Cascading Failures in Finite-Size Random Geometric Networks

Ali Eslami, Member, IEEE, Chuan Huang, Member, IEEE, Junshan Zhang, Fellow, IEEE, and Shuguang Cui, Fellow, IEEE

Abstract

The problem of cascading failures in cyber-physical networks is garnering much attention for different network models underlining various applications. While a variety of analytic results has been reported for the case of large networks, very few of them are readily applicable to finite-size networks. This paper studies cascading failures in finite-size geometric networks where the number of nodes is on the order of tens or hundreds as in many real-life networks. As an initial step, the impact of tolerance parameter on network resiliency has been investigated. We quantified network reaction to initial disturbances of different sizes by measuring the damage imposed on the network. Lower and upper bounds on the number of failures have been provided to characterize such damages. In addition to the finite analysis, an asymptotic analysis of both bounds has been carried out, pointing to a threshold behavior of the network as the tolerance parameter changes. The critical value of tolerance parameter in the asymptotic regime has been derived. Findings of this paper, in particular, shed light on choosing the tolerance parameter appropriately such that a cascade of failures could be avoided.

Index Terms

Cascading Failure, Finite-Size Complex Networks, Random Geometric Graphs.

I. INTRODUCTION

A cascading failure in a complex network is a phenomenon in which the failure of a small set of nodes triggers the failure of successive nodes, leading to the failure of a large fraction of nodes eventually. There have been many types of cascading failure events occurring in natural and man-made systems, from power grid and computer networks to political, economic, and ecological systems. Cascading failure is common in power grids where a single failure of a fully loaded or slightly overloaded node (component) can set
off more overloads, thereby taking down the entire system in a very short time. A few examples of power outages caused by a cascading failure are the blackouts in northeast America in 2003, Italy in 2003, London in 2003, and northern India in 2012. Cascading failures can also occur in computer networks (such as the Internet), when a crucial router or node becomes overloaded. Network traffic then needs to be re-routed through an alternative path. This alternative path, as a result, may become overloaded, causing it to go down, and so on.

The problem of cascading failures in complex networks has been studied extensively [1]–[8], especially for large networks. For analytical purposes, different types of random graphs have been used as models of complex networks, including Bernoulli random graphs, random geometric graphs, and scale-free graphs [9]–[13]. Also, depending on the underlying application, different models of failure propagation have been considered. The two major categories of propagation rules are the degree-based and load-based propagation. In a degree-based propagation, the state of each node is determined by the states of all or part of its neighbors in the network [1], [4], [9], [12], [13]. For example, in [9], each node is assigned a random threshold $\phi$, and it fails if at least a fraction $\phi$ of its neighbors fail. On the other hand, in a load-based propagation, the state of a node is defined based on the amount of load it carries [2], [10]. For instance in [10], each node can carry a load up to its capacity, above which it becomes overloaded. An overloaded node fails and redistributes its load to its neighbors.

While the vast majority of the existing analytical work is focused on large-scale networks, their findings can hardly be applied to the small or moderate size networks that we face in the real world. In this paper, we are concerned with providing rigorous analytical results for finite-size networks. Furthermore, we are interested in studying cascading failures in networks with geometric characteristics such as electrical power grids and wireless communication networks, which are well-modeled as random geometric graphs. Indeed, random geometric graphs are widely used in studying wireless networks (see [14] and references therein). Also, it is shown that geometry plays an important role in quantifying the topology of the smart grid communication and control networks [15].

We adopt load-based failure propagation in this paper as it makes sense in a group of important applications such as the power grid and the wireless networks. We assume that each node has a certain capacity, part of which is used to carry a load in normal conditions. If, for any reason, a node receives more load than its capacity, it fails and redistributes its load to its neighbors. A node here can be a component in a power grid, such as a transmission line or a regional transformer, which usually operates
in normal conditions but is able to handle higher loads up to a certain capacity. A node could also be a device in a wireless distributed storage network, or a routing hub in the internet. In all these cases, a node can be assumed to operate under a certain load in normal situations, while it is able to handle a higher load up to a limit, if necessary.

The difference between the capacity and the normal load of a node is specified by the tolerance parameter \( [2], [10] \). Tolerance parameter is a design parameter that plays an important role in network resiliency against a cascade. When resiliency is the priority, a larger tolerance parameter is desired as it enables the network to handle more severe operation disturbances. However, a larger tolerance parameter leads to a larger unused capacity that imposes higher costs. Therefore, it is crucial to maintain a clear understanding of the impact of tolerance parameter on network reaction to disruptions of different scales. In this paper, we characterize such reaction through analytical means in both finite and asymptotic regimes.

The rest of this paper is organized as follows. In the next section, we explain the network model and notations, while formally stating the problem. Sections III, IV, and V provide the main results and the bulk of the analysis. Finally, Section VI concludes the paper.

**II. Network Model and Problem Formulation**

In this paper, we consider networks modeled by random geometric graphs \( G(\lambda, R) \), whose nodes are deployed in a region \( S \) according to a Poisson point process with density \( \lambda \). There is an edge between each pair of nodes if they are less than \( R \) apart. We assume that \( S \) is a circular region with diameter \( D \) and centered at the origin, \( 0 \). However, the results presented in this paper can be extended to other deployment regions with minimal changes. Initially, all the nodes carry the same amount of load \( l \), and have the same capacity \( c = \alpha l \), where \( \alpha \geq 1 \) is the tolerance parameter. While the load of each node may change over time, the capacity remains the same. A node is called “healthy” if it carries a load less than its capacity.

A dish attack on \( S \) is modeled by a circle \( A \) of radius \( R_a < D/2 \) centered at the origin. This is shown in Fig. 1. After the attack, all the nodes located at a distance \( r < R_a \) from the center of attack will fail, and their load will be redistributed to their neighbors, which in turn may lead to a propagation of failure throughout the network. Note that a dish attack only affects the nodes inside the dish and not the ones located on its border at \( r = R_a \). We focus on the set of conditions under which a cascading failure is realized and the corresponding damage caused by such a cascade. We assume the following model for the
propagation of failures. At any stage of cascade, when a node fails, its load will be redistributed *equally* among its *healthy* neighbors. A node that carries a total load greater than its capacity will fail.

The number of failures at each stage is clearly a random variable (r.v.). In this paper, we focus on characterizing the impact of an attack on the network. To measure such an impact, we use the number of failures, outside the attacked region, caused by a limited dish attack. Let $F$ denote this number. We define *failure ratio* as

$$f \triangleq \frac{F}{|S \setminus A|},$$

where $|S \setminus A|$ is the total number of nodes outside the attack region. We use the average value of the random variable $f$, taken over all realizations of $G(\lambda, R)$, to measure the impact of an attack outside the attack region. We are particularly interested in the variation of this value as the design parameter—here the tolerance parameter—changes. Fig. 2 shows $\bar{f}$ versus $\alpha$ for a typical attack on a network where $R_a = R = 0.1$, $D = 1$, for different values of $\lambda$. The contributions of this paper can be summarized as follows:

- While finding a closed form expression for $\bar{f}$ in the finite regime could be very difficult, we analytically derive lower and upper bounds for $\bar{f}$ with manageable computational complexities.
These bounds provide us with valuable intuitions into network resiliency when designing real-world networks. We also present an asymptotic analysis of our bounds.

