Optimal Nonpreemptive Scheduling in a Smart Grid Model

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Abstract

We study a scheduling problem arising in demand response management in smart grid. Consumers send in power requests with a flexible feasible time interval during which their requests can be served. The grid controller, upon receiving power requests, schedules each request within the specified interval. The electricity cost is measured by a convex function of the load in each timeslot. The objective is to schedule all requests with the minimum total electricity cost. Previous work has studied cases where jobs have unit power requirement and unit duration, and offline setting was considered. We extend the study to arbitrary power requirement and duration, which has been shown to be NP-hard. We give the first online algorithm for the problem, and prove the competitive ratio is optimal. We also prove that the problem is fixed parameter tractable.

1 Introduction

We study a scheduling problem arising in “demand response management” in smart grid. The electrical smart grid is one of the major challenges in the 21st century. The smart grid is a power grid system that makes power generation, distribution and consumption more efficient through information and communication technologies against the traditional power system. Peak demand hours happen only for a short duration, yet makes existing electrical grid less efficient. It has been noted in that in the US power grid, 10% of all generation assets and 25% of distribution infrastructure are required for less than 400 hours per year, roughly 5% of the time.

Demand response management attempts to overcome this problem by shifting users’ demand to off-peak hours in order to reduce peak load. The electricity grids supports demand response mechanism and obtains energy efficiency by organizing customer consumption of electricity in response to supply conditions. It is demonstrated in that demand response is of remarkable advantage to consumers, utilities, and society. Effective demand load management brings down the cost of operating the grid, as well as energy generation and distribution. Demand response management is not only advantageous to the supplier but also to the consumers as well. It is common that electricity supplier charges according to the generation cost, i.e., the higher the generation cost the higher the electricity price. Therefore, it is to the consumers’ advantage to reduce electricity consumption at high price and hence reduce the electricity bill.

The smart grid operator and consumers communicate through smart metering devices. A consumer sends in a power request with the power requirement, required duration of service, and the time interval that this request can be served (giving some flexibility). For example, a consumer may want the dishwasher to operate for one hour during the periods from 8am to 11am. The grid operator upon receiving requests has to schedule them in their respective time intervals using the minimum energy cost. The load of the grid at each timeslot is the sum of the power requirements of all requests allocated to that timeslot. The electricity cost is modeled by a convex function on the load, in particular we consider the cost to be the α-th power of the load, where α > 1 is some constant.
Table 1: Summary of our online results.

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<tr>
<td>Unit</td>
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<td>$2^{\alpha} \cdot (8e^\alpha + 1)$-competitive $2^{\alpha+1}$-approximate</td>
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<td>Uniform</td>
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<td>Unit</td>
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<td>min((4\alpha)^{\alpha}/2 + 1, 2^{\alpha} \cdot (8e^\alpha + 1))-competitive agreeable deadline</td>
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<tr>
<td>Arbitrary</td>
<td>Uniform</td>
<td>((12\alpha)^{\alpha}/2 + 1)-competitive same release time or same deadline</td>
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**Previous work.** Koutsopoulos and Tassiulas [24] has formulated a similar problem to our problem where the cost function is piecewise linear. They show that the problem is NP-hard, and their proof can be adapted to show the NP-hardness of the general problem studied in this paper [6]. Burcea et al. [6] gave polynomial time optimal algorithms for the case of unit height (c.f. unit power requirement) and unit width (c.f. duration of request). Salinas et al. [38] considered a multi-objective problem to minimize energy consumption cost and maximize some utility. A closely related problem is to manage the load by changing the price of electricity over time [7, 33, 35, 13]. Heuristics have also been developed for demand side management [30]. Other aspects of smart grid have also been considered, e.g., communication [26, 8, 27, 28], security [32, 28]. Reviews of smart grid can be found in [19, 20, 31, 46, 14].

The combinatorial problem we defined in this paper has analogy to the traditional load balancing problem [3] and machine minimization problem [9, 10, 11] but the main differences are the objective being maximum load and jobs are unit height [9, 10, 11]. Minimizing maximum load has also been looked at in the context of smart grid [11, 23, 59, 14, 55], some of which further consider allowing reshaping of the jobs [11, 23]. As to be discussed in Section 2, our problem is more difficult than minimizing the maximum load. Our problem also has resemblance to the dynamic speed scaling problem [2, 43, 5] and our algorithm has employed some techniques there.

**Our contribution.** In this paper, we consider a demand response optimization problem. We propose the first online algorithm with worst case competitive ratio, which is logarithm in the max-min ratio of the duration of jobs (Theorem 23 in Section 4); and show that the competitive ratio is asymptotically optimal. The algorithm is based on an $O(1)$-competitive online algorithm for jobs with uniform duration (Section 3). We also propose $O(1)$-competitive online algorithms for some special cases (Section 5). In addition, we show that the Grid problem is fixed parameter tractable by proposing the first fixed parameter exact algorithms for the problem; and derive lower bounds on the running time (Section 6). Table 1 gives a summary of our results.

Technically speaking, our online algorithms are based on identifying a relationship with a related problem called dynamic speed (voltage) scaling (DVS) problem [43]. Roughly speaking, we construct a schedule based on a schedule for DVS with some necessary modifications. On the other hand, in time regions that the load is low, the optimal DVS schedule may have much lower cost than the optimal grid schedule, in which case, it is not sufficient to simply compare with the optimal DVS schedule. Yet we find a way to work around this technical difficulty (Lemma 5 and Theorem 27). The exact algorithms we developed are based on dynamic programming where we identify properties to reduce the search space of the feasible schedules.

## 2 Definitions and preliminaries

**The input.** The time is labeled from 0 to $\tau$ and we consider events (release time, deadlines) occurring at integral time. We call the unit time $[t, t+1)$ timeslot $t$. We denote by $J$ a set of input jobs in
which each job \( J \) comes with release time \( r(J), \) deadline \( d(J), \) width \( w(J) \) representing the duration required by \( J, \) and height \( h(J) \) representing the power required by \( J. \) We assume \( r(J), d(J), w(J), \) and \( h(J) \) are integers. The feasible interval, denoted by \( I(J), \) is defined as the interval \([r(J), d(J))\) and we say that \( J \) is available during \( I(J). \) We define the density of \( J, \) denoted by \( \text{den}(J), \) to be \( \frac{h(J)\text{\normalfont\text{\textwidth}(J)}}{d(J) - r(J)}.\) Roughly speaking, the density signifies the average load required by the job over its feasible interval. We then define the “average” load at any time \( t \) as \( \text{avg}(t) = \sum_{J \in I(J)} \text{\normalfont\text{\textwidth}(J)} / \text{\normalfont\text{\textwidth}}(J). \) In our analysis, we have to distinguish timeslots with high and low average load. Therefore, for any \( h > 0, \) we define \( I_{>h} \) and \( I_{\leq h} \) to be set of timeslots where the average load \( \text{avg}(t) \) is larger than \( h \) and at most \( h, \) respectively. Note that \( I_{>h} \) and \( I_{\leq h} \) do not need to be contiguous.

In Section \[4\] we consider an algorithm that classifies jobs according to their widths. To ease discussion, we let \( w_{\text{max}} \) and \( w_{\text{min}} \) be the maximum and minimum width over all jobs, respectively. We further define the logarithm of the max-min ratio of width, denoted by \( k_w, \) to be \( k_w = \lceil \log \frac{w_{\text{max}}}{w_{\text{min}}} \rceil, \) which is the number of classes to be defined. Without loss of generality, we assume that \( w_{\text{min}} = 1. \) We say that a job \( J \) is in class \( C_p \) if and only if \( 2^{p-1} < w(J) \leq 2^p \) for any \( 0 \leq p \leq k_w. \)

**Schedule.** A schedule \( S \) has to assign for each job \( J \) a start time \( st(S, J) \in \mathbb{Z} \) meaning that \( J \) runs from \([st(S, J), et(S, J))\), where the end time \( et(S, J) = st(S, J) + w(J). \) Note that this means preemption is not allowed. The load of \( S \) at time \( t, \) denoted by \( \ell(S, t) = \sum_{J \in I(t)} h(J), \) is the sum of the height (power request) of all jobs running at \( t, \) i.e., \( \ell(S, t) = \sum_{J \in I(t)} \text{\normalfont\text{\textwidth}(J)} h(J). \) We drop \( S \) and use \( \ell(t) \) when the context is clear. For any algorithm \( A, \) we use \( A(\mathcal{J}) \) to denote the schedule of \( A \) on \( J. \) We denote by \( O \) the optimal algorithm.

The cost of a schedule \( S \) is the sum of the \( \alpha \)-th power of the load over all time, for a constant \( \alpha > 1, \) i.e., \( \text{cost}(S) = \sum_{t} \ell(t)^{\alpha}. \) For a set of timeslots \( \mathcal{T} \) (not necessarily contiguous), we denote by \( \text{cost}(S, \mathcal{T}) = \sum_{t \in \mathcal{T}} \ell(t)^{\alpha}. \) We call this the Grid problem.

**Online algorithms.** In this paper, we consider online algorithms, where the job information is only revealed at the time the job is released; the algorithm has to decide which jobs to run at the current time without future information and decisions made cannot be changed later. Let \( A \) be an online algorithm. We say that \( A \) is \( c \)-competitive if for all input job sets \( \mathcal{J}, \) we have \( \text{cost}(A(\mathcal{J})) \leq c \cdot \text{cost}(O(\mathcal{J})). \) In particular, we consider non-preemptive algorithms where a job cannot be preempted to resume/restart later.

**Special input instances.** We consider various special input instances. A job \( J \) is said to be unit-width (resp. unit-height) if \( w(J) = 1 \) (resp. \( h(J) = 1 \)). A job set is said to be uniform-width (resp. uniform-height) if the width (resp. height) of all jobs are the same. A job set is said to have agreeable deadlines if for any two jobs \( J_1 \) and \( J_2, \) \( r(J_1) \leq r(J_2) \) implies \( d(J_1) \leq d(J_2). \)

**Relating to the speed scaling problem.** The Grid problem resembles the dynamic speed scaling (DVS) problem \([33]\) and we are going to refer to three algorithms for the DVS problem, namely, the YDS algorithm which gives an optimal algorithm for the DVS problem, the online algorithms called BK\(P\) and \( A\mathcal{V}\mathcal{R}. \) We first recall the DVS problem and the associated algorithms. In the DVS problem, jobs come with release time \( r(J), \) deadline \( d(J), \) and a work requirement \( p(J). \) A processor can run at speed \( s \in [0, \infty) \) and consumes energy in a rate of \( s^\alpha, \) for some \( \alpha > 1. \) The objective is to complete all jobs by their deadlines using the minimum total energy. The main differences of DVS problem to the Grid problem include (i) jobs in DVS can be preempted while preemption is not allowed in our problem; (ii) as processor speed in DVS can scale, a job can be executed for varying time duration as long as the total work is completed while in our problem a job must be executed for a fixed duration given as input; (iii) the work requirement \( p(J) \) of a job \( J \) in DVS can be seen as \( w(J) \times h(J) \) for the corresponding job in Grid.

