\)

Noted that \(P_{ij} > 0\) if

\[
\gamma_{ij} > -f_{ij}^x \hat{h}_{ij} \alpha_{ij}/f_{ij}^x, \quad \hat{\gamma}_{ij} > -f_{ij}^x \hat{h}_{ij} \hat{\alpha}_{ij}/f_{ij}^x.
\]

Similarly, it can be shown that \(Q_{ij} > 0\) if

\[
\gamma_{ij} > -f_{ij}^y \hat{h}_{ij} \alpha_{ij}/f_{ij}^y, \quad \hat{\gamma}_{ij} > -f_{ij}^y \hat{h}_{ij} \hat{\alpha}_{ij}/f_{ij}^y.
\]

\[
\gamma_{ij} > \left\{ \begin{array}{l}
-\frac{h_{ij} \left( f_{ij}^x h_{ij}^2 + f_{ij}^y h_{ij}^2 \right)}{\delta_i^2} \quad (\delta_i > 0) \\
\left( h_{ij} \left( f_{ij}^x h_{ij}^2 + f_{ij}^y h_{ij}^2 \right) \right) \quad (\delta_i = 0)
\end{array} \right.
\]

\[
\hat{\gamma}_{ij} > \left\{ \begin{array}{l}
\left( h_{ij} \left( f_{ij}^x h_{ij}^2 + f_{ij}^y h_{ij}^2 \right) \right) \quad (\delta_i > 0) \\
\left( h_{ij} \left( f_{ij}^x h_{ij}^2 + f_{ij}^y h_{ij}^2 \right) \right) \quad (\delta_i = 0)
\end{array} \right.
\]
Consider $R_{ij} > 0$, such that

\[
R_{ij} > 0
\]

\[
= \frac{\beta_{ij}(f_{i+1,j} - f_{i,j+1})}{f_{i+1,j+1} - f_{i,j+1}}(1 - s)^3
\]

\[
+ ((\gamma_{ij} + \mu_{ij})(f_{i+1,j} - f_{i,j+1}) + h_{ij} \beta_{ij} + f_{i+1,j} - f_{i,j+1})(\gamma_{ij} + \mu_{ij}))s(1 - s)^2
\]

\[
+ ((\gamma_{ij} + \mu_{ij})(f_{i+1,j} - f_{i,j+1}) + h_{ij} \beta_{ij} + f_{i+1,j} - f_{i,j+1})(\gamma_{ij} + \mu_{ij}))s(1 - s)^2
\]

\[
> 0
\]

Noted that $R_{ij} > 0$, if

\[
\gamma_{ij} > \frac{f_{i+1,j} h_{ij}}{f_{i,j+1}}, \quad \frac{f_{i+1,j+1} h_{ij}}{f_{i+1,j+1}}
\]

\[
\gamma_{ij} > \frac{(h_{ij} + \beta_{ij} + h_{ij})(\gamma_{ij} + \mu_{ij})}{(f_{i+1,j} + \gamma_{ij} + \mu_{ij})(\gamma_{ij} + \mu_{ij})}
\]

\[
\gamma_{ij} > \frac{(h_{ij} + \beta_{ij} + h_{ij})(\gamma_{ij} + \mu_{ij})}{(f_{i+1,j} + \gamma_{ij} + \mu_{ij})(\gamma_{ij} + \mu_{ij})}
\]

Finally $T_{ij} > 0$ if

\[
\gamma_{ij} > \frac{f_{i+1,j} h_{ij}}{f_{i,j+1}}, \quad \frac{f_{i+1,j+1} h_{ij}}{f_{i+1,j+1}}
\]

\[
\gamma_{ij} > \frac{(h_{ij} + \beta_{ij} + h_{ij})(\gamma_{ij} + \mu_{ij})}{(f_{i+1,j} + \gamma_{ij} + \mu_{ij})(\gamma_{ij} + \mu_{ij})}
\]

\[
\gamma_{ij} > \frac{(h_{ij} + \beta_{ij} + h_{ij})(\gamma_{ij} + \mu_{ij})}{(f_{i+1,j} + \gamma_{ij} + \mu_{ij})(\gamma_{ij} + \mu_{ij})}
\]

Table 2

A 3D positive data set produced from function (23).