As it can be seen in Fig. 2, the failure ratio changes rather quickly over a short interval of $\alpha$. We will show that this interval tends to zero as $\lambda$ goes to infinity. Particularly, in such an asymptotic case, there exists a threshold value of $\alpha$, denoted as $\alpha_U$, such that $\bar{f} = 1$ if $\alpha < \alpha_U$, and $\bar{f} = 0$ if $\alpha \geq \alpha_U$. We will find $\alpha_U$ in terms of other network parameters.

**Connected vs. Disconnected Graphs:** By definition, in a connected network, there exists a path between any two arbitrary nodes in the network. For $G(\lambda, R)$, connectivity is only guaranteed when $\lambda \to \infty$. In practice however, the probability of connectivity can be arbitrarily close to 1 if $\lambda$ is chosen large enough. In our analysis, whenever connectivity is required, we will consider the subspace $G_c(\lambda, R)$ of connected members of the probability space $G(\lambda, R)$. In simulations, however, it is extremely time-consuming to check the connectivity of each realization. Therefore, in order to have a connectivity probability close to 1, we assume $\lambda$ is chosen such that $\lambda \pi R^2 \geq 6$. 

Fig. 2. Figure shows the average failure ratio versus $\alpha$ in both finite and large-scale networks. In this paper, we provide analytical bounds for finite-size networks, and find the threshold value for large-scale networks.
A. Preliminaries

Here, we provide some notations and preliminaries required to understand the results. We denote the number of nodes in the attack region $A$ by $a$. Note that $a$ is a Poisson r.v. with parameter

$$\bar{a} = \lambda \pi R_a^2.$$  

(2)

It makes sense to assume that a dish attack is large enough to affect at least one node. Therefore, we assume $\bar{a} = \lambda \pi R_a^2 \geq 3$ in this paper. This way, we will have $\Pr(a \geq 1) > 0.95$.

Consider the rings (annuli) of width $R$ around the attacked region, as depicted in Fig. 1. For $i \geq 1$, we denote an annulus with inner radius $R_{i-1} = R_a + (i-1)R$ and outer radius $R_i = R_a + iR$ by $A_i$, and the set of nodes in $A_i$ by $A_i$. We denote the cardinality of $A_i$ by $a_i$. Note that $a_i$, which is the number of nodes in the ring $A_i$, is simply a Poisson random variable with parameter

$$\bar{a}_i = \lambda \pi (R_i^2 - R_{i-1}^2).$$

(3)

The following lemma will help us in our analysis. All the proofs in this paper are provided in the Appendix.

**Lemma 1.** Let $\bar{a} = \lambda \pi R_a^2 \geq 3$ and $\lambda \pi R^2 \geq 6$, as it is assumed in this paper. We will then have

$$\bar{a}_i > 14, \quad i \geq 1.$$  

(4)

Since $\bar{a}_i$ is greater than 10, Lemma 1 implies that the Poisson r.v. $a_i$ could be well approximated by a Gaussian r.v. for $i \geq 1$ [16]. We will make it clear when we use this assumption in our analysis. Consider two circles, one with radius $r_1$ centered at a distance $a$ from the origin, and one with radius $r_2$ centered at a distance $b$. We denote by $I(a, r_1, b, r_2)$ the intersection region of these two circles, while we use $I(a, r_1, b, r_2)$ to show the area of this region, which could be obtained as follows [16].

$$I(a, r_1, b, r_2)) = r_2 \cos^{-1} \left( \frac{(b - a)^2 + r_2^2 - r_1^2}{2(b - a)r_2} \right) + r_1^2 \cos^{-1} \left( \frac{(b - a)^2 + r_1^2 - r_2^2}{2(b - a)r_1} \right)$$

$$- \frac{1}{2} \sqrt{(-|b - a| + r_1 + r_2)(|b - a| + r_2 - r_1)(|b - a| - r_2 + r_1)(|b - a| + r_2 + r_1)}.$$  

(5)

The following lemma proves helpful in our analysis.

**Lemma 2.** Let $u$ be a node located randomly and uniformly on $I(0, R_a, r_v, R)$ where $r_v \geq R_a$, as shown
in Fig. 3. Also let $r$ be the random variable representing $u$'s distance from the center of attack (i.e. origin). Then the probability distribution function (PDF) of $r$ is given as below.

$$
\psi(r) = \begin{cases} 
\frac{2r}{I(0, R_a, r_v, R)} \arccos \left( \frac{r_v^2 - R^2 + r^2}{2r_v r} \right) & \text{if } r + r_v > R \\
\frac{2\pi r}{I(0, R_a, r_v, R)} & \text{if } r + r_v \leq R.
\end{cases}
$$

III. AN UPPER BOUND ON THE FAILURE RATIO

In this section, we turn our attention to a necessary condition for having a cascade. A cascade of failures is possible only if at least one node outside the attack region fails due to the load redistribution. Otherwise, if the load of the attacked nodes in $A$ is completely absorbed by the rest of the network, a propagation of failure does not occur. By finding the probability of this event, we will derive an upper bound on the average failure ratio.

We start our analysis for the finite-size networks by investigating the load received by nodes outside the attack region, immediately after the attack. This is the load received by immediate neighbors of the attacked nodes after the very first load redistribution. Note that this load is a random variable. Also recall that the neighbors of the attacked region are all located in $A_1$. We will find the average and standard
deviation of the load received by these nodes. Having the statistics of this random variable, we will show that its distribution is well approximated by a Gaussian random variable. Using such an approximation, we find the probability of an overload for the nodes in $A_1$, which later helps us find an upper bound on the average failure ratio. Recall that “average” here stands for an average taken over all graph realizations. Before presenting the main result of this section, we need to state the following lemmas for which the proofs can be found in the Appendix.