With the resemblance of the two problems, we make an observation about their optimal algorithms. Let \( O_D \) and \( O_G \) be the optimal algorithm for the DVS and Grid problem, respectively. Given a job set \( \mathcal{J}_G \) for the Grid problem, we can convert it into a job set \( \mathcal{J}_D \) for DVS by keeping the release time and deadline for each job and setting the work requirement of a job in \( \mathcal{J}_D \) to the product of the width and height of the corresponding job in \( \mathcal{J}_G. \) Then we have the following observation.

**Observation 1.** Given any schedule \( S_G \) for \( \mathcal{J}_G, \) we can convert \( S_G \) into a feasible schedule \( S_D \) for \( \mathcal{J}_D \) such that \( \text{cost}(S_D(\mathcal{J}_D)) \leq \text{cost}(S_G(\mathcal{J}_G)); \) implying that \( \text{cost}(O_D(\mathcal{J}_D)) \leq \text{cost}(O_G(\mathcal{J}_G)). \)
Proof. Consider any feasible schedule \( S_G \). At timeslot \( t \), suppose there are \( k \) jobs scheduled and their sum of heights is \( H \). The schedule for \( S_D \) during timeslot \( t \) can be obtained by running the processor at speed \( H \) and the jobs time-share the processor in proportion to their height. This results in a feasible schedule with the same cost and the observation follows.

It is known that the online algorithm \( AVR \) for the DVS problem is \( \frac{(2\alpha)^\alpha}{2} \)-competitive \[18\]. Basically, at any time \( t \), \( AVR \) runs the processor at a speed which is the sum of the densities of jobs that are available at \( t \). By Observation \[1\] we have the following corollary. Note that it is not always possible to convert a feasible schedule for the DVS problem to a feasible schedule for the Grid problem easily. Therefore, the corollary does not immediately solve the Grid problem but as to be shown it provides a way to analyze algorithms for Grid.

**Corollary 2.** For any input \( J_G \) and the corresponding input \( J_D \), \( \text{cost}(AVR(J_D)) \leq \frac{(2\alpha)^\alpha}{2} \cdot \text{cost}(O_G) \).

The online algorithm \( BKP \) proposed by Bansal et al. \[1\] for DVS problem is \( 8e\alpha \)-competitive. Let \( \ell(BKP, t) \) denote the speed of \( BKP \) at time \( t \). \( \ell(BKP, t) = \max_{t' < t} \frac{p(t, t', J)}{t'-t} \) where \( p(t, I) \) denotes the total work of jobs \( J \) with \( I(J) \subseteq I \) and \( r(J) \leq t \). That is, \( BKP \) chooses the interval \( I^* = [t_1, t_2] \) which has maximal released average load and \( |[t_1, t_2]| = e : 1 \) and uses \( e : \frac{p(t, J)}{|I^*|} \) as speed at \( t \). By Observation \[1\] we have the following corollary:

**Corollary 3.** For any input \( J_G \) and the corresponding input \( J_D \), \( \text{cost}(BKP(J_D)) \leq 8e\alpha \cdot \text{cost}(O_G) \).

**Relating to minimizing maximum cost.** The problem of minimizing maximum cost over time (min-max) has been studied before \[16\]. We note that since \( \alpha \) is large enough \( \alpha \) for a large enough \( \alpha \), the maximum load would dominate the load in other timeslots and we would be able to solve the min-max problem if we have a solution for the min-sum problem on \( \alpha \).

On the other hand, minimizing the maximum cost does not necessarily minimize the total cost. For example, consider an input of three jobs \( J_1, J_2 \) and \( J_3 \) where \( I(J_1) = [0, 2\alpha] \), \( h(J_1) = 1 \), \( w(J_1) = 2\alpha \); \( I(J_2) = [2\alpha, 2\alpha + 1] \), \( h(J_2) = 3 \), \( w(J_2) = 1 \); and \( I(J_3) = [0, 2\alpha + 1] \), \( h(J_3) = 1 \), \( w(J_3) = 2\alpha \). Note that only \( J_3 \) has flexibility where it can be scheduled. To minimize the maximum cost over time, we would schedule \( J_3 \) to start at time 0 and achieve a maximum load of 3. This gives a total cost of \( 2\alpha \cdot 2\alpha + 3\alpha = 4\alpha + 3\alpha \). However, to minimize the total cost, we would schedule \( J_3 \) to start at time 2\alpha giving a total cost of \( 2\alpha + 4\alpha + (2\alpha - 1) = 4\alpha + 2\alpha + 1 - 1 \), which is smaller than \( 4\alpha + 3\alpha \) when \( \alpha > 1 \).

### 3 Online algorithm for uniform width jobs

To handle jobs of arbitrary width and height, we first study the case when jobs have uniform width (all jobs have the same width \( w \geq 1 \)). The proposed algorithm \( UV \) (Section \[3.2\]) is based on a further restricted case of unit width, i.e., \( w = 1 \) (Section \[3.1\]).

#### 3.1 Unit width and arbitrary height

In this section, we consider jobs with unit width and arbitrary height. We present an online algorithm \( V \) which makes reference to an arbitrary feasible online algorithm for the DVS problem, denoted by \( R \). In particular, we require that the speed of \( R \) remains the same during any integral timeslot, i.e., in \([t, t + 1] \) for all integers \( t \). Note that when jobs have integral release times and deadlines, many known DVS algorithms satisfy this criteria, including \( VDS, BKP \), and \( AVR \).

Recall in Section \[2\] how a job set for the Grid problem is converted to a job set for the DVS problem. We simulate a copy of \( R \) on the converted job set and denote the speed used by \( R \) at \( t \) as \( \ell(R, t) \). Our algorithm makes reference to \( \ell(R, t) \) but not the jobs run by \( R \) at \( t \).

**Algorithm V.** For each timeslot \( t \), we schedule jobs to start at \( t \) such that \( \ell(V, t) \) is at least \( \ell(R, t) \) or until all available jobs have been scheduled. Jobs are chosen in an EDF manner.

**Analysis.** We note that since \( V \) makes decision at integral time and jobs have unit width, each job is completed before any further scheduling decision is made. In other words, \( V \) is non-preemptive.
To analyze the performance of $V$, we first note that $V$ gives a feasible schedule (Lemma 4), and then analyze its competitive ratio (Theorem 6).

**Lemma 4.** $V$ is feasible.

**Proof.** Let $\ell(S, I)$ denote the total work done by schedule $S$ in $I$. That is, $\ell(S, I) = \sum_{t \in I} \ell(S, I)$. According to the algorithm, for all $I = [0, t)$, $\ell(V, I) \leq \ell(R, I)$.

Suppose on the contrary that $V$ has a job $J_m$ missing deadline at $t$. That is, $d(J_m) = t$ but $J_m$ is not assigned before $t$. By the algorithm, for all $t' \in [0, t)$, $\ell(V, t') \geq \ell(R, t')$ unless there are less than $\ell(R, t')$ available jobs at $t'$ for $V$. Let $t_0$ be the last timeslot in $[0, t)$ such that $\ell(V, t_0) < \ell(R, t_0)$, $r(J_m) > t_0$ since all jobs released at or before $t_0$ have been assigned. For all $t' \in (t_0, t)$, $\ell(V, t') \geq \ell(R, t')$. Also, all jobs $J$ with $r(J) \leq t_0$ are finished by $t_0 + 1$ and jobs executed in $(t_0, t)$ are those released after $t_0$. Consider set $J$ of jobs with feasible interval completely inside $I = (t_0, t)$ (note that $J_m \in J$), $\ell(S, I) \geq \sum_{j \in J} h(J)$ for any feasible schedule $S$. Since $V$ assigns jobs in EDF manner and is not feasible, $\ell(V, I) \leq \sum_{j \in J} h(J)$. It follows that $\sum_{j \in J} h(J) > \ell(V, I) \geq \ell(R, I)$. It contradicts to the fact that $R$ is feasible. Hence $V$ finishes all jobs before their deadlines.

Let $h_{\max}(V, t)$ be the maximum height of jobs scheduled at $t$ by $V$; we set $h_{\max}(V, t) = 0$ if $V$ assigns no job at $t$. We first classify each timeslot $t$ into two types: (i) $h_{\max}(V, t) < \ell(R, t)$, and (ii) $h_{\max}(V, t) \geq \ell(R, t)$. We denote by $I_1$ and $I_2$ the union of all timeslots of Type (i) and (ii), respectively. Notice that $I_1$ and $I_2$ can be empty and the union of $I_1$ and $I_2$ covers the entire time line. The following lemma bounds the cost of $V$ in each type of timeslots. Recall that cost($S, I$) denotes the cost of the schedule $S$ over the interval $I$ and cost($S$) denotes the cost of the entire schedule.

**Lemma 5.** The cost of $V$ satisfies the following properties. (i) $\text{cost}(V, I_1) \leq 2^a \cdot \text{cost}(R)$; and (ii) $\text{cost}(V, I_2) \leq 2^a \cdot \text{cost}(O)$.

**Proof.** (i) By the algorithm, $\ell(V, t) < \ell(R, t)$ and $h_{\max}(V, t) \leq 2 \cdot \ell(R, t)$ for $t \in I_1$. It follows that $\text{cost}(V, I_1) \leq 2^a \cdot \sum_{t \in I_1} \ell(R, t) = 2^a \cdot \text{cost}(R, I_1) \leq 2^a \cdot \text{cost}(R)$.

(ii) By convexity, $\text{cost}(O) \geq \sum_{J \in J} h(J)^a$. It is easy to see that $\text{cost}(O) \geq \sum_{t \in I_2} h_{\max}(V, t)^a$. According to the algorithm, $\ell(V, t) < \ell(R, t) + h_{\max}(V, t) \leq 2 \cdot h_{\max}(V, t)$ for $t \in I_2$. Hence, $\text{cost}(V, I_2) = \sum_{t \in I_2} \ell(V, t)^a \leq 2^a \cdot \sum_{t \in I_2} h_{\max}(V, t)^a \leq 2^a \cdot \text{cost}(O)$.

Notice that cost($V$) = cost($V, I_1$) + cost($V, I_2$) since $I_1$ and $I_2$ have no overlap. Together with Lemma 5 and Observation 4, we obtain the competitive ratio of $V$ in the following theorem.

**Theorem 6.** Algorithm $V$ is $2^a \cdot (R + 1)$-competitive, where $R$ is the competitive ratio of the reference DVS algorithm $R$.

As mentioned in Section 2 the BKP algorithm is $8 \cdot e^a$-competitive. On the other hand, $V$ can take an offline DVS algorithm, e.g., the optimal YDS algorithm, as reference and returns an offline schedule. Therefore, we have the following corollary.

**Corollary 7.** $V$ is $2^a \cdot (8 \cdot e^a + 1)$-competitive and $2^a \cdot 2$-approximate when the algorithm BKP and YDS are referenced, respectively.

### 3.2 Uniform width and arbitrary height

In this section, we consider jobs with uniform width $w$ and arbitrary height. The idea of handling uniform width jobs is to treat them as if they were unit width, however, this would mean that jobs may have release times or deadlines at non-integral time. To remedy this, we define a procedure ALIGNFI to align the feasible intervals (precisely, release times and deadlines) to the new time unit of duration $w$.