<table>
<thead>
<tr>
<th>$y$</th>
<th>$x$</th>
<th>$x$</th>
<th>$x$</th>
<th>$x$</th>
<th>$x$</th>
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</tr>
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<td>2</td>
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<td>-3</td>
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<td>4.9</td>
<td>0.1</td>
<td>0.1923</td>
<td>2.5</td>
</tr>
<tr>
<td>-2</td>
<td>0.1923</td>
<td>2.5</td>
<td>3.7</td>
<td>1.3</td>
<td>2.5</td>
<td>4.8077</td>
</tr>
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<td>-1</td>
<td>0.1</td>
<td>1.3</td>
<td>2.5</td>
<td>2.5</td>
<td>2.5</td>
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<td>2.5</td>
<td>2.5</td>
<td>1.3</td>
<td>0.1923</td>
</tr>
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<td>2.5</td>
<td>1.3</td>
<td>3.7</td>
<td>2.5</td>
<td>4.8077</td>
</tr>
<tr>
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<td>2.5</td>
<td>0.1923</td>
<td>0.1</td>
<td>4.9</td>
<td>4.8077</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Fig. 7. (a) Bi-cubic Hermite surface (b) in xz-view.
Fig. 8. (a) Positive rational bi-cubic surface with shape parameters $x_{ij} = \beta_{ij} = \bar{x}_{ij} = \bar{\beta}_{ij} = 1.75$ (b) in $xz$-view.

Fig. 9. (a) Positive rational bi-cubic surface with shape parameters $x_{ij} = \beta_{ij} = \bar{x}_{ij} = \bar{\beta}_{ij} = 0.25$ (b) in $xz$-view.

Table 3
A positive 3D data set obtained from function (24).

<table>
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<th></th>
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<td>0.1169</td>
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<td>3</td>
<td>1.0262</td>
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<td>4</td>
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<td>1.4447</td>
<td>0.7353</td>
</tr>
<tr>
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<td>0.4318</td>
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</tr>
<tr>
<td>6</td>
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</tr>
</tbody>
</table>
Fig. 10. (a) Surface using bi-cubic Hermite interpolant (b) bi-cubic Hermite surface in yz-view.

Fig. 11. (a) Positive surface using developed rational bi-cubic interpolation with shape parameters $\alpha_{ij} = \beta_{ij} = 3.25$ (b) in xz-view.

Fig. 12. (a) Positive surface using developed rational bi-cubic interpolation with shape parameters $\alpha_{ij} = \beta_{ij} = 2.5$ (b) in xz-view.
From (15) and (20),
\[ y_{ij} > \max \left( 0, -\frac{f''_{ij} h_{ij} z_{ij}}{f_{ij}}, -\frac{f''_{i+1,j} h_{i+1,j}}{f_{i+1,j}}, \frac{f''_{i,j+1} h_{i,j+1}}{f_{i,j+1}} \right). \]

The above result can be rewritten as
\[ y_{ij} = a_{ij} + \max \left( 0, -\frac{f''_{ij} h_{ij} z_{ij}}{f_{ij}}, -\frac{f''_{i+1,j} h_{i+1,j}}{f_{i+1,j}}, \frac{f''_{i,j+1} h_{i,j+1}}{f_{i,j+1}} \right), \quad a_{ij} > 0 \] (21)

From (16)–(19),
\[ y_{ij} > \max \left( 0, -\frac{f''_{ij} h_{ij} z_{ij}}{f_{ij}}, -\frac{f''_{i+1,j} h_{i+1,j}}{f_{i+1,j}}, \frac{f''_{i,j+1} h_{i,j+1}}{f_{i,j+1}} \right) \]
\[ \left( \frac{h_{ij} \left( f''_{ij} h_{ij} z_{ij} + f''_{i+1,j} h_{i+1,j} - f''_{i,j+1} h_{i,j+1} \right)}{f_{ij} f_{i+1,j} f_{i,j+1}} \right) \]
\[ \left( \frac{h_{i+1,j} \left( f''_{i+1,j} h_{i+1,j} z_{i+1,j} + f''_{i,j+1} h_{i,j+1} - f''_{i+1,j} h_{i+1,j} \right)}{f_{i+1,j} f_{i,j+1} f_{i+1,j}} \right) \]
\[ \left( \frac{h_{i,j+1} \left( f''_{i,j+1} h_{i,j+1} z_{i,j+1} + f''_{i,j+1} h_{i,j+1} - f''_{i,j+1} h_{i,j+1} \right)}{f_{i,j+1} f_{i,j+1} f_{i,j+1}} \right) \]

The above expression can be represented as

<table>
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<th>( x )</th>
</tr>
</thead>
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<tr>
<td>(-3)</td>
<td>0.2082</td>
</tr>
<tr>
<td>(-2)</td>
<td>0.2823</td>
</tr>
<tr>
<td>(-1)</td>
<td>0.1956</td>
</tr>
<tr>
<td>(1)</td>
<td>0.1956</td>
</tr>
<tr>
<td>(2)</td>
<td>0.2823</td>
</tr>
<tr>
<td>(3)</td>
<td>0.2082</td>
</tr>
</tbody>
</table>

Table 4
A 3D positive data set.