**Lemma 3.** Let $d_u$ be a Poisson random variable with density $\lambda_u$. We then have

$$E\left[ \frac{1}{d_u} \mid d_u > 0 \right] = \frac{e^{-\lambda_u} g(\lambda_u)}{1 - e^{-\lambda_u}},$$

$$E\left[ \frac{1}{d_u^2} \mid d_u > 0 \right] = \frac{e^{-\lambda_u}}{1 - e^{-\lambda_u}} \int_{-\infty}^{\lambda_u} \frac{1}{x} g(x) dx,$$

where

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{k \cdot k!} = \int_{-\infty}^{x} \frac{e^z - 1}{z} dz.$$

**Lemma 4.** Consider a node $v$ located at a distance $r_v \in [R_a, R_a + R)$ from the center of attack. Also consider a node $u$, a neighbor of $v$, located inside the attacked region at a distance $r < R_a$ from the center, as shown in Fig. 4. The average load $l_u$ redistributed to $v$ from $u$ can be obtained as follows.

$$E[l_u | r] = \frac{e^{-\lambda J(r)} g(\lambda J(r))}{1 - e^{-\lambda J(r)}} \triangleq h^{(1)}(r),$$

where $g(\cdot)$ is defined in (9) and $J(r) = \pi R^2 - I(r, R, 0, R_a)$. Moreover, we have

$$E[l^2_u | r] = \frac{e^{-\lambda J(r)}}{1 - e^{-\lambda J(r)}} \int_{-\infty}^{\lambda J(r)} \frac{g(x)}{x} dx \triangleq h^{(2)}(r).$$

If $u$ is located randomly and uniformly on $I(r_v, R, 0, R_a)$, then

$$E[l_u] = \int_{r_v - R}^{R_a} h^{(1)}(r) \times \psi(r) \, dr,$$

$$E[l^2_u] = \int_{r_v - R}^{R_a} h^{(2)}(r) \times \psi(r) \, dr.$$
If \( R - r_v \geq R_a \), then \( J(r) = J^* \triangleq \pi R^2 - \pi R_a^2 \), and (12) and (13) can be reduced to

\[
E[l_u] = \frac{e^{-\lambda J^*} g(\lambda J^*)}{1 - e^{-\lambda J^*}}, \\
E[l_u^2] = \frac{e^{-\lambda J^*}}{1 - e^{-\lambda J^*}} \int_{-\infty}^{\lambda J^*} \frac{g(x)}{x} dx. \tag{14}
\]

Finally, given \( E[l_u] \) and \( E[l_u^2] \), the variance \( \sigma^2_{l_u} \) can be obtained as

\[
\sigma^2_{l_u} = E[l_u^2] - E[l_u]^2. \tag{15}
\]

The following theorem employs the results of Lemma 2, 3, and 4 to find the average value and variance of the load redistributed to a node at distance \( r_v \) from the center of attack, right after the attack.

**Theorem 1.** Consider a node \( v \) located at a distance \( r_v \in [R_a, R_a + R) \) from the center of attack. Let \( L_v \) be the load redistributed to \( v \) by its neighbors inside the attacked region. We then have

\[
E[L_v] = \lambda I(r_v, R, 0, R_a) \int_{r_v-R}^{R_a} h^{(1)}(r) \times \psi(r) \, dr, \tag{16}
\]

\[
\sigma^2_{L_v} = \lambda I(r_v, R, 0, R_a) \times \sigma^2_{l_u}, \tag{17}
\]
where \( h^{(1)}(r) \) is defined in (10), and \( \sigma^2_{l_u} \) is given by (15). If \( R - r_v \geq R_a \), (16) and (17) are reduced to

\[
E[L_v] = \lambda \pi R_a^2 \frac{e^{-\lambda J^*} g(\lambda J^*)}{1 - e^{-\lambda J^*}},
\]

\[
\sigma^2_{L_v} = \lambda \pi R_a^2 \sigma^2_{l_u},
\]

(18)

where \( g(\cdot) \) is given by (9), and \( J^* = \pi R^2 - \pi R_a^2 \).

Now that we have the average value and variance of \( L_v \), an approximation of \( L_v \)'s PDF can be obtained using the central limit theorem as follows. Note that

\[
L_v = \sum_{u=1}^{N} l_u,
\]

(19)

where \( N \) is the number of nodes inside \( \mathcal{I}(r_v, R, 0, R_a) \), a Poisson r.v. with mean \( \lambda \mathcal{I}(r_v, R, 0, R_a) \). Given that \( N = n \), the nodes \( u = 1, \ldots, n \) would be distributed randomly and independently on \( \mathcal{I}(r_v, R, 0, R_a) \), making \( l_u \)'s i.i.d. random variables. Therefore, for large values of \( n \), the central limit theorem asserts that the probability distribution of \( L_v \) is well-approximated by a Gaussian random variable. In practice, however, \( n \geq 5 \) is large enough to ensure a PDF very close to the normal random variable [16]. The following corollary is a formal statement of what we just explained.

**Corollary 1.** If \( \lambda \times I(r_v, R, 0, R_a) \gg 1 \), the load received by a node \( v \) at \( r_v \) can be approximated by a Gaussian r.v. \( L_v \sim \mathcal{N}(\bar{L}_v, \sigma^2_{L_v}) \), where \( \bar{L}_v \) and \( \sigma^2_{L_v} \) are given by (16) and (17). Particularly, we have

\[
Pr\{v \text{ fails}\} = Pr\{L_v > \alpha - 1\} \approx 1 - \Phi\left(\frac{\alpha - 1 - \bar{L}_v}{\sigma_{L_v}}\right),
\]

(20)

where \( \Phi(\cdot) \) is the cumulative distribution function (CDF) of the standard normal distribution with mean 0 and variance 1.

When \( \lambda I(r_v, R, 0, R_a) \) is small due to either \( \lambda \) or \( I(r_v, R, 0, R_a) \), the load received by \( v \) becomes very small. Since the Gaussian approximation for \( L_v \) also turns small values in this case, the error in approximation becomes negligible anyways. The following theorem employs this fact along with Corollary 1 to find the probability of survival for the nodes in \( \mathcal{A}_1 \) after the very first round of load redistribution.

**Theorem 2.** Let \( v \) be a node located randomly and uniformly on \( \mathcal{A}_1 \). Also let \( p_1 \) be the probability that the load received by \( v \) is less than or equal to \( \alpha - 1 \), i.e., \( p_1 \triangleq Pr\{L_v \leq \alpha - 1\} \). Then, \( p_1 \) is obtained as
follows,

\[ p_1 \approx \int_{R_a}^{R_a+R} \Phi\left( \frac{\alpha - 1 - \bar{L}_v}{\sigma_{L_v}} \right) \times \frac{2\pi r_v}{|A_1|} dr_v. \]  

(21)

Note that \( \bar{L}_v \) and \( \sigma_{L_v} \) are functions of \( r_v \), given by Theorem 1.

Using the finding of Theorem 2, an upper bound on the average failure ratio can be obtained for finite values of \( \lambda \).

**Theorem 3.** The average failure ratio due to a dish attack of radius \( R_a \) applied to a network \( G(\lambda, R) \) is upper-bounded as

\[ \bar{f} \leq 1 - e^{-\lambda_1(1-p_1)}, \]  

(22)

where

\[ \lambda_1 = \lambda \pi \left( (R_a + R)^2 - R_a^2 \right) \]  

(23)

is the density of nodes in \( A_1 \), and \( p_1 \) is given by Theorem 2.