Let $J$ be a uniform width job set. We first define the notion of “tight” and “loose” jobs. A job $J$ is said to be tight if $|I(J)| \leq 2w$; otherwise, it is loose. Let $J_T$ and $J_L$ be the disjoint subsets of tight and loose jobs of $J$, respectively. We design different strategies for tight and loose jobs. As to be shown, tight jobs can be handled easily by starting them at their release times. For any loose job, we modify it via Procedure ALIGNFI such that its release time and deadline is a multiple of $w$. With this alternation, we can treat the jobs as unit width and make scheduling decisions at time multiple of $w$. 


The following properties hold: (i) For any job $J$ we define the procedure

\textbf{Procedure AlignFI.} Given a loose job set $\mathcal{J}_L$ in which $w(J) = w$ and $|I(J)| > 2 \cdot w \forall J \in \mathcal{J}_L$. We define the procedure AlignFI to transform each loose job $J \in \mathcal{J}_L$ into a job $J'$ with release time and deadline “aligned” as follows. We denote the resulting job set by $\mathcal{J}'$.

- $r(J') \leftarrow \min_{i \geq 0}\{i \cdot w \mid i \cdot w \geq r(J)\}$;
- $d(J') \leftarrow \max_{i \geq 0}\{i \cdot w \mid i \cdot w \leq d(J)\}$.

\textbf{Observation 8.} For any job $J \in \mathcal{J}_L$ and the corresponding $J'$, (i) $\frac{1}{3} \cdot |I(J)| < |I(J')| \leq |I(J)|$; (ii) $|I(J')| \geq w$; (iii) $I(J') \subseteq I(J)$.

Notice that after AlignFI, the release time and deadline of each loose job are aligned to timeslot $i_1 \cdot w$ and $i_2 \cdot w$ for some integers $i_1 < i_2$. By Observation 8, a feasible schedule of $J'$ is also a feasible schedule of $J$. Furthermore, after AlignFI all jobs are released at time which is a multiple of $w$. Hence, the job set $\mathcal{J}'$ can be treated as job set with unit width, where each unit has duration $w$ instead of 1.

As a consequence of altering the feasible intervals, we introduce two additional procedures that convert associated schedules. Given a schedule $S$ for job set $\mathcal{J}_L$, AlignSch converts it to a schedule $S'$ for the corresponding job set $\mathcal{J}'$. The other procedure FreeSch takes a schedule $S'$ for a job set $\mathcal{J}'$ and converts it to a schedule $S$ for $\mathcal{J}_L$.

\textbf{Transformation AlignSch.} AlignSch transforms $S$ into $S'$ by shifting the execution interval of every job $J \in \mathcal{J}_L$.

- $st(S', J') \leftarrow \min\{d(J') - w(J'), \min_{i \geq 0}\{i \cdot w \mid i \cdot w \geq st(S, J)\}\}$;
- $et(S', J') \leftarrow st(S', J') + w(J')$.

\textbf{Observation 9.} Consider any schedule $S$ for $\mathcal{J}_L$ and the schedule $S'$ for $\mathcal{J}'$ constructed by AlignSch. The following properties hold: (i) For any job $J \in \mathcal{J}_L$ and the corresponding $J'$, $st(J') > st(J) - w$ and $et(J') < et(J) + w$; (ii) $S'$ is a feasible schedule for $\mathcal{J}'$; and (iii) At any time $t$, $\ell(S', t) \leq \ell(S, t) + \ell(S, t - (w - 1)) + \ell(S, t + (w - 1))$.

\textbf{Proof.} (ii) By AlignSch, $st(S', J') \leq d(J') - w(J')$. Also, $|[st(S', J'), et(S', J')]| = w(J')$. Hence $[st(S', J'), et(S', J')] \subseteq I(J')$. That is, $S'$ is a feasible schedule for both $\mathcal{J}'$ and $J$.

(iii) By (i), $st(J') > st(J) - w$ and $et(J') < et(J) + w$ for each $J$. Hence, for any timeslot $t$, for each job $J$ with $[st(S, J), et(S, J)) \cap [t - (w - 1), t + (w - 1)) = \emptyset$, $t \notin [st(S', J'), et(S', J')]$. On the other hand, consider the jobs $J$ that $[st(J), et(J)) \cap [t - (w - 1), t + (w - 1)) \neq \emptyset$. Since $|[st(J), et(J))| = w$, at least one of the timeslots $t - (w - 1), t$, or $t + (w - 1)$ is in $[st(J), et(J))$. Hence we can capture $\ell(S', t)$ by $\ell(S, t) + \ell(S, t - (w - 1)) + \ell(S, t + (w - 1))$.

\textbf{Corollary 10.} Using AlignSch to generate $S'$ given $S$, we have $\text{cost}(S') \leq 3^\alpha \cdot \text{cost}(S)$.

\textbf{Proof.} By Lemma 9 (iii), $\text{cost}(S') = \sum \ell(S', t)^\alpha \leq \sum (3 \cdot \ell(S, t))^\alpha = 3^\alpha \cdot \text{cost}(S)$.

\textbf{Lemma 11.} $\text{cost}(O(J')) \leq 3^\alpha \cdot \text{cost}(O(J_L))$.

\textbf{Proof.} Consider set of loose jobs $\mathcal{J}_L$ with uniform width and the corresponding $\mathcal{J}'$. Given $O(J_L)$, there exists schedule $S(J')$ generated by AlignSch. By Lemma 10, $\text{cost}(S(J')) \leq 3^\alpha \cdot \text{cost}(O(J_L))$. Hence, $\text{cost}(O(J')) \leq \text{cost}(S(J')) \leq 3^\alpha \cdot \text{cost}(O(J_L))$.

\textbf{Transformation FreeSch.} FreeSch transforms $S'$ into $S$.

- $st(S, J) \leftarrow st(S', J')$;
- $et(S, J) \leftarrow et(S', J')$.

The feasibility of $S'$ can be easily proved by Observation 8.

\textbf{Lemma 12.} Using FreeSch, we have $\text{cost}(S) = \text{cost}(S')$.

\textbf{Proof.} It is easy to see that $\ell(S, t) = \ell(S', t)$ for all $t$. Hence $\text{cost}(S) = \text{cost}(S')$. 


Online algorithm $UV$. The algorithm takes a job set $J$ with uniform width $w$ as input and schedules the jobs in $J$ as follows. Let $J_T$ be the set of tight jobs in $J$ and $J_L$ be the set of loose jobs in $J$.

1. For any tight job $J \in J_T$, schedule $J$ to start at $r(J)$.

2. Loose jobs in $J_L$ are converted to $J'$ by $\text{ALIGNFI}$. For $J'$, we run Algorithm $\mathcal{V}$, which is defined in Section 3.1 with $\text{BKP}$ as the reference DVS algorithm. Jobs are chosen in an earliest deadline first (EDF) manner.

Note that the decisions of $UV$ can be made online.

Analysis of Algorithm $UV$. We analyze the tight jobs and loose jobs separately. We first give an observation.

Observation 13. For any two job sets $J_x \subseteq J_y$, $\text{cost}(\mathcal{O}(J_x)) \leq \text{cost}(\mathcal{O}(J_y))$.

Proof. Assume on the contrary that $\text{cost}(\mathcal{O}(J_x)) < \text{cost}(\mathcal{O}(J_y))$, we can generate a schedule $S(J_x)$ by removing jobs from $\mathcal{O}(J_y)$ which are not in $J_x$. It follows that $\text{cost}(S(J_x)) \leq \text{cost}(\mathcal{O}(J_y)) < \text{cost}(\mathcal{O}(J_x))$, contradicting to the fact that $\mathcal{O}(J_x)$ is optimal for $J_x$.

In the following analysis we say that interval $I = [t_1,t_2)$ is a $\text{BKP}$ interval of $\mathcal{T}$ if $t_1 \in t$ and $(t_2 - t_1) : (t_2 - t) = \varepsilon : 1$. The next lemma proves the competitive ratio separately for $J_T$ and $J_L$.

Lemma 14. (i) $\text{cost}(UV(J_T)) \leq 2^\alpha \cdot 8e^\alpha \cdot \text{cost}(\mathcal{O}(J))$; (ii) $\text{cost}(UV(J_L)) \leq 6^\alpha \cdot (8e^\alpha + 1) \cdot \text{cost}(\mathcal{O}(J))$.

Proof. (i) We prove the competitive ratio by comparing to the cost of $\text{BKP}$. For any time $t$, assume that $\ell(UV(J_T),t) = k$ for some $k$. Let the set of jobs contributing to the load at $t$ be $J_k$. That is, $\sum_{J \in J_k} h(J) = k$. For each $J \in J_k$, $r(J) > t \cdot 2 \cdot w$ and $d(J) \leq t \cdot 2 \cdot w$ since $J$ is a tight job. Let $I$ be the minimum $\text{BKP}$ interval of $t$ such that $I(J) \subseteq I$ for all $J \in J_k$. It is easy to see that $|I| \leq e \cdot 2w$. Also, $p(t,I) \geq \sum_{J \in J_k} h(J) \cdot w(J) = \sum_{J \in J_k} h(J) \cdot w$. Hence $p(t,I) \geq \frac{1}{2^\alpha} \sum_{J \in J_k} h(J) = \frac{1}{2^\alpha} \cdot k$. Suppose that $\text{BKP}$ chooses $I^*$ as the $\text{BKP}$ interval of $t$, $\ell(\text{BKP}, t) = e \cdot \frac{p(t,I^*)}{|I^*|} \geq e \cdot \frac{p(t,I)}{|I|} > \frac{k}{2^\alpha}$. So $\ell(UV(J_T), t) = k \leq 2 \cdot e \cdot \frac{p(t,I^*)}{|I^*|} = 2 \cdot \ell(\text{BKP}, t)$. It follows that $\text{cost}(UV(J_T)) \leq 2^\alpha \cdot \text{cost}(\text{BKP}(J_T))$.

(ii) For $J_L$, we perform ALIGNFI and get $J'_L$. We then run $\mathcal{V}$ and get $\mathcal{V}(J'_L)$, which can be viewed as a schedule for unit width jobs. We get $S(J'_L) = \mathcal{V}(J'_L)$ by $\text{FREESch}$. Hence, $\text{cost}(UV(J_L)) = \sum_{I} \ell(UV(J'_L),t)^{a} = \sum_{I} \ell(\mathcal{V}(J'_L),t)^{a} = \sum_{I} \ell(\mathcal{V}(J'_L),t)^{a} = \text{cost}(\mathcal{V}(J'_L))$. According to Corollary 7, $\text{cost}(\mathcal{V}(J'_L)) \leq 2^\alpha \cdot (8e^\alpha + 1) \cdot \text{cost}(\mathcal{O}(J'_L))$ by choosing $\text{BKP}$ as reference algorithm. Since $J'_L$ is a set of loose jobs with uniform width, $\text{cost}(\mathcal{O}(J'_L)) \leq 3^\alpha \cdot \text{cost}(\mathcal{O}(J'_L)) \leq 3^\alpha \cdot \text{cost}(\mathcal{O}(J))$ by Lemma 11 and Observation 13. Hence, $\text{cost}(UV(J_L)) \leq 2^\alpha \cdot (8e^\alpha + 1) \cdot 3^\alpha \cdot \text{cost}(\mathcal{O}(J))$.