Fig. 13. (a) Bi-cubic Hermite surface (b) xz-view of Hermite surface.
\[ \gamma_{ij} = b_{ij} + \text{Max} \left( 0, \frac{f_{x}^{i} b_{x}^{i} x_{i}}{x_{j}}, \frac{f_{y}^{j} b_{y}^{j} y_{j}}{y_{i}}, -\frac{\left( b_{x}^{i} \left( f_{y}^{i} b_{y}^{i} y_{i} + b_{y}^{i} \right) + f_{y}^{j} b_{y}^{j} y_{j} \right)}{(f_{y}^{j} b_{y}^{j} y_{j} + b_{y}^{j})}, \frac{f_{x}^{i} b_{x}^{i} x_{i}}{x_{j}} \right), \frac{f_{x}^{i} b_{x}^{i} x_{i}}{x_{j}}, \frac{f_{y}^{j} b_{y}^{j} y_{j}}{y_{i}}, -\frac{\left( b_{x}^{i} \left( f_{y}^{i} b_{y}^{i} y_{i} + b_{y}^{i} \right) + f_{y}^{j} b_{y}^{j} y_{j} \right)}{(f_{y}^{j} b_{y}^{j} y_{j} + b_{y}^{j})}, \frac{f_{x}^{i} b_{x}^{i} x_{i}}{x_{j}} \right) \right), \quad b_{ij} > 0. \]

Algorithm

1. Take \((n + 1) \times (m + 1)\) positive data points \(\{(x_i, y_j, f_{ij}) : i = 0, 1, 2, \ldots; n; j = 0, 1, 2, \ldots, m\}\) such that \(f_{ij} > 0\) \forall i, j.
2. Compute the step lengths \(h_i, h_j\) and \(\Delta_{ij}, \Delta_{ij}\) at data points.
3. Approximate the partial derivatives \(f_{ij}^{x}, f_{ij}^{y}, f_{ij}^{xy}\) at given data points.

Fig. 14. (a) Positive rational bi-cubic surface with shape parameters \(x_{ij} = \hat{x}_{ij} = \hat{y}_{ij} = 1.5\) (b) positive surface in xz-view.

Fig. 15. (a) Positive rational bi-cubic surface with shape parameters \(x_{ij} = \hat{x}_{ij} = \hat{y}_{ij} = 0.5\) (b) positive surface in xz-view.
4. Choose any positive value for free parameters $\alpha_{ij}, \tilde{\beta}_{ij}$, $\tilde{\alpha}_{ij}, \tilde{\beta}_{ij}$.
5. Take the maximum value of shape parameters $\gamma_{ij}$ and $\tilde{\gamma}_{ij}$ from developed set of constraints given in Theorem 5.1.
6. Compute the blending functions $b_i(t), \tilde{b}_i(s), i = 0(3)$ using Steps 1 to 5.
7. Insert the values of data points, partial derivatives, shape parameters and blending functions of Steps 1 to 6 in the proposed rational bi-cubic function (5) to get the positivity-preserving surface.

Table 5
A positive surface data.

<table>
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<th>y</th>
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<th>−2</th>
<th>−1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
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<td>0.1755</td>
<td>1.0401</td>
<td>0.1755</td>
<td>0.0404</td>
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</tr>
<tr>
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<td>0.1936</td>
<td>1.0583</td>
<td>0.1936</td>
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</tr>
<tr>
<td>−1</td>
<td>0.4078</td>
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<td>0.5432</td>
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<td></td>
</tr>
<tr>
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<td></td>
</tr>
</tbody>
</table>

Fig. 16. (a) Bi-cubic Hermite surface (b) xz-view of Hermite surface (c) yz-view of Hermite surface.
7. Demonstration

In this section, the efficiency of the proposed positivity-preserving surface scheme through several numerical examples is presented. Experimental evidence suggests that the rational bi-cubic function dominates over the rational bi-cubic partially blended function [10–13] not only because it generates positivity preserving surface everywhere in the domain but also produces smoother graphical results. A comparison of proposed scheme with bi-cubic Hermite spline and existing schemes are also part of this section.