Fig. 5 depicts the upper bound from Theorem 3 for different values of network parameters, where we also include the simulation results for the exact value of \( \bar{f} \). As it can be seen, the proposed upper bound is especially helpful when it comes to picking a value of \( \alpha \) to avoid a cascade. For example, for the network \( G(\lambda = 400, R = 0.1) \), the upper bound suggests that \( \alpha = 3 \) is a good choice to contain dish attacks of radius \( R_a = 0.1 \) or smaller.

**IV. Asymptotic Analysis of Upper Bound and the Threshold Behavior of Failure Ratio**

While Theorem 3 provides an upper bound on the failure ratio in finite-size networks, an asymptotic analysis of the upper bound could provide intuition regarding the behavior of a large network under attacks. As we will see, such an analysis reveals the threshold behavior of the failure ratio in terms of the tolerance parameter. For the case when \( \lambda \to \infty \), it will be shown that as the tolerance parameter increases above 1, the failure ratio drops from 1 to 0 at a critical value of the tolerance parameter. We will find this critical value, which could be very helpful when studying large networks. We start our analysis by finding what happens to the load \( L_v \) in Theorem 1 when \( \lambda \to \infty \). In this section, in order to explicitly show the dependence of \( L_v \) on \( r_v \), we use the notation \( L(r_v) \) instead of \( L_v \) for the load received by the
node \( v \) located at \( r_v \). This slight modification will prove helpful in understanding the analysis.

**Theorem 4.** Consider a dish attack of radius \( R_a \) applied to a network \( G(\lambda, R) \). Let \( L(r_v) \) be the load received by a node \( v \) located at a distance \( r_v \in [R_a, R_a + R] \) from the center of attack, right after the attack. When \( \lambda \rightarrow \infty \), \( L(r_v) \) will no longer be a random variable, and is given as

\[
L(r_v) \rightarrow 2 \int_{r_v-R}^{R_a} \frac{r}{J(r)} \arccos \left( \frac{r^2 - R^2 + r_v^2}{2r_v r} \right) dr,
\]

(24)

where \( J(r) = \pi R^2 - I(r, R, 0, R_a) \).

Having the asymptotic value of \( L(r_v) \) from Theorem 4, a sufficient condition for a cascade of failures in the asymptotic case can be obtained as below.

**Theorem 5.** Consider a healthy node \( v \) located at \( r_v \geq R_a \) after a dish attack of radius \( R_a \) on \( G(\lambda, R) \) where \( \lambda \rightarrow \infty \). Let

\[
\alpha_U \triangleq 1 + L(R_a) = 1 + 2 \int_{R_a-R}^{R_a} \frac{r}{J(r)} \arccos \left( \frac{R_a^2 - R^2 + r^2}{2R_a r} \right) dr.
\]

(25)

If \( \alpha < \alpha_U \) and all the nodes located at \( r < r_v \) have failed, then \( v \) will fail as well. Hence, a cascade of failures occurs throughout the network, resulting in \( \bar{f} = 1 \).

The following theorem combines the sufficient condition from Theorem 5 with a necessary condition for a cascade, proving a threshold behavior for the average failure ratio in the asymptotic regime.

**Theorem 6.** Consider a dish attack of radius \( R_a \) applied to a network \( G(\lambda, R) \) where \( \lambda \rightarrow \infty \). Let \( \alpha_U \)
be the value of $\alpha$ given by (25). Then, $\bar{f} = 0$ if $\alpha \geq \alpha_U$, and $\bar{f} = 1$ if $\alpha < \alpha_U$.

Fig. 6 demonstrates the evolution of the average failure ratio $\bar{f}$ as $\lambda$ grows larger. It also shows the value of $\alpha_U$ given by Theorem 5 for the asymptotic case. As it can be seen, a threshold behavior around $\alpha_U$ becomes clear as $\lambda$ increases.

V. A LOWER BOUND ON THE FAILURE RATIO

In this section, we derive a lower bound on the failure ratio by analyzing a sufficient condition for propagation of failure throughout the network. This condition is based on the fact that if a cascade cannot be stopped in the presence of a) full cooperation between nodes, and b) the most favorable connectivity condition, then for sure it cannot be stopped without them. We first provide the lower bound for finite-size networks.

**Theorem 7.** Consider the subspace $G_c(\lambda, R)$ of the probability space $G(\lambda, R)$, formed by all the connected realizations in $G(\lambda, R)$. Suppose that a dish attack of radius $R_a$ is applied to $G_c(\lambda, R)$. Also let $q$ denote the ratio $R_a/R$. If

$$\alpha < 3/2 + q,$$

(26)
then we have

\[ \tilde{f} \geq e^{-\bar{a}} \sum_{k=1}^{\infty} \Phi\left( \frac{\frac{k}{\bar{a}} - \bar{a}_1}{\sqrt{\bar{a}_1}} \right) \bar{a}_1^k, \]  

(27)

where \( \bar{a} = \lambda \pi R^2_a, \bar{a}_1 = \lambda \pi [(R_a + R)^2 - R^2_a], \) and \( \Phi(\cdot) \) is the CDF of standard normal distribution. \[\blacksquare\]

Note that \( \Phi(\cdot) \leq 1, \) and \( \bar{a}_1^k/k! \) decays very fast as \( k \) grows above \( \bar{a} \). Hence, in practice, the summation in (27) needs to be calculated only for \( 2\bar{a} \) or \( 3\bar{a} \) terms. Before proving Theorem 7, we need to state a few lemmas. Recall the rings \( \mathcal{A}_i, i \geq 1, \) in Fig. 1. A cascade of failures, at each stage of its progress, goes through one of these rings. Let us look at how the failure propagates after the attack. After an attack on \( \mathcal{A} \), all the nodes that may potentially fail in the next step are the neighbors of \( \mathcal{A} \) located in \( \mathcal{A}_1 \). If some of the nodes in \( \mathcal{A}_1 \) fail, the next step of propagation includes some nodes in \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). In general, if the failures have already been spread through \( \mathcal{A}_1, ..., \mathcal{A}_i, \) potential failures of the next step are all in \((\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_i) \cup \mathcal{A}_{i+1}\). We know that \( a_i \)'s are Poisson random variables with parameter \( \bar{a}_i \) given by (3). Lemmas 5 and 6 below establish a connection between \( \bar{a}_i \)'s and \( \alpha \). We use this connection later in Lemma 7 to prove a useful property for finding the lower bound. The proofs of all lemmas can be found in the Appendix.