Theorem 15. $\text{cost}(UV(J)) \leq 12^\alpha \cdot (8e^\alpha + 1) \cdot \text{cost}(\mathcal{O}(J))$.

Proof. By definition, $\text{cost}(UV(J)) \leq \sum_{I} \ell(UV(J),t)^{a} = \sum_{I} \ell(UV(J_T),t) + \ell(UV(J_L),t))^{a} \leq 2^{\alpha - 1} \cdot \sum_{I} \ell(UV(J_T),t)^{a} + \ell(UV(J_L),t))^{a} = 2^{\alpha - 1} \cdot (\text{cost}(UV(J_T)) + \text{cost}(UV(J_L)))$. By Lemma 14, $\text{cost}(UV(J)) \leq 2^{\alpha - 1} \cdot (2^\alpha \cdot 8e^\alpha + 6^\alpha \cdot (8e^\alpha + 1)) \cdot \text{cost}(\mathcal{O}(J)) \leq 2^\alpha \cdot 6^\alpha \cdot (8e^\alpha + 1) \cdot \text{cost}(\mathcal{O}(J))$.

4 Online algorithm for general case

In this section, we present an algorithm $G$ for jobs with arbitrary width and height. We first transform job set $J$ to a “nice” job set $J^*$ (to be defined) and show that such a transformation only increases the cost modestly. Furthermore, we show that for any nice job set $J^*$, we can bound $\text{cost}(G(J^*))$ by $\text{cost}(\mathcal{O}(J^*))$ and in turn by $\text{cost}(\mathcal{O}(J))$. Then we can establish the competitive ratio of $G$. 
4.1 Nice job set and transformations

A job $J$ is said to be a nice job if $w(J) = 2^p$, for some non-negative integer $p$ and a job set $J^*$ is said to be a nice job set if all its jobs are nice jobs. In other words, the nice job $J$ is in class $C_p$.

**Procedure Convert.** Given a job set $J$, we define the procedure $\text{Convert}$ to transform each job $J \in J$ into a nice job $J^*$ as follows. We denote the resulting nice job set by $J^*$. Suppose $J$ is in class $C_p$. We modify its width, release time and deadline.

- $w(J^*) \leftarrow 2^p$;
- $r(J^*) \leftarrow r(J)$;
- $d(J^*) \leftarrow r(J^*) + \max\{d(J) - r(J), 2^p\}$.

Modifications to $r(J^*)$ and $d(J^*)$ are due to rounding up the width. The observation below follows directly from the definition.

**Observation 16.** For any job $J$ and its nice job $J^*$ transformed by Convert, (i) $I(J) \subseteq I(J^*)$; (ii) $I(J) \neq I(J^*)$ if and only if $|I(J)| < 2^p$; in this case, $\text{den}(J) > \frac{1}{2}$ and $\text{den}(J^*) = 1$.

We then define two procedures that transform schedules related to nice job sets. $\text{RelaxSch}$ takes a schedule $S$ for a job set $J$ and converts it to a schedule $S^*$ for the corresponding nice job set $J^*$. On the other hand, $\text{ShrinkSch}$ takes a schedule $S^*$ for a nice job set $J^*$ and converts it to a schedule $S$ for $J$.

**Transformation RelaxSch.** $\text{RelaxSch}$ transforms $S$ into $S^*$ by moving the start and end time of every job $J$.

- $\text{st}(S^*, J^*) = \min\{d(J^*) - w(J^*), \text{st}(S, J)\}$
- $\text{et}(S^*, J^*) = \text{st}(S^*, J^*) + w(J^*)$.

**Observation 17.** Consider any schedule $S$ for $J$ and the schedule $S^*$ constructed by $\text{RelaxSch}$ for the corresponding $J^*$. We have $[\text{st}(S^*, J^*), \text{et}(S^*, J^*)] \subseteq [\text{r}(J^*), \text{d}(J^*)]$; in other words, $S^*$ is a feasible schedule for $J^*$.

To analyze the load of the schedule $S^*$, we consider partial schedule $S_p^* \subseteq S^*$ (resp. $S_p \subseteq S$) which is for all the jobs of $J^*$ (resp. $J$) in class $C_p$. Intuitively, the load of $S_p$ at any time is at most the sum of the load of $S_p$ at the current time and $2^{p-1} - 1$ timeslots before and after the current time.

**Lemma 18.** At any time $t$, $\ell(S_p^*, t) \leq \ell(S_p, t) + \ell(S_p, t - (2^{p-1} - 1)) + \ell(S_p, t + (2^{p-1} - 1))$.

**Proof.** We prove that for any job $J$, $J^*$ contributes to $\ell(S_p^*, t)$ only if $J$ contributes to either $\ell(S_p, t)$, $\ell(S_p, t - (2^{p-1} - 1))$, or $\ell(S_p, t + (2^{p-1} - 1))$. There are two cases that $J$ does not contribute to $\ell(S_p, t - (2^{p-1} - 1))$ nor $\ell(S_p, t + (2^{p-1} - 1))$: (i) $\text{et}(J) < t - (2^{p-1} - 1)$ or $\text{et}(J) < t + (2^{p-1} - 1)$, and (ii) $[\text{st}(J), \text{et}(J)] \subseteq (t - (2^{p-1} - 1), t + (2^{p-1} - 1))$.

Consider case (i), $\text{et}(J^*) \leq \text{et}(J) + (2^{p-1} - 1)$ and $\text{st}(J^*) \geq \text{st}(J) - (2^{p-1} - 1)$. Hence, $t \notin [\text{st}(J^*), \text{et}(J^*)]$ if $\text{et}(J) < t - (2^{p-1} - 1)$ or $\text{et}(J) > t + (2^{p-1} - 1)$. That is, $J^*$ does not contribute to $\ell(S_p, t)$. Notice that if $\text{et}(J) = t - (2^{p-1} - 1)$ or $\text{et}(J) = t + (2^{p-1} - 1)$, $J$ does not necessarily contribute to $\ell(S_p, t)$. We count the contribution for worst case analysis.

For case (ii), consider job $J$ with $[\text{st}(J), \text{et}(J)] \subseteq (t - (2^{p-1} - 1), t + (2^{p-1} - 1))$. Since $2^{p-1} < w(J) \leq 2^p$, $t \in [\text{st}(J), \text{et}(J)]$. That is, $J$ contributes to $\ell(S_p, t)$ no matter if $J^*$ contributes to $\ell(S_p^*, t - (2^{p-1} - 1))$ or $\ell(S_p^*, t + (2^{p-1} - 1))$.

By case (i) and (ii), for any job $J$ with $[\text{st}(J), \text{et}(J)] \cap [t - (2^{p-1} - 1), t + (2^{p-1} - 1)] = \emptyset$, $J^*$ does not contribute to $\ell(S_p^*, t)$. And for any job $J$ with $[\text{st}(J), \text{et}(J)] \subseteq (t - (2^{p-1} - 1), t + (2^{p-1} - 1))$, $J$ contributes to $\ell(S_p, t)$. Hence, by assuming all jobs at timeslot $t - (2^{p-1} - 1)$ or $t + (2^{p-1} - 1)$ contribute to $\ell(S_p, t), \ell(S_p^*, t)$ is bounded by $\ell(S_p, t) + \ell(S_p, t - (2^{p-1} - 1)) + \ell(S_p, t + (2^{p-1} - 1))$. \qed
Lemma 19. Using RELAXSCH, we have \( \text{cost}(S_p^*) \leq 3^\alpha \cdot \text{cost}(S_p) \).

Proof. By Lemma 18, \( \text{cost}(S_p^*) = \sum_i \ell(S_p^*, t)^\alpha \leq \sum_i (\ell(S_p, t) + \ell(S_p, t - (2^p-1) - 1) + \ell(S_p, t + (2^p-1) - 1))^\alpha \leq \sum_i (3 \cdot \ell(S_p, t))^\alpha = 3^\alpha \cdot \text{cost}(S_p) \).

**Transformation ShrinkSCH.** On the other hand, ShrinkSCH converts a schedule \( S^* \) for a nice job set \( J^* \) to a schedule \( S \) for the corresponding job set \( J \). We set

- \( st(S, J) \leftarrow st(S^*, J^*) \);
- \( et(S, J) \leftarrow st(S, J) + w(J) \), therefore, \( et(S, J) \leq et(S^*, J^*) \).

Observation 20 asserts that the resulting schedule \( S \) is feasible for \( J \) and Lemma 21 analyzes the cost of the schedule.

**Observation 20.** Consider any schedule \( S^* \) for \( J^* \) and schedule \( S \) constructed by ShrinkSCH for the corresponding \( J \). For any \( J^* \) and the corresponding \( J \), (i) \( [\text{st}(S, J), \text{et}(S, J)] \subseteq [\text{st}(S^*, J^*), \text{et}(S^*, J^*)] \); (ii) \( [\text{st}(S, J), \text{et}(S, J)] \subseteq [r(J), d(J)] \).

By Observation 20, we have the following lemma.

**Lemma 21.** Using ShrinkSCH, we have \( \text{cost}(S_p) \leq \text{cost}(S_p^*) \).

4.2 The online algorithm

**Online algorithm \( G \).** We are now ready to describe the algorithm \( G \) for an arbitrary job set \( J \). When a job \( J \) is released, it is converted to \( J^* \) by CONVERT and classified into one of the classes \( C_p \). Jobs in the same class after CONVERT (being a uniform-width job set) are scheduled by UV independently of other classes. We then modify the execution time of \( J^* \) in UV to the execution time of \( J \) in \( G \) by Transformation ShrinkSCH. Note that all these procedures can be done in an online fashion.

Using the results in Sections 3 and 4.1, we can compare the cost of \( G(J) \) with \( \text{cost}(J^*_p) \) for each class \( C_p \) (see Theorem 23). It remains to analyze the cost of \( \text{cost}(J^*_p) \) and \( \text{cost}(J) \) in the next observation.

**Observation 22.** Consider any job set \( J \), its corresponding job set \( J^* \) and the corresponding job set of each class \( J^*_p \) and \( J^*_p \). (i) \( \text{cost}(O(J^*_p)) \leq 3^\alpha \cdot \text{cost}(O(J_p)) \); (ii) \( \text{cost}(O(J^*_p)) \leq \text{cost}(O(J^*_p)) \).

Proof. (i) Given \( O(J_p) \), there exists schedule \( S(J^*_p) \) generated by RELAXSCH. By Lemma 19, \( \text{cost}(S(J^*_p)) \leq 3^\alpha \cdot \text{cost}(O(J_p)) \). Hence, \( \text{cost}(O(J^*_p)) \leq \text{cost}(S(J^*_p)) \) \( \leq 3^\alpha \cdot \text{cost}(O(J_p)) \).

(ii) Assume on the contrary that \( \text{cost}(O(J)) < \text{cost}(O(J_p)) \), we can generate a schedule \( S(J_p) \) by removing jobs from \( O(J) \) which are not in \( J_p \). It follows that \( \text{cost}(S(J_p)) \leq \text{cost}(O(J)) \) \( < \text{cost}(O(J_p)) \), contradicting to the fact that \( O(J_p) \) is optimal for \( J_p \).