Example 6.1. A positive data set with four decimal places taken in Table 2 is generated by the following mathematical function

\[
f(x, y) = \frac{xy(x^2 - y^2)}{(x^2 + y^2)^2} + 2.5, \quad -3 \leq x, y \leq 3 \quad (x, y \neq 0).
\] (23)

Fig. 7(a) is generated by bi-cubic Hermite spline [6] that does not conserve the positivity of positive data. A more clear view of surface that does not conserve the positivity can be seen in Fig. 7(b) which is an xz-view of Fig. 7(a). Fig. 8(a) and Fig. 9(a) are drawn by shape preserving rational bi-cubic interpolation for positive data developed in Section 5, with different values of shape parameters, to conserve the positive surface of positive data. A more clear view of developed positive surfaces can be seen in Fig. 8(b) and Fig. 9(b) which are xz-view of Fig. 8(a) and Fig. 9(a), respectively. A prominent difference in the

![Graphical representation of the data set](image-url)

Fig. 17. (a) Positivity-preserving rational bi-cubic surface with shape parameters \(a_{ij} = \beta_{ij} = z_{ij} = \bar{a}_{ij} = 2.0\) (b) positive surface in xz-view (c) positive surface in yz-view.
smoothness with a visually pleasant view of positive surfaces can be seen in Fig. 8(a) and Fig. 9(a) due to choice on the values of shape parameters given to designer.

**Example 6.2.** A positive data set up to 4 decimal places taken in Table 3 is generated by the following function,

\[ f(x, y) = \sin(x) \cos(y) + 0.95, \quad 1 \leq x, y \leq 6. \]  

Fig. 10(a) (with yz-view in Fig. 10(b)) is produced by using bi-cubic Hermite spline [6] that loses the inherent shape feature of positive data. On the other hand, the efficiency of the proposed scheme is shown in Fig. 11(a) (with yz-view in Fig. 11(b)) and Fig. 12(a) (with yz-view in Fig. 12(b)). A remarkable difference in the smoothness with pleasant graphical view is visible in these figures due to the freedom granted to the designer on the values of shape parameters.

**Example 6.3.** A data set up to four decimal places taken in Table 4 is obtained by following mathematical function

\[ f(x, y) = \frac{\sin(x^2 + y^2)}{(x^2 + y^2)} + 0.25, \quad -3 \leq x, y \leq 3 \quad (x, y \neq 0). \]  

Fig. 13(a) (with xz-view in Fig. 13(b)) is generated by bi-cubic Hermite spline scheme [6] which does not conserve the positivity of positive data. On the other hand, Fig. 14(a) (with xz-view in Fig. 14(b)) and Fig. 15(a) (with xz-view in Fig. 15(b)) are

![Fig. 18](image)

(a) Positivity-preserving rational bi-cubic surface with shape parameters \( a_{ij} = b_{ij} = \bar{a}_{ij} = \bar{b}_{ij} = 0.5 \)  
(b) positive surface in xz-view  
(c) positive surface in yz-view.
drawn by the positivity preserving rational bi-cubic interpolation developed in Section 5 to conserve the positivity through same positive surface data with the different values of shape parameters. A more clear view of developed positive surfaces can be seen in Fig. 14(b) and Fig. 15(b). Fig. 15(a) is more visually pleasant and smooth as compared to Fig. 14(a) due to liberty granted to designer on the values of shape parameters.

**Example 6.4.** A positive data set up to 4 decimal places taken in Table 5 is generated by the following function,

\[ f(x, y) = e^{-x^2} + e^{-2y^2} + 0.04, \quad -3 \leq x, y \leq 3. \]

Fig. 16(a) (with xz-view in Fig. 16(b) and yz-view in Fig. 16(c)) is generated by bi-cubic Hermite spline [6] that does not conserve the positivity through positive data. Fig. 17(a) and Fig. 18(a) are drawn by shape preserving rational bi-cubic interpolation for positive data developed in Section 5, with different values of shape parameters, to conserve the positive surface through positive data. A more clear view of developed positive surfaces can be seen in Fig. 17(b), (c) and Fig. 18(b), (c). A prominent difference in the smoothness with a visually pleasant view of positive surfaces can be seen in Fig. 17(a) and Fig. 18(a) due to choice on the values of shape parameters given to designer.

### 8. Conclusion

In this paper, we have extended the rational cubic function [1] to rational bi-cubic function with six shape parameters in each rectangular patch to conserve the positivity of surface through 3D positive data. The developed surface scheme has been demonstrated through several numerical examples and it is concluded that the scheme is not only $C^1$, local and computationally economical but also visually pleasant. The proposed scheme is equally applicable for the data with derivative or without derivatives. There is no need of additional information about derivatives because they are calculated directly from data points.

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### References