**Lemma 5.** Let \( q = R_a/R \). Given that \( a = a_0 \), if \( \alpha - 1 < 1/2 + q \), then

\[ \frac{\bar{a}_2}{\bar{a}_1} < \frac{\alpha}{\alpha - 1} \]  

(28)

\[\blacksquare\]

**Lemma 6.** For \( i \geq 2 \), we have

\[ \frac{\bar{a}_{i+1}}{\bar{a}_i} \leq \frac{\bar{a}_i}{\bar{a}_{i-1}}. \]  

(29)

Particularly, if \( \alpha - 1 < 1/2 + q \) as in Lemma 5, then

\[ \frac{\bar{a}_{i+1}}{\bar{a}_i} \leq \frac{\bar{a}_2}{\bar{a}_1} < \frac{\alpha}{\alpha - 1}, \]  

(30)

for \( i \geq 2 \). \[\blacksquare\]

**Gaussian approximation for \( a_i, i \geq 1 \):** Recall Lemma 1 and the discussion afterwards where we explained how our assumptions for \( \lambda \pi R^2_a \) and \( \lambda \pi R^2 \) lead to \( \bar{a}_i > 14, \) for \( i \geq 1 \). This means that the
Poisson r.v. \( a_i \) is well approximated by a Gaussian r.v. with the same mean and variance \( \bar{a}_i \) given by (3). We use this approximation in proving the following lemma.

**Lemma 7.** Given that \( a = a_0 \) and \( \alpha - 1 < 1/2 + q \), the following property holds for \( a_i, i \geq 0 \).

\[
Pr\{a_0 + a_1 + \ldots + a_i > a_{i+1}(\alpha - 1)\} \geq Pr\{a_0 > a_1(\alpha - 1)\}.
\]  (31)

Using the preliminary results stated above, we obtain the following theorem regarding an all-out cascade in a finite-size network. The proof is given in the Appendix.

**Theorem 8.** Suppose a dish attack of radius \( R_a \) is applied to \( G_c(\lambda, R) \). Given that \( a = a_0 \), if \( \alpha - 1 < 1/2 + q \), then the probability that all the nodes fail is lower-bounded by \( Pr\{a_0 > a_1(\alpha - 1)\} \).

Now we are ready to prove Theorem 7.

**Proof of Theorem 7:** Let us define an identity random variable \( X_a \) as

\[
X_a = \begin{cases} 
1 & \text{if } a \text{ causes a total failure,} \\
0 & \text{otherwise.}
\end{cases}
\]

We can write

\[
\bar{f} \geq \sum_{a \geq 0} X_a Pr\{X_a = 1\} \overset{(a)}{=} \sum_{k \geq 1} Pr\{a > (\alpha - 1)a_1|a = k\} \times Pr\{a = k\}
\]  (32)

\[
= e^{-\bar{a}} \sum_{k=1}^{\infty} \Phi\left(\frac{k}{\sqrt{\bar{a}_1}}\right) \frac{\bar{a}_1^k}{k!}
\]  (33)

where \((a)\) is due to Theorem 8, and \((b)\) follows from the Gaussian distribution of \( a_1 \) and Poisson distribution of \( a \).

Fig. 7 depicts the lower bound from Theorem 7 along with the simulation result for the average failure ratio. The upper bound from Theorem 3 is also shown for the comparison. As we see, the two bounds together successfully predict the interval within which the failure ratio decreases from 1 to 0.

**A. Asymptotic Analysis of the Lower Bound**

Here, we look at the lower bound obtained in the last section from an asymptotic point of view. As \( \lambda \) grows very large, similar to what was observed for the upper bound in Section IV, the lower bound takes
the shape of a step function. That is, there exists a value of $\alpha$, denoted as $\alpha_L$, such that $\vec{f}$ takes the value of 1 for $\alpha < \alpha_L$, and it takes the value of 0 for $\alpha \geq \alpha_L$. The following theorem derives the value of $\alpha_L$.

**Theorem 9.** Consider the probability space $G(\lambda, R)$ when $\lambda \to \infty$. Suppose that a dish attack of radius $R_a$ is applied to $G(\lambda, R)$. Also let $q$ denote the ratio $R_a/R$. If

$$\alpha < \alpha_L \triangleq 1 + \frac{q^2}{1+2q},$$

all the nodes would fail.

Fig. 8 depicts the variation of $\alpha_L$ over $q = R_a/R$. As it can be seen, $\alpha_L$ grows sub-linearly with $R_a/R$.

The following lemmas help us prove Theorem 9. While Lemma 6 holds for the asymptotic case, Lemma 8 below is the asymptotic version of Lemma 5. Also, Lemma 9 below can be interpreted as the asymptotic version of Lemma 7. The proofs can be found in the Appendix.

**Lemma 8.** Consider a dish attack of radius $R_a$ applied to $G(\lambda, R)$. Let $q = R_a/R$. If $\alpha < 1 + \frac{q^2}{1+2q}$, then:

- $\bar{a}_1(\alpha - 1) < \bar{a}$,
- $\bar{a}_2(\alpha - 1) < \bar{a}_1 + \bar{a}$.
Lemma 9. Consider the setting of Lemma 8. If $\alpha - 1 < \frac{q^2}{1 + 2q}$, then for every $i \geq 1$ we have

$$\bar{a} + \bar{a}_1 + ... + \bar{a}_i > \bar{a}_{i+1}(\alpha - 1).$$  

(35)

Now we are ready to prove Theorem 9.

Proof of Theorem 9: The proof is mostly along the same lines as in Theorem 8 with a few minor changes. First note that as $\lambda$ grows very large, the network become connected. So there is no need to consider the subspace $G_c(\lambda, R)$ here. Second, $a$ is given as the initial number of failed nodes due to the attack. However, when $\lambda \to \infty$, $a$ tends to $\bar{a}$. Similarly, $a_i$ tends to $\bar{a}_i$ for $i \geq 1$. Just like Theorem 8, in the best scenario, $\bar{a}_1(\alpha - 1)$ is the excess capacity available to absorb the load of the $\bar{a}$ failed nodes. If $\alpha - 1 < \frac{q^2}{1 + 2q}$, then $\bar{a} > \bar{a}_1(\alpha - 1)$. In this case, $A_1$ cannot absorb the load of $A$, and the aggregate load of $\bar{a} + \bar{a}_1$ needs to be absorbed by the rest of the nodes. However, Lemma 9 asserts that such an absorbtion will not be realized as the failure propagates through $A_2$, $A_3$, and the outer rings until it causes a failure of the whole network.

VI. Conclusion

This paper investigates the problem of cascading failures in finite-size networks modeled by random geometric graphs. Rigorous analytical results have been provided for network resiliency under a dish
attack of arbitrary size. In particular, the average failure ratio due to the attack was studied in terms of the tolerance parameter, which is a critical design consideration in real-life networks. By deriving lower and upper bounds on the average failure ratio, we were able to track the network reaction to different attacks. The asymptotic analysis of both bounds has been presented. Particularly, the asymptotic analysis of the upper bound revealed the threshold behavior of the network reaction to the changes in the tolerance parameter. Our findings can be exploited to choose appropriate values of the tolerance parameter to avoid a cascade.