**Theorem 23.** For any job set \( J \), we have \( \text{cost}(G(J)) \leq (36k_w)^\alpha \cdot (8e^a + 1) \cdot \text{cost}(O(J)) \), where \( k_w \) is \( \log(\frac{w_{\text{max}}}{w_{\text{min}}}) \).

Proof. By definition, \( \text{cost}(G(J)) = \sum_t \ell(G(J), t)^\alpha = \sum_t (\sum_{p=1}^{k_w} \ell(G(J), t))^\alpha \). The latter is at most \( (k_w)^{\alpha-1} \sum_{p=1}^{k_w} (\ell(G(J), t))^\alpha \). For each group of jobs \( J_p \), we CONVERT it to \( J^*_p \), perform algorithm UV on it, and transform the schedule into a schedule for \( J_p \) by ShrinkSCH. Hence, \( \ell(G(J), t) \leq \ell(UV(J^*_p), t) \) for each \( t \). It follows that \( \text{cost}(G(J)) \leq (k_w)^{\alpha-1} \sum_{p=1}^{k_w} \text{cost}(UV(J^*_p)) \leq (k_w)^{\alpha-1} \sum_{p=1}^{k_w} \text{cost}(O(J^*_p)) \).

By Lemma 15 and Observation 13, \( \text{cost}(UV(J^*_p)) \leq 12^\alpha \cdot (8e^a + 1) \cdot \text{cost}(O(J^*_p)) \leq 12^\alpha \cdot (8e^a + 1) \cdot 3^\alpha \cdot \text{cost}(O(J^*_p)) \leq 36^\alpha \cdot (8e^a + 1) \cdot \text{cost}(O(J^*_p)) \). Hence \( \text{cost}(G(J)) \leq 36^\alpha \cdot (k_w)^{\alpha-1} \cdot (8e^a + 1) \cdot \sum_{p=1}^{k_w} \text{cost}(O(J^*_p)) = (36k_w)^\alpha \cdot (8e^a + 1) \cdot \text{cost}(O(J^*_p)) \).
4.3 Lower bound

In this section, we show a lower bound of competitive ratio for Grid problem with unit height and arbitrary width by designing an adversary for the problem. This lower bound is immediately a lower bound for the general case of Grid problem.

The adversary constructs a set of jobs with a low cost of offline optimal schedule but a high cost of any online algorithm $A$. It generates jobs one by one and assigns release times, deadlines and widths of jobs based on the previously generated jobs. The start times of jobs scheduled by algorithm $A$ will be used for the job generations later. This ensures that algorithm $A$ has to put a job on top of all existing jobs and results in a high energy cost for algorithm $A$. Meanwhile, the adversary will choose an appropriate feasible interval for each job such that an optimal offline algorithm can schedule the job set with low energy cost. The following is the description of the adversary.

**Adversary $A$ and job instance $J$.** Given an online algorithm $A$, a constant $\alpha > 1$ and a large number $x$, adversary $A$ outputs a set of jobs $J$ with $[\alpha] + 1$ jobs. Let $J_i$ denote the $i$th job of $J$. The adversary first computes a width for each job before running algorithm $A$. It sets $w(J_{[\alpha]}) = x$, $w(J_{[\alpha]+1}) = x - 1$, and $w(J_i) = 3w(J_{i+1}) + 1$ for $1 \leq i \leq [\alpha] - 1$. Then adversary $A$ computes a release time and deadline for each job through a interaction with algorithm $A$. For the first job $J_1$, adversary $A$ chooses any release time and deadline such that $d(J_1) - r(J_1) \geq 3w(J_1)$. For the $i$th job $J_i \in J$ for $2 \leq i \leq [\alpha] + 1$, adversary $A$ sets $r(J_i) = st(A, J_{i-1}) + 1$ and $d(J_i) = et(A, J_{i-1})$. This limits algorithm $A$ to fewer choices of start times for scheduling a new job. A job can only be scheduled in the execution interval of the previous job by algorithm $A$. On the other hand, no two jobs have the same release time. Algorithm $A$ shall schedule the jobs accordingly from $J_1$ to $J_{[\alpha]+1}$ and one job at a time.

Let $w_{\text{max}}$ and $w_{\text{min}}$ denote by the maximum and minimum width of jobs respectively, and let $O$ be an optimal offline algorithm for Grid problem. We have the following results.

**Lemma 24.** $\text{cost}(O(J)) \leq x \cdot 3^{\alpha}.$

*Proof.* By the setting of adversary $A$, we show that $O$ can schedule all jobs in $J$ without overlapping, and the cost of an optimal schedule is just the summation of widths of all the jobs.

For any job $J_i \in J$ and $i \geq 2$, the length of its feasible interval is $d(J_i) - r(J_i) = et(A, J_{i-1}) - (st(A, J_{i-1}) + 1) = w(J_{i-1}) - 1 = 3w(J_i)$. This means no matter where we schedule a job, one of the length $[r(J_i), st(J_i))$ and $[et(J_i), d(J_i)]$ is at least $w(J_i)$, and algorithm $O$ can schedule the remaining jobs in the interval with length at least $w(J_i)$ because the summation of widths of all the remaining jobs does not exceed $w(J_i)$. The remaining jobs will not overlap to $J_i$. Since this argument can be applied on all the jobs, this implies that all the jobs do not overlap to each other in an optimal schedule. Thus the cost of an optimal schedule is the summation of widths of all the jobs. More precisely,

$$\text{cost}(O(J)) = (x - 1) + x + (3x + 1) + (3(3x + 1) + 1) + \ldots + w_{\text{max}} \leq 2x + 2 \cdot 3x + 2 \cdot 9x + \ldots + 2 \cdot 3^{[\alpha] - 1}x = 2x \cdot \frac{3^{[\alpha]} - 1}{2} \leq x \cdot 3^{[\alpha]}.$$ \hfill \qed

**Theorem 25.** For any deterministic online algorithm $A$ for Grid problem with unit height and arbitrary width, adversary $A$ constructs an instance $J$ such that

$$\frac{\text{cost}(A(J))}{\text{cost}(O(J))} \geq \left( \frac{1}{3} \log \frac{w_{\text{max}}}{w_{\text{min}}} \right)^{\alpha}.$$ 

*Proof.* We first give a lower bound of cost$(A(J))$ and then give the lower bound of the ratio by combining cost$(A(J))$ with Lemma 24.

By the setting of adversary $A$, all the jobs scheduled by algorithm $A$ overlap to each other. For ease of the computation, we only consider the timeslots contained by the execution interval of the last job $J_{[\alpha]+1}$ when we compute the cost of $A$. Thus cost$(A(J)) \geq (x - 1) \cdot (\alpha + 1)^{\alpha}$. Now we use $w_{\text{max}}$ and $w_{\text{min}}$ to bound $\alpha + 1$. According to Lemma 24, we have $w_{\text{max}} \leq \text{cost}(O(J)) \leq x \cdot 3^{[\alpha]}$, and thus

$$\alpha \geq \log_3 \frac{w_{\text{max}}}{x} \geq \log_3 \frac{w_{\text{max}}}{3(x - 1)} = \log_3 \frac{w_{\text{max}}}{w_{\text{min}}} - 1.$$ 

10
We first consider jobs with uniform-height and unit-width (Section 5.2) and secondly consider jobs with agreeable deadlines (Section 5.3).

The main idea is to make reference to the online algorithm AVR and consider two types of intervals, $I_{>h}$ where the average load is higher than $h$ and $I_{\leq h}$ where the average load is at most $h$. For the former, we show that we can base on the competitive ratio of AVR and in such case, we compare directly to the optimal algorithm. Combining the two cases, we have Lemma 27 which holds for any job set. In Sections 5.2 and 5.3, we show how we can use this lemma to obtain algorithms for the special cases. Notice that the number $\left\lceil \frac{\text{avg}(t)}{h} \right\rceil$ is the minimum number of jobs needed to make the load at $t$ at least avg$(t)$.

**Lemma 27.** Suppose we have an algorithm $A$ for any job set $J$ such that (i) $\ell(A, t) \leq c \cdot \left\lfloor \text{avg}(t) \right\rfloor$ for all $t \in I_{>h}$, and (ii) $\ell(A, t) \leq c' \cdot \text{avg}(t)$ for all $t \in I_{\leq h}$. Then we have $\text{cost}(A(J)) \leq \left(\frac{4(c' + 1)^\alpha}{2} + c\alpha\right) \cdot \text{cost}(O(J))$.

**Proof.** (i) We denote the speed of AVR at $t$ as $\ell(AVR, t)$. We claim that $\text{cost}(A(J)) \leq \left(\frac{4c\alpha}{2}\right) \cdot \text{cost}(O(J))$. We prove the claim by comparing $\ell(A, t)$ to $\ell(AVR, t)$ for each timeslot $t$. The assumption of $A$ means that $\ell(A, t) \leq c \cdot h \cdot \left\lfloor \frac{\text{avg}(t)}{h} \right\rfloor + c \cdot (\text{avg}(t) + 1) = c \cdot (\text{avg}(t) + h) \leq 2c \cdot \text{avg}(t)$ since avg$(t) > h$.

Combining the two cases, we have Lemma 27 which holds for any job set. In Sections 5.2 and 5.3, we show how we can use this lemma to obtain algorithms for the special cases. Notice that the number $\left\lceil \frac{\text{avg}(t)}{h} \right\rceil$ is the minimum number of jobs needed to make the load at $t$ at least avg$(t)$.

**5 Online algorithm for uniform height jobs**

In this section we focus on uniform-height jobs of height $h$ and consider two special cases of the width. We first consider jobs with uniform-height and unit-width (Section 5.2) and secondly consider jobs with agreeable deadlines (Section 5.3).

To ease the discussion, we refine a notation we defined before. For any algorithm $A$ for a any job set $J$ and a time interval $I$, we denote by $A(J, I)$ the schedule of $A$ on $J$ over the time interval $I$.

**5.1 Main ideas**

The main idea is to make reference to the online algorithm AVR and consider two types of intervals, $I_{>h}$ where the average load is higher than $h$ and $I_{\leq h}$ where the average load is at most $h$. For the former, we show that we can base on the competitive ratio of AVR directly; for the latter, our load could be much higher than that of AVR and in such case, we compare directly to the optimal algorithm. Combining the two cases, we have Lemma 27 which holds for any job set. In Sections 5.2 and 5.3, we show how we can use this lemma to obtain algorithms for the special cases. Notice that the number $\left\lceil \frac{\text{avg}(t)}{h} \right\rceil$ is the minimum number of jobs needed to make the load at $t$ at least avg$(t)$.

**Lemma 27.** Suppose we have an algorithm $A$ for any job set $J$ such that (i) $\ell(A, t) \leq c \cdot \left\lfloor \text{avg}(t) \right\rfloor$ for all $t \in I_{>h}$, and (ii) $\ell(A, t) \leq c' \cdot \text{avg}(t)$ for all $t \in I_{\leq h}$. Then we have $\text{cost}(A(J)) \leq \left(\frac{4(c' + 1)^\alpha}{2} + c\alpha\right) \cdot \text{cost}(O(J))$.

**Proof.** (i) We denote the speed of AVR at $t$ as $\ell(AVR, t)$. We claim that $\text{cost}(A(J)) \leq \left(\frac{4c\alpha}{2}\right) \cdot \text{cost}(O(J))$. We prove the claim by comparing $\ell(A, t)$ to $\ell(AVR, t)$ for each timeslot $t$. The assumption of $A$ means that $\ell(A, t) \leq c \cdot h \cdot \left\lfloor \frac{\text{avg}(t)}{h} \right\rfloor + c \cdot (\text{avg}(t) + 1) = c \cdot (\text{avg}(t) + h) \leq 2c \cdot \text{avg}(t)$ since avg$(t) > h$.

Combining the two cases, we have Lemma 27 which holds for any job set. In Sections 5.2 and 5.3, we show how we can use this lemma to obtain algorithms for the special cases. Notice that the number $\left\lceil \frac{\text{avg}(t)}{h} \right\rceil$ is the minimum number of jobs needed to make the load at $t$ at least avg$(t)$.

**5.2 Uniform-height and unit-width**

In this section we consider job sets where all jobs have uniform-height and unit-width, i.e., $w(J) = 1$ and $h(J) = h \forall J$. Note that such case is a subcase discussed in Section 3.1. Here we illustrate a different approach using the ideas above and describe the algorithm $\mathcal{U} \mathcal{U}$ for this case. The competitive ratio of $\mathcal{U} \mathcal{U}$ is better than that of Algorithm $V$ in Section 3.1 when $\alpha < 3.22$.

**Algorithm $\mathcal{U} \mathcal{U}$.** At any time $t$, choose $\left\lceil \frac{\text{avg}(t)}{h} \right\rceil$ jobs according to the EDF rule and schedule them to start at $t$. If there are fewer jobs available, schedule all available jobs.

The next theorem asserts that the algorithm gives feasible schedule and states its competitive ratio.

**Theorem 28.** (i) The schedule constructed by Algorithm $\mathcal{U} \mathcal{U}$ is feasible. (ii) Algorithm $\mathcal{U} \mathcal{U}$ is $\left(\frac{(4c' + 1)^\alpha}{2} + c\alpha\right)$-competitive.
Proof. (i) The feasibility can be proved by comparing to $AVR$. At any time $t$, the total work done by $AVR$ in interval $[0, t)$ is $\sum_{t' < t} \text{avg}(t')$. On the other hand, the total work done by $UU$ in the same interval is $\sum_{t' < t} h \cdot \left\lceil \text{avg}(t') \right\rceil \geq \sum_{t' < t} h \cdot \text{avg}(t) = \sum_{t' < t} \text{avg}(t)$ if there are enough available jobs in this interval. If the number of available jobs is less than $\sum_{t' < t} h \cdot \left\lceil \text{avg}(t') \right\rceil$, $UU$ will execute all these jobs. The work done by $UU$ within interval $[0, t)$ is at least the work done by $AVR$ in both cases. Hence, $UU$ is feasible since $AVR$ is feasible.

(ii) We note that $t(UU, t) \leq h \cdot \left\lceil \text{avg}(t) \right\rceil$. To use Lemma 27, we can set $c' = 1$ for $t \in \mathcal{I}_{\leq h}$ by the definition of $\mathcal{I}_{\leq h}$. Furthermore, we can set $c = 1$ for $t \in \mathcal{I}_{> h}$.

5.3 Uniform-height, arbitrary width and agreeable deadlines

In this section we consider jobs with agreeable deadlines. We first note that simply scheduling $\left\lceil \frac{\text{avg}(t)}{h} \right\rceil$ number of jobs may not return a feasible schedule.

Example 29. Consider four jobs each job $J$ with $r(J) = 0$, $d(J) = 5$, $h(J) = h$, $w(J) = 3$. Note that $\text{avg}(t) = 2.4 \cdot h$ for all $t$. If we schedule at most $\left\lceil \frac{\text{avg}(t)}{h} \right\rceil = 3$ jobs at any time, we can complete three jobs but the remaining job cannot be completed. To schedule all jobs feasibly, we need at least two timeslots where all jobs are being executed.

To schedule these jobs, we first observe in Lemma 30 that for a set of jobs with total densities at most $h$, it is feasible to schedule them such that the load at any time is at most $h$. Roughly speaking, we consider jobs in the order of release, and hence in EDF manner since the jobs have agreeable deadlines. We keep the current ending time of all jobs that have been considered. As a new job is released, if its release time is earlier than the current ending time, we set its start time to the current ending time (and increase the current ending time by the width of the new job); otherwise, we set its start time to be its release time. Lemma 30 asserts that such scheduling is feasible and maintains the load at any time to be at most $h$.

Using this observation, we then partition the jobs into “queues” each of which has sum of densities at most $h$. Each queue $Q_i$ is scheduled independently and the resulting schedule is to “stack up” all these schedules. The queues are formed in a Next-Fit manner: (i) the current queue $Q_i$ is kept “open” and a newly arrived job is added to the current queue if including it makes the total densities stays at most 1; (ii) otherwise, the current queue is “closed” and a new queue $Q_{i+1}$ is created as open.

Lemma 30. Given any set of jobs of uniform-height, arbitrary-width and agreeable deadlines. If the sum of densities of all jobs is at most $h$, then it is feasible to schedule all of them using a maximum load $h$ at any time. That is, there is no stacking up among these jobs.

Proof. Suppose there are $k$ jobs $J_1, J_2, \cdots, J_k$ such that $\sum_{1 \leq i \leq k} \text{den}(J_i) \leq h$. Without loss of generality, we assume that $d(J_i) \leq d(J_j)$ and $r(J_i) \leq r(J_j)$ for $1 \leq i < j \leq k$. We claim that it is feasible to set $[st(J_i), et(J_i))$ to $[\max\{r(J_i), et(J_{i-1})\}, st(J_i) + w(J_i))$ for all $1 < i \leq k$ and $[st(J_1), et(J_1)) = [r(J_1), r(J_1) + w(J_1))$. We observe that $\text{den}(J_1) \leq h$ since $\sum_i \text{den}(J_i) \leq h$. It is feasible to set $[st(J_1), et(J_1))$ to $[r(J_1), r(J_1) + w(J_1))$ since the input is feasible. Then we have to prove that $[st(J_i), et(J_i)) = [\max\{r(J_i), et(J_{i-1})\}, st(J_i) + w(J_i)) \subseteq [r(J_i), d(J_i))$. It is easy to see that $st(J_i) \geq r(J_i)$. Assume that $\cup_{g \leq i} I(g)$ is a contiguous interval. Since $\sum_{g \leq i} \text{den}(J_g) \leq h$, $\sum_{g \leq i} \frac{w(J_g)}{\text{den}(J_g)} \leq 1$. From the ordering of jobs we have $1 \geq \sum_{g \leq i} \frac{w(J_g)}{\text{den}(J_g)} \geq \sum_{g \leq i} \frac{w(J_g)}{\text{den}(J_g) - r(J_g)}$. Hence, $\sum_{g \leq i} \frac{w(J_g)}{d(J_g) - r(J_g)} \leq 1$. From the ordering of jobs we have $1 \geq \sum_{g \leq i} \frac{w(J_g)}{\text{den}(J_g) - r(J_g)} \geq \sum_{g \leq i} \frac{w(J_g)}{\text{den}(J_g) - r(J_g)}$. Hence, $\sum_{g \leq i} \frac{w(J_g)}{d(J_g) - r(J_g)} \leq 1$. From the ordering of jobs we have $1 \geq \sum_{g \leq i} \frac{w(J_g)}{\text{den}(J_g) - r(J_g)} \geq \sum_{g \leq i} \frac{w(J_g)}{\text{den}(J_g) - r(J_g)}$. Hence, $\sum_{g \leq i} \frac{w(J_g)}{d(J_g) - r(J_g)} \leq 1$. Therefore $J_i$ can be finished before $d(J_i)$. On the other hand, if $\cup_{g \leq i} I(g)$ is not contiguous. The proof above shows that for each contiguous, each of the involving jobs can be finished by its deadline.

Algorithm $AD$. The algorithm consists of the following components: InsertQueue, SetStartTime and ScheduleQueue.

InsertQueue: We keep a counter $q$ for the number of queues created. When a job $J$ arrives, if $\text{den}(J) + \sum_{J' \in Q_q} \text{den}(J') \leq h$, then job $J$ is added to $Q_q$; otherwise, job $J$ is added to a new queue $Q_{q+1}$ and we set $q \leftarrow q + 1$. 

For jobs with uniform height, arbitrary width and agreeable deadlines, an algorithm with parameters \( p \) and \( \ell \) is fixed-parameter tractable with respect to a few small parameters. According to our algorithm, \( AD \) is feasible. We then analyze its load and hence derive its competitive ratio. Recall the definition of \( I_{>h} \) and \( I_{\leq h} \).

**Lemma 31.** Using \( AD \), we have (i) \( \ell(AD,t) \leq 3 \cdot h \cdot \lceil \frac{\text{avg}(t)}{h} \rceil \) for \( t \in I_{>h} \); (ii) \( \ell(AD,t) \leq h \) for \( t \in I_{\leq h} \).

**Proof.** For timeslot \( t \), suppose there are \( k \) queues \((Q_1,Q_2,\ldots,Q_k)\) which contains jobs available at \( t \). According to our algorithm, \( \ell(AD,t) \leq k \cdot h \).

Let \( D_i \) be the sum of densities of jobs in \( Q_i \). Consider \( t \in I_{>h} \). By the InsertQueue procedure, \( D_i + D_{i+1} > h \) for \( 1 \leq i < k - 1 \). Therefore, \( \text{avg}(t) = \sum_{1 \leq i < k} D_i > \sum_{2 \leq i \leq k-1} D_i \geq \left\lfloor \frac{k-2}{2} \right\rfloor \cdot h = \left( \left\lfloor \frac{k}{2} \right\rfloor - 1 \right) \cdot h \). It can be shown that \( k \leq 3 \cdot \lceil \text{avg}(t) \rceil \) since \( \text{avg}(t) > 1 \).

That is, \( \ell(AD,t) = k \cdot h \leq 3 \cdot h \cdot \lceil \text{avg}(t) \rceil \) for \( t \in I_{>h} \).

For \( t \in I_{\leq h} \), \( \text{avg}(t) \leq 1 \) by definition. That is, the sum of densities of all available jobs at \( t \) is no more than \( h \). By the InsertQueue procedure all jobs will be in the same queue. Hence \( \ell(AD,t) \leq h \) for \( t \in I_{\leq h} \).

By Lemma 31 and Lemma 27, we have Theorem 32 by setting \( \epsilon = 3 \) and \( \epsilon' = 1 \).

**Theorem 32.** For jobs with uniform height, arbitrary width and agreeable deadlines, \( AD \) is \((\frac{12a}{2})^n + 1)\)-competitive.

**Corollary 33.** For jobs with uniform height, arbitrary width and same release time or same deadline, the competitive ratio can be improved to \((\frac{8a}{2})^n + 1\) by using first-fit instead of next-fit for InsertQueue.

## 6 Exact Algorithms

In this section, we propose exact algorithms and derive lower bounds on the running time of exact algorithms. Table 2 gives a summary of our results.