**APPENDIX**

**Proof of Lemma 1:** Note that

\[
\bar{a}_i = \pi \lambda [(R_a + R)^2 - R_a^2] = \pi \lambda R^2 + 2 \pi \lambda RR_a
\]

\[
\geq 6 + 2\sqrt{\pi^2 \lambda^2 R^2 R_a^2} \geq 6 + 2\sqrt{3 \times 6} = 6(1 + \sqrt{2}) > 14.
\]

It is easy to see that \( \bar{a}_{i+1} \geq \bar{a}_i \) when \( i \geq 1 \).

**Proof of Lemma 2:** Consider the angle \( \hat{\theta} \) in Fig. 3 and the arc of radius \( r \) associated with \( \hat{\theta} \) crossing over the node \( i \). Let us denote the length of this arc by \( \omega \). The PDF of \( r \) can be obtained by considering the probability of \( i \) being located on this arc and its tiny vicinity shown with the dashed line. We have

\[
\psi(r) = \frac{2\omega}{I(0, R_a, r_v, R)}.
\]

Given \( \hat{\theta}, \omega \) can be found as \( \omega = \hat{\theta} \times r \). So we only need to find \( \hat{\theta} \). Note that \( \hat{\theta} \) is an angle in a triangle with sides \( r, r_v, \) and \( R \). Particularly, \( \hat{\theta} \) is opposite to the side of length \( R \). Therefore, we have

\[
\hat{\theta} = \arccos \left( \frac{r_v^2 - R^2 + r^2}{2r_v r} \right).
\]

The equation above holds when \( r + r_v > R \). When \( r + r_v \leq R \), we simply have \( \hat{\theta} = \pi \).

**Proof of Lemma 3:** We have

\[
E[1/d_u] = \frac{1}{k} \times \Pr\{d_u = k \mid d_u \geq 1\}
\]

\[
E[1/d_u^2] = \frac{1}{k^2} \times \Pr\{d_u = k \mid d_u \geq 1\}
\]
Since $d_u \sim \text{Poi}(\lambda_u)$, we have
\[
\Pr\{d_u = k \mid d_u \geq 1\} = \frac{e^{-\lambda_u} \lambda_u^k}{k!(1 - e^{-\lambda_u})}. \quad (42)
\]
Therefore,
\[
E[1/d_u] = \frac{e^{-\lambda_u}}{1 - e^{-\lambda_u}} \sum_{k=1}^{\infty} \frac{1}{k} \lambda_u^k, \quad (43)
\]
\[
E[1/d_u^2] = \frac{e^{-\lambda_u}}{1 - e^{-\lambda_u}} \sum_{k=1}^{\infty} \frac{1}{k^2} \lambda_u^k. \quad (44)
\]
To find a closed form for the summations above note that we have
\[
\sum_{k=0}^{\infty} \frac{1}{k+1} \frac{\lambda_u^{k+1}}{(k+1)!} = \int \sum_{k=0}^{\infty} \frac{\lambda_u^k}{(k+1)!} \ d\lambda_u. \quad (45)
\]
For the expression under the integral, we can use the Taylor expansion of an exponential function as follows.
\[
\sum_{k=0}^{\infty} \frac{\lambda_u^k}{(k+1)!} = \frac{1}{\lambda_u} \sum_{k=1}^{\infty} \frac{\lambda_u^k}{k!} = e^{\lambda_u} - 1. \quad (46)
\]
Substituting (46) in (45) yields (9). Note that integral in (9) can be found numerically. Along the same lines, we will find the following for the summation in (44).
\[
\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{\lambda_u^k}{k!} = \int_{-\infty}^{\lambda_u} \sum_{k=1}^{\infty} \frac{1}{k} \frac{x^{k-1}}{k!} \ dx = \int_{-\infty}^{\lambda_u} \frac{1}{x} \int_{-\infty}^{x} \sum_{k=0}^{\infty} \frac{y^k}{(k+1)!} \ dy = \int_{-\infty}^{\lambda_u} \frac{g(x)}{x} \ dx. \quad (47)
\]
Substituting (47) in (44) gives us (8).

**Proof of Lemma 4:** The load of $i$ will be redistributed equally among its neighbors outside $A$. Let us denote the number of such neighbors by $d_u$. The average load received by $v$ is $h^{(1)}(r) = E[1/d_u | r, d_u > 0]$ where the average is taken over $d_u$. Given that $i$ is located at $r$, $d_u$ is distributed as a Poisson($\lambda_u = \lambda(\pi R^2 - I(r, R, 0, R_a))$). As a result, by using Lemma 3 we have
\[
h^{(1)}(r) = E[1/d_u | r, d_u > 0] = \frac{e^{-\lambda J(r)} g(\lambda J(r))}{1 - e^{-\lambda J(r)}}. \quad (48)
\]
If $i$ is located randomly and uniformly on $I(v, R, 0, R_a)$, then the probability density function of its distance from the center is given by $\psi(r)$ at (6). Using $\psi(r)$ to find $E[h^{(1)}(r)]$ and $E[h^{(2)}(r)]$ by taking
average over \( r \) leads to (12) and (13). When \( R - r_v \geq R_a \), the attacked region entirely resides in \( v \)'s neighborhood; hence, the averaging of (12) or (13) over \( r \) will not be needed, and we will get (14).

**Proof of Theorem 1:** Let us denote the number of neighbors of \( v \) inside \( \mathcal{I}(r_v, R, 0, R_a) \) by \( N \). Note that \( N \sim \text{Poi}(\lambda I(r_v, R, 0, R_a)) \). Let \( l_u, u = 1, ..., N \) denote the sequence of r.v.'s corresponding to the load redistributed to \( v \) by its neighbors. As it has been shown in the proof of Lemma 4, \( l_u \) completely depends on a Poisson point process outside \( \mathcal{I}(r_v, R, 0, R_a) \), whereas \( N \) is given by a Poisson point process inside \( \mathcal{I}(r_v, R, 0, R_a) \). Hence, the random variables \( l_u, u = 1, ..., N \) and \( N \) are independent. Therefore, we can write

\[
E[L_v] = E[\sum_{u=1}^{N} l_u] = E[E[\sum_{u=1}^{N} l_u | N = n]]
\]

\[
= a E[N] \times E[l_u] = b \lambda I(r_v, R, 0, R_a) \times \int_{r_v - R}^{R_a} h(r) \times \psi(r) \ dr,
\]

where \(a\) follows from Wald's identity [16] and the fact that a Poisson point process, given the number of points, will become a uniform point process. Also, \(b\) follows from substituting \( E[l_u] \) with (12) from Lemma 4. Similarly, for \( \sigma_L \) we have

\[
\sigma_{L_v}^2 = E[(L_v - \bar{L}_v)^2] = E[E[(\sum_{u=1}^{N} (l_u - \bar{l}_u))^2 | N = n]]
\]

\[
= E[N] \times E[(l_u - \bar{l}_u)^2] = \lambda I(r_v, R, 0, R_a) \times \sigma_{l_u}^2.
\]

When \( R - r_v \geq R_a \), \( \mathcal{A} \) is entirely included in \( v \)'s neighborhood. Thus, \( E[N] = \lambda \pi R_a^2 \), and \( E[l_u] \) is given by (14), together leading to (18).