### 6.1 Fixed parameter algorithms

In parameterized complexity theory, the complexity of a problem is not only measured in terms of the input size, but also in terms of parameters. The theory focuses on situations where the parameters can be assumed to be small, and the time complexity depends mainly on these small parameters. The problems having such small parameters are captured by the concept “fixed-parameter tractability”. An algorithm with parameters \( p_1,p_2,\ldots \) is said to be an fixed parameter algorithm if it runs in \( f(p_1,p_2,\ldots) \cdot O(g(N)) \) time for any function \( f \) and any polynomial function \( g \), where \( N \) is the size of input. A parameterized problem is fixed-parameter tractable if it can be solved by a fixed parameter algorithm. In this section, we show that the general case of Grid problem, jobs with arbitrary release times, deadlines, width and height, is fixed-parameter tractable with respect to a few small parameters.

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1. Given \( \frac{k}{2} - 1 < a \) where \( a \) is a real number and \( k \) is integral, we have \( \frac{k}{2} - 1 < \frac{k}{2} < a + 1 \leq \lceil a \rceil + 1 \). Hence, \( k < 2\lceil a \rceil + 2 \). Because both sides are integral, \( k \leq 2\lceil a \rceil + 1 \). Furthermore, \( k \leq 3\lceil a \rceil \) if \( a > 0 \).
We design two fixed parameter algorithms that are based on a dynamic programming fashion. Roughly speaking, we divide the timeline into \( k \) contiguous windows in a specific way, where each window \( W_i \) represents a time interval \([b_i, b_{i+1})\) for \( 1 \leq i \leq k \). The algorithm visits all windows accordingly from the left to the right and maintains a candidate set of schedules for the visited windows that no optimal solution is deleted from the set. In the first fixed parameter algorithm, the parameters of the algorithm are the maximum width of jobs, the maximum number of overlapped feasible intervals and the maximum size of windows, where the latter two can be parameterized if we interpret the input job set as an “interval graph”. We will drop out the last parameter in the second algorithm. All parameters do not increase necessarily as the number of jobs grows, and can be assumed to be small in practice. For example, a width of a job is a requested amount of time to run an appliance, and the running time is usually a few hours, which is small when we make a timeslot to be an hour. And the number of overlapped feasible intervals is at most the number of appliances.

**6.1.1 Key notions**

We propose two exact algorithms, both of which runs in \( \mathcal{O}(n^2 \cdot \max \{w, \sqrt{|B|}\}) \) time, where \( n \) is the number of jobs, \( w \) is the maximum width of jobs, \( |B| \) is the maximum number of overlapped feasible intervals, and \( \sqrt{|B|} \) is the maximum size of windows. We maintain a table \( T \) of configurations in windows. We maintain a table \( T \) row in the table consists of the configurations of all the jobs. In addition, for each window \( W_i \), we compute a table \( T_i \) to store all possible configurations of start and end time of jobs available in \( W_i \). The configurations in \( T_i \) would then be “concatenated” to some configurations in \( T_{i-1} \) that are “compatible” with each other. These merged configurations will be filtered to remove those non-optimal ones. The remaining configurations will become the new \( T_i \) for the next window. To describe the details of the algorithm, we explain several notions below. We denote by \( W_i \) the union of the windows corresponding to \( T_i \). More formally, in the \( i \)-th stage, \( W_i = \cup_{j<i} W_j \). And we use \( T_{i+1} \) to denote \( T_{i+1} \) when the context is clear.

**6.1.2 Framework of the algorithms**

The configurations in \( T \) are executed completely before \( W_i \). Similarly, setting \( st_i(J) = b_{i+1} \) and \( et_i(J) = b_{i+1} + 1 \) means \( J \) starts
execution after $W_i$. Also, setting $st_i(J) = b_i - 1$ and $et_i(J) = b_{i+1} + 1$ means $J$ starts execution before $W_i$, crosses the whole window $W_i$, and ends execution after $W_i$. We say that $st_i(J) \in W_i$ or $et_i(J) \in W_i$ if $st_i(J) \in [b_i, b_{i+1})$ or $et_i(J) \in (b_i, b_{i+1}]$ respectively. And $J$ is executed in $W_i$ if both $st_i(J) \in W_i$ and $et_i(J) \in W_i$ hold. For a collection $C$ of jobs, we use $F_i(C)$ to denote the set of configurations of all jobs in $C$, and $F_{\text{left}}(J)$ and $F_{\text{right}}(C)$ for the counterparts corresponding to $T_{\text{left}}$. The cost of $F_i(C)$ is the cost corresponding to the execution segments in $F_i(C)$. That is, $\text{cost}(F_i(C)) = \sum_{t \in W_j} (\sum_{i \in C : t \in F_i(J)} h(J))^\alpha$.

**Validity.** A configuration $F_i(J)$ is *invalid* if one of the following conditions hold: (i) $st_i(J) \geq et_i(J)$; (ii) $et_i(J) > st_i(J) + w(J)$ meaning that the length of execution segment of $J$ is larger than the width of $J$; (iii) $(et_i(J) < st_i(J) + w(J)) \land (st_i(J) \geq b_i) \land (et_i(J) \leq b_{i+1})$ meaning that the length of execution segment of $J$ is smaller than the width of $J$; (iv) $(st_i(J) < r(J)) \land (st_i(J) < b_{i+1})$ meaning that the start time of $J$ is earlier than the release time of $J$; (v) $(et_i(J) > d(J)) \land (et_i(J) > b_i)$ meaning that the end time of $J$ exceeds the deadline of $J$. Note that for $F_{\text{left}}(J)$, the validity is defined on the boundaries $b_1$ (instead of $b_i$) and $b_{i+1}$. And for $T_{\text{left}}$, $F_{\text{left}}(J)$ is also invalid if $st_{\text{left}}(J) = b_i - 1$ since there is no window on the left of $W_{\text{left}}$. Similarly, $F_{\text{right}}(J)$ is invalid if $et_{\text{right}}(J) = b_{i+1} + 1$. A configuration $F_i(C)$ is *invalid* if there exists $J \in C$ such that $F_i(J)$ is invalid.

**Compatibility.** For job $J$, the two configurations $F_{\text{left}}(J)$ and $F_i(J)$ are *compatible* if (i) $J$ is executed in $W_{\text{left}}$ for $F_{\text{left}}(J)$, and $J$ is executed before $W_i$ for $F_i(J)$; (ii) $J$ starts execution in $W_{\text{left}}$ and ends execution after $W_i$ for $F_{\text{left}}(J)$, and $J$ starts execution before $W_i$ and ends execution either in $W_i$ or after $W_i$ for $F_i(J)$; (iii) $J$ is executed completely after $W_{\text{left}}$ for $F_{\text{left}}(J)$, and $J$ does not start before $W_i$ for $F_i(J)$.

**Concatenating configurations.** To concatenate two configurations $F_{\text{left}}(J)$ and $F_i(J)$, we create a new $F_{\text{left}}(J)$ by the following setting based on the three types of compatible configurations described in the previous paragraph: for type (i), $st_{\text{left}}(J)$ and $et_{\text{left}}(J)$ leave unchanged; for type (ii), $st_{\text{left}}(J)$ leaves unchanged and set $et_{\text{left}}(J) \leftarrow et_i(J)$; and for type (iii), set $st_{\text{left}}(J) \leftarrow st_i(J)$ and $et_{\text{left}}(J) \leftarrow et_i(J)$. Concatenating $F_{\text{left}}(C)$ and $F_i(C)$ is to concatenate the configurations of each job in $C$. The corresponding cost is simply adding the cost of the two configurations.

**Uncertainty and identity.** A configuration $F_i(J)$ is *uncertain* if $et_i(J) = b_{i+1} + 1$ meaning that the end time of $J$ is not determined yet, and we are not sure at the $i$-th stage whether $F_i(J)$ will be valid after concatenating $F_i(J)$ and $F_{i+1}(J)$. Two configurations $F_i(C)$ and $F_i'(C)$ are *identical* if (i) $F_i(J)$ is uncertain if and only if $F_i'(J)$ is uncertain for all job $J \in C$; and (ii) the start time of $F_i(J)$
is equal to the start time of $F'_i(J)$ for all uncertain configuration $F_i(J)$ and $J \in C$. That is, we only consider the differences among the start times of those jobs with uncertain configurations when we distinguish two configurations of a set of jobs.

6.1.3 An algorithm with three parameters

**Algorithm $\mathcal{E}$.** The algorithm consists of three components: ListConfigurations, ConcatenateTables and FilterTable. In the algorithm, we first transform the input job set $\mathcal{J}$ to an interval graph, and obtain the maximal cliques $C_i$ for $1 \leq i \leq k$ and the corresponding windows $W_i$. We start with $T_{\text{left}}$ containing the only configuration, which sets $st_0(J) = b_1$ and $et_0(J) = b_1 + 1$ for all jobs $J$. That is, the configuration treats all the jobs to be not yet executed. Then we visit the windows from the left to the right.

**ListConfigurations:** For window $W_i$ and jobs in $C_i$, we construct $T_{\text{right}}$ storing all configurations of $J \in C_i$. We enumerate all $st_i(J) \in W_i$ and $et_i(J) \in W_i$ for each job $J \in C_i$, list all the combinations of all the jobs $J$ with all of its start times and end times, and store the results in $T_{\text{right}}$ in the way that one row is for one configuration $F_i(C_i)$. In another words, $T_{\text{right}}$ stores all the combinations of execution segments in $W_i$ for all jobs $J \in C_i$. Note that the jobs with release time later than $W_i$ are considered to execute after $W_i$ and the jobs with deadline earlier than $W_i$ are considered to execute before $W_i$. For each configuration $F_i(C_i)$, we also store its cost contribution $\text{cost}(F_i(C_i))$ together. We also check each of the configurations and delete those invalid ones.

**ConcatenateTables:** We then concatenate compatible configurations in $T_{\text{left}}$ and $T_{\text{right}}$. The resulting table is the new $T_{\text{left}}$. More specifically, for each configuration $F_{\text{left}}(C)$ in $T_{\text{left}}$ and each configuration $F_{\text{right}}(C)$ in $T_{\text{right}}$, we concatenate $F_{\text{left}}(C)$ and $F_{\text{right}}(C)$ if they are compatible, and store the result to a new row in $T_{\text{left}}$. We also check each of the configurations in the new $T_{\text{left}}$ and delete those invalid ones.

**FilterTable:** After concatenation, we filter non-optimal configurations. We classify all the configurations in $T_{\text{left}}$ into groups such that the configurations in a group are identical and no two configurations from different groups are identical. For each group, we only leave the configuration with the lowest cost (choosing anyone to break tie if any) and remove the others in the group. In the current $T_{\text{left}}$, no two configurations are identical.

After processing all windows, the only configuration in the final $T_{\text{left}}$ is returned as the solution. Algorithm 1 is the pseudocode of this algorithm.