**Proof of Theorem 2:** Since \( v \) is located randomly and uniformly in \( \mathcal{A}_1 \), the PDF of \( r_v \) is obtained as \( \frac{2 \pi r_v}{|\mathcal{A}_1|} \). Thus we have

\[
p_1 = \int_{R_a}^{R_a+R} \Pr\{L_v \leq \alpha - 1\} \times \frac{2 \pi r_v}{|\mathcal{A}_1|} \ dr_v.
\]

It remains to find \( \Pr\{L_v \leq \alpha - 1\} \). According to Corollary 1, if \( \lambda \times I(r_v, R, 0, R_a) \gg 1 \) then \( \Pr\{L_v \leq \alpha - 1\} \approx \Phi\left(\frac{\alpha - 1 - L_v}{\sigma_{L_v}}\right) \). However, this holds when either \( r_v \) is close to \( R_a \) or \( \lambda \) is large. If \( v \) is located far from the edge of the attack, then \( I(r_v, R, 0, R_a) \) might be small, and \( \lambda \times I(r_v, R, 0, R_a) \gg 1 \) may not hold for moderate values of \( \lambda \). However, as \( I(r_v, R, 0, R_a) \) becomes smaller, \( J(\cdot) \) in (10) and (11) becomes larger for neighbors of \( v \) in \( \mathcal{A} \), causing \( E[l_u] \) and \( E[l^2_u] \) to drop quickly for these neighbors. At the
same time, the number of such neighbors, which is a Poisson r.v. with mean \( \lambda \times I(r_v, R, 0, R_a) \), becomes smaller. Therefore, as \( r_v \) grows larger, both \( \bar{L}_v \) and \( \sigma_{L_v} \) in (16) drop quickly until they become zero when \( r_v = R_a + R \). As a result, \( \Pr\{L_v \leq \alpha - 1\} \) grows rapidly as \( r_v \) increases, and becomes very close to 1.

This can also be verified using numerical methods. Now, since

\[
(1 - \frac{1}{L_v})
\]

also takes values very close to one in such cases, approximating \( \Pr\{L_v \geq \alpha - 1\} \) by

\[
(1 - \frac{1}{L_v})
\]

even when \( \lambda \times I(r_v, R, 0, R_a) \) is not large will have a negligible effect on the value of \( p_1 \). Using such an approximation leads to (21).

**Proof of Theorem 3:** Since \( 0 \leq f \leq 1 \), we have

\[
\bar{f} \leq \Pr\{f = 0\} \times 0 + \Pr\{f > 0\} \times 1 = \Pr\{f > 0\} = \Pr\{\text{at least 1 failure in } S \setminus A\}
\]

\[
= \Pr\{\text{at least 1 failure in } A_1\} = 1 - p_0,
\]

where \( p_0 \overset{\Delta}{=} \Pr\{\text{no failure in } A_1\} \). Note that \( p_0 \) is the probability that the load received by every node in \( A_1 \) is less than \( \alpha - 1 \). Let us denote by \( P(k) \) the probability that there are \( k \) nodes in \( A_1 \). Also recall that \( p_1 \) is the probability that the load received by a node located randomly and uniformly in \( A_1 \) is less than \( \alpha - 1 \). We have

\[
p_0 = \sum_{k=0}^{\infty} P(k)p_1^k = \sum_{k=0}^{\infty} e^{-\lambda_1} \frac{\lambda_1^k}{k!} p_1^k = e^{-\lambda_1} \sum_{k=0}^{\infty} \frac{(\lambda_1 p_1)^k}{k!} = e^{-\lambda_1} e^{\lambda_1 p_1} = e^{-\lambda_1 (1 - p_1)}.
\]

(55)

Substituting \( p_1 \) above by its value given by Theorem 2, and then substituting (55) into (54) gives us (22).

**Proof of Theorem 4:** In order to prove the theorem, we will show that, as \( \lambda \to \infty, \sigma_{L_v} \to 0 \), and \( E[L(r_v)] \) takes the righthand side of (24). First note that when \( \lambda \to \infty \), by applying the L’Hopital’s rule to (10) and (11) we get

\[
\bar{h}^{(1)}(r) = E[l_u | r] \to_{\lambda \to \infty} \frac{1}{\lambda J(r)},
\]

(56)

\[
\bar{h}^{(2)}(r) = E[l_u^2 | r] \to_{\lambda \to \infty} \frac{1}{[\lambda J(r)]^2}.
\]

(57)

Substituting (56) into (16) gives us the asymptotic average of \( L(r_v) \) in (24). Now let us show that the asymptotic value of \( \sigma_{L_v} \) tends to 0. Using (12), (13), and (15), we find that

\[
\sigma_{L_u}^2 \to_{\lambda \to \infty} \frac{m(r_v)}{\lambda^2},
\]

(58)
where

$$m(r_v) = \frac{2}{I(r_v, R, 0, R_a)} \int_{r_v - R}^{R_a} r \arccos \left( \frac{r_v^2 - R^2 + r^2}{2r_v r} \right) dr$$

$$- \frac{4}{[I(r_v, R, 0, R_a)]^2} \left[ \int_{r_v - R}^{R_a} r \arccos \left( \frac{r_v^2 - R^2 + r^2}{2r_v r} \right) dr \right]^2$$

(59)

is a function of \(r_v\), taking only finite values. Now from (17) for \(\sigma^2_{L_v}\) we have

$$\sigma^2_{L_v} = \lambda I(r_v, R, 0, R_a) \times \sigma^2_{\mu} \xrightarrow{\lambda \to \infty} \frac{I(r_v, R, 0, R_a)}{\lambda} m(r_v) \xrightarrow{\lambda \to \infty} 0.$$  

(60)

Proof of Theorem 5: In the best case, let us assume that all the nodes located at \(r > r_v\) are healthy and have received no load so far. We will show that \(v\) still fails in this case. Note that this assumption is equivalent to having a dish attack of radius \(r_v \geq R_a\). Let us denote by \(L(r)\) and \(L'(r)\) the asymptotic load distribution right after the attack for attacks of radius \(R_a\) and \(r_v\), respectively. Since the latter is a larger attack, we have

$$L'(r_v) \geq L(R_a) = \alpha_U - 1 > \alpha - 1.$$  

(61)

Therefore \(v\) fails. Applying the same procedure to nodes located at \(r > r_v\) results in a propagation of failures throughout the network, leading to \(\bar{f} = 1\).

Proof of Theorem 6: Clearly, the closer a node is to the attack region, the larger is the load it receives right after the attack. Therefore, we have \(L(R_a) > L(r_v)\), for \(r_v > R_a\). Consequently, for \(\alpha \geq \alpha_U\), we have

$$L(r_v) < L(R_a) = \alpha_U - 1 \leq \alpha - 1,$$  

(62)

which means that none of the nodes in \(A_1\) would fail. Hence, there will not be any propagation of failures, resulting in \(\bar{f} = 0\). It remains to prove that a cascade of failures is assured if \(\alpha < \alpha_U\). In this case we have \(L(R_a) = \alpha_U - 1 > \alpha - 1\), which means that every node located at \(R_a\) would fail. Now, by simply applying Theorem 5 for \(r_v = R_a\) we will have \(\bar{f} = 1\).
Proof of Lemma 5: We need to show that
\[
\frac{\bar{a}_2}{\bar{a}_1} = \frac{(R_a + 2R)^2 - (R_a + R)^2}{(R_a + R)^2 - R_a^2} = 1 + \frac{2R}{R + 2R_a} < 1 + \frac{1}{\alpha - 1},
\]
\[\Rightarrow \alpha - 1 < \frac{R + 2R_a}{2R} = 1/2 + \frac{R_a}{R}\]
\[\Rightarrow \alpha - 1 < 1/2 + q. \tag{63}\]

However, (70) is given by lemma’s assumption, which completes the proof. □

Proof of Lemma 6: We need to show that
\[
\bar{a}_{i+1} \times \bar{a}_{i-1} \leq \bar{a}_i^2
\]
\[\Rightarrow (R_{i+1}^2 - R_i^2)(R_{i-1}^2 - R_{i-2}^2) \leq (R_i^2 - R_{i-1}^2)^2\]
\[\Rightarrow (2R_a + (2i + 1)R)(2R_a + (2i - 3)R) \leq (2R_a + (2i - 1)R)^2. \tag{64}\]

If we set \(x_1 = 2R_a + (2i + 1)R\) and \(x_2 = 2R_a + (2i - 3)R\), then (64) can be deduced from the “inequality of arithmetic and geometric means” [17] asserting that for two non-negative numbers \(x_1\) and \(x_2\) we have
\[
\frac{x_1 + x_2}{2} \geq \sqrt{x_1x_2} \Rightarrow \left(\frac{x_1 + x_2}{2}\right)^2 \geq x_1x_2. \tag{65}\]

Now that (29) is proved, (30) can be obtained by simply applying the second item of Lemma 5. □

Proof of Lemma 7: We prove the theorem by induction. For \(i = 0\) the equality holds. Let us assume the theorem holds for \(i = k - 1, k > 1\); we will prove that it needs to hold for \(i = k\) as well. We have
\[
\Pr\{a_0 + a_1 + \ldots + a_k > a_{k+1}(\alpha - 1)\} \overset{(a)}{\geq} \Pr\{a_0 + a_1 + \ldots + a_{k-1} + a_k > a_k\alpha\}
\]
\[\overset{(b)}{=} \Pr\{a_0 + a_1 + \ldots + a_{k-1} > a_k(\alpha - 1)\} \geq \Pr\{a_0 > a_1(\alpha - 1)\}, \tag{66}\]
where (a) holds due to Lemma 6 and using the Gaussian distribution for \(a_i, i \geq 1\). In the above, (b) holds because of the induction assumption made for \(i = k - 1\). □

Proof of Theorem 8: Here \(a_0\) is given as the initial number of failed nodes due to the attack. In its best, \(a_1(\alpha - 1)\) is the excess capacity available to absorb the load of these \(a_0\) failed nodes. Now consider a best-case load distribution strategy where all the nodes in \(\mathcal{A}\) can collaborate and all the nodes in \(\mathcal{A}_1\) are connected to \(\mathcal{A}\). Then, nodes in \(\mathcal{A}\) can distribute their loads equally among the set \(\mathcal{A}_1\) in order to use all the excess capacity and avoid a cascade. If \(a_0 > a_1(\alpha - 1)\), \(\mathcal{A}_1\) cannot absorb the load of \(\mathcal{A},\)
and the aggregate load of $a_0 + a_1$ needs to be absorbed by rest of the nodes. However, Lemma 7 asserts that such an absorption becomes even less likely, and the failure propagates through $A_2$, $A_3$, and outer rings until it takes out the whole network. Therefore, the probability of a total failure is lower-bounded by $\Pr\{a_0 > a_1(\alpha - 1)\}$.

**Proof of Lemma 8:** In order to prove the first item, note that we have

$$\bar{a}_1(\alpha - 1) < \bar{a} \Rightarrow \alpha - 1 < \frac{\bar{a}}{\bar{a}_1} = \frac{R_a^2}{R^2 + 2R_aR} = \frac{q^2}{1 + 2q}. \tag{68}$$

Moving on to prove the second item, given that $a_1(\alpha - 1) < a_0$, it suffices to show that

$$\bar{a}_2(\alpha - 1) < \alpha \bar{a}_1 = \bar{a}_1 + \bar{a}_1(\alpha - 1) < \bar{a}_1 + \bar{a}. \tag{69}$$

From Lemma 5, we already know that

$$\alpha - 1 < 1/2 + q \Rightarrow \bar{a}_1 \times \alpha > \bar{a}_2(\alpha - 1). \tag{70}$$

To complete the proof, we only need to show that (68) leads to (70). That is we need to have

$$\frac{q^2}{1 + 2q} < 1/2 + q. \tag{71}$$

Inequality (71) can be rewritten as

$$q^2 + 2q + 1/2 > 0 \Rightarrow (q + 1)^2 - 1/2 > 0 \Rightarrow (q + 1 - \frac{1}{\sqrt{2}})(q + 1 + \frac{1}{\sqrt{2}}) > 0, \tag{72}$$

which always holds since $q > 0$. This completes the proof.

**Proof of Lemma 9:** The proof is simple and is given by induction on $i$. First, note that for $i = 1$, the case is proven by Lemma 8. Now suppose the statement is true for $i = k - 1$. We show that it holds for $i = k$, $k \geq 2$, as well. For $i = k$ we have

$$(a_0 + \bar{a}_1 + \ldots + \bar{a}_{k-1}) + \bar{a}_k > \bar{a}_k(\alpha - 1) + \bar{a}_k$$

$$= \bar{a}_k \times \alpha \geq \bar{a}_{k+1}(\alpha - 1), \tag{73}$$

where the last inequality holds due to Lemma 6.
REFERENCES