**Algorithm 1** The fixed parameter algorithm $\mathcal{E}$

| Input: | a set of job $\mathcal{J}$ |
| Output: | an optimal configuration of $\mathcal{J}$ |
| $\{(W_i, C_i)\}_{i=1}^{k}$ | the windows and their corresponding cliques of $\mathcal{J}$ |
| $T_{\text{left}}$ | a configuration that sets all jobs $J \in \mathcal{J}$ to be not yet executed |
| for $i$ from 1 to $k$ do |
| $T_{\text{right}}$ | ListConfigurations($W_i$, $C_i$) |
| $T_{\text{left}}$ | ConcatenateTables($T_{\text{left}}$, $T_{\text{right}}$) |
| $T_{\text{left}}$ | FilterTable($T_{\text{left}}$) |
| return | any configuration in $T_{\text{left}}$ |

**Lemma 34.** Algorithm $\mathcal{E}$ outputs an optimal solution.

**Proof.** In each stage, we list all possible configurations. A configuration is deleted only when it is invalid or it is identical to another configuration with lower cost. It is easy to see an invalid configuration cannot be optimal. So we focus on the other case. Given two identical configurations $F_{\text{left}}(C)$ and $F'_{\text{left}}(C)$ with $\text{cost}(F_{\text{left}}(C)) < \text{cost}(F'_{\text{left}}(C))$, we show that $F'_{\text{left}}(C)$ cannot be optimal. Suppose there is an optimal solution $F^*$ containing $F'_{\text{left}}(C)$, which means each execution segment $F'_i(J)$ in $F'_{\text{left}}(C)$ is completely contained by the corresponding execution interval of $J$ in $F^*$. Since $F_{\text{left}}(C)$ and $F'_{\text{left}}(C)$ are identical, the start times of $J$ are the same in the two configurations for all uncertain jobs $J$. In $W_{\text{left}}$, this means the uncertain jobs do not make the costs of the two configurations to be different,
and the jobs \( J \) that are not uncertain do. Note that \( J \) is consisted of the jobs with their end times being determined. This means we can replace the configurations of \( J \) in \( F'_{\text{left}}(C) \) by the configurations of \( J \) in \( F_{\text{left}}(C) \) and this action will not affect the procedures in the algorithm thereafter. However, this also results in a solution of lower cost and contradicts the assumption that \( F^* \) is optimal. Thus \( F'_{\text{left}}(C) \) cannot be optimal. Therefore, none of the deleted configuration can be part of an optimal schedule. That is, no optimal schedule would be removed through out the whole process. \( \square \)

**Theorem 35.** Algorithm \( \mathcal{E} \) computes an optimal solution in \( O(k \cdot w_{\text{max}}^{2m} \cdot (W_{\text{max}} + 1)^{4m} \cdot n) \) time, where \( n \) is the number of jobs, \( w_{\text{max}} \) is the maximum width of jobs, \( m \) is the maximum size of cliques, \( W_{\text{max}} \) is the maximum length of windows, and \( k \) is the number of windows.

**Proof.** We first compute the time complexities for the three components of the algorithm, and then compute the total time complexity. For ListConfigurations, there are at most \( (W_{\text{max}} + 1)^{2m} \) configurations in the outputted table \( T_{\text{right}} \), since there are at most \( W_{\text{max}} + 1 \) possible start times and end times respectively and at most \( m \) jobs that should be considered in the current window. For each configuration, it takes \( O(n) \) time for construction and validity checking. It also takes \( O(nW_{\text{max}}) \) to compute the cost of a configuration. So, the time complexity for ListConfigurations is

\[
O((W_{\text{max}} + 1)^{2m} \cdot nW_{\text{max}}) = O((W_{\text{max}} + 1)^{2m+1} \cdot n).
\]

Before computing the time complexities of the other components, we focus on the number of configurations of \( T_{\text{left}} \) at the end of each iteration in the algorithm. Since \( T_{\text{left}} \) is filtered to have no identical configurations, the number of configurations can be upper bounded. This number depends on the number of different start times of uncertain jobs. There are at most \( m \) uncertain jobs, and for each such job, the number of start times is at most \( w_{\text{max}} \). Note that the end times of these jobs are all set to be later than the current window and will not affect the number of configurations. So the number of configurations of \( T_{\text{left}} \) at the end of each iteration is at most \( w_{\text{max}}^m \).

For ConcatenateTables, there are at most \( W_{\text{max}}^m \cdot (W_{\text{max}} + 1)^{2m} \) configurations in the outputted table \( T_{\text{left}} \). This is because for each configuration in the input \( T_{\text{left}} \), we need to compare it with all the configurations in \( T_{\text{right}} \) for compatibility checking. For each configuration, it takes \( O(n) \) time for compatibility checking, concatenation and validity checking. Thus the time complexity for ConcatenateTables is \( O(w_{\text{max}}^m \cdot (W_{\text{max}} + 1)^{2m} \cdot n) \).

For FilterTable, the number of configurations in the outputted table \( T_{\text{left}} \) is at most the number of configurations outputted by ConcatenateTables. Also, the number of groups is at most its number of configurations. Thus it takes

\[
O([w_{\text{max}}^m \cdot (W_{\text{max}} + 1)^{2m}]^2 \cdot n) = O(w_{\text{max}}^{2m} \cdot (W_{\text{max}} + 1)^{4m} \cdot n)
\]

time for classification. And it takes \( O(w_{\text{max}}^m \cdot (W_{\text{max}} + 1)^{2m}) \) time for deletion. So the time complexity for FilterTable is \( O(w_{\text{max}}^{2m} \cdot (W_{\text{max}} + 1)^{4m} \cdot n) \). Since there are \( k \) iterations, the total time complexity is \( O(k \cdot w_{\text{max}}^{2m} \cdot (W_{\text{max}} + 1)^{4m} \cdot n) \).

In the worst case, there are at most \( O(n) \) windows. So algorithm \( \mathcal{E} \) also runs in \( f(w_{\text{max}}, m, W_{\text{max}}) \cdot O(n^2) \) time where \( f(w_{\text{max}}, m, W_{\text{max}}) = w_{\text{max}}^{2m} \cdot (W_{\text{max}} + 1)^{4m} \).

**Corollary 36.** Grid problem is fixed parameter tractable with respect to the maximum width of jobs, the maximum number of overlapped feasible intervals, and the maximum length of windows.

### 6.1.4 An algorithm with two parameters

This section describes how to drop out the parameter \( W_{\text{max}} \) in the previous algorithm by generalizing the definitions of windows and boundaries.

At the beginning of Algorithm \( \mathcal{E} \), we transform a set of jobs to its corresponding interval graph and obtain a sequence of windows by the set of maximal cliques in the interval graph. We require in the algorithm that all the cliques should be maximal. However, the algorithm is still optimal and has parameterized bound of time complexity if we divide a maximal clique into multiple non-maximal
cliques in a specific way. Given a maximal clique $C_i$ and its corresponding window $W_i$, we divide $W_i$ into a set of contiguous windows $W_{i1}, W_{i2}, \ldots$ such that $W_i = \bigcup_j W_{ij}$. Note that the set of jobs $C_{ij}$ corresponding to $W_{ij}$ is a clique in the interval graph since $C_i$ is a clique and $C_{ij} \subseteq C_i$. In this way, the number of jobs in the window $W_{ij}$ is still at most $m$. Furthermore, since this window division does not affect the proof of lemma 34, the algorithm is still optimal. Thus we have the following observation.

Observation 37. Algorithm $E$ outputs an optimal solution if it receives a set of contiguous windows containing all the jobs such that each window represents a clique (not necessarily maximal) in the interval graph of the input jobs. And we have the number of jobs in each window is at most the maximum number of overlapped feasible intervals.

To drop out the parameter $W_{\max}$ in the previous algorithm, we divide windows into smaller ones such that the number of configurations in a window can be bounded by $w_{\max}$ and $m$. In the new algorithm, we set the locations of boundaries at the release times and deadlines of all the jobs and construct the windows bases on these boundaries. In this way, there is no job being released or attaining its deadline in the middle of a window, and all the jobs in the window can be put anywhere in the window. Thus the number of used timeslots is at most $m \cdot w_{\max} + 2(w_{\max} - 1)$. This is because in the worst case, all jobs in a window are scheduled such that no job overlaps to another and these jobs consume at most $m \cdot w_{\max}$ timeslots. In addition, we need to consider the cases that a job’s start time is earlier than the window or its deadline is later than the window. Both cases consume at most $w_{\max} - 1$ timeslots respectively. Note that this window division results in a set of windows that their sizes are smaller than their original counterparts, and thus observation 37 can be applied. Based on this new window division, we have the following algorithm.

Algorithm $E^+$. This algorithm is similar to algorithm $E$ except the definitions of boundaries and the component ListConfigurations. Given a set of jobs $J$, the algorithm uses the set of boundaries $\{r(J) \mid J \in \mathcal{J}\} \cup \{d(J) \mid J \in \mathcal{J}\}$ to construct the windows and obtain the corresponding cliques. Let $k$ denotes the number of windows. There are $k$ stages for the algorithm. At the $i$-th stage, the algorithm runs ListConfigurations, ConcatenateTables and FilterTable accordingly as algorithm 1 does. It finally outputs the only configuration in $T_{\text{left}}$. For the component ListConfigurations, we only consider to schedule jobs on the timeslots used instead of enumerating all possibilities of start times and end times. The algorithm tries all $m \cdot w_{\max}$ timeslots (the worst case described in the previous paragraph) as the start time of a job, and also the $2(w_{\max} - 1)$ schedules that a job is partially executed in the window. In addition, the component shall includes the cases that either a job is completely executed before the window, it is completely executed after the window, or it crosses the window.

Theorem 38. Algorithm $E^+$ computes an optimal solution in $f(w_{\max}, m) \cdot O(n^2)$ time, where $n$ is the number of jobs, $w_{\max}$ is the maximum width of jobs, $m$ is the maximum size of cliques, and $f(w_{\max}, m) = (4m \cdot w_{\max})^{2^m} \cdot m$.

Proof. As in the proof of theorem 35 we compute the running time of the three components and then the total time complexity. For the component ListConfigurations, there are at most $(m \cdot w_{\max} + 2(w_{\max} - 1) + 3)^m$ outputted configurations, since there are at most $m \cdot w_{\max} + 2(w_{\max} - 1) + 3$ schedules for a job (see the description in the previous paragraph) and at most $m$ jobs in a window. It takes $O(n(m \cdot w_{\max} + 2(w_{\max} - 1))) \leq O(n \cdot m \cdot w_{\max})$ time to compute the cost for each configuration. Thus the time complexity for ListConfigurations is at most

$$O((m \cdot w_{\max} + 2(w_{\max} - 1) + 3)^m \cdot (n \cdot m \cdot w_{\max})) \leq O((4m \cdot w_{\max})^{m+1} \cdot n).$$

The time complexities of ConcatenateTables and FilterTable are similar to that in the proof of theorem 35 except the number of outputted configurations. For ConcatenateTables and FilterTable, both the number of outputted configurations are at most $w_{\max}^m \cdot (4m \cdot w_{\max})^m$. Thus their running time are at most $O(w_{\max}^{2m} \cdot (4m \cdot w_{\max})^{2m} \cdot n)$. Since there are $k = O(n)$ iterations, the total time complexity of the algorithm is at most

$$O((4m \cdot w_{\max}^{2m})^2 \cdot n^2) = f(w_{\max}, m) \cdot O(n^2).$$

Corollary 39. Grid problem is fixed parameter tractable with respect to the maximum width of jobs, and the maximum number of overlapped feasible intervals.
6.2 An exact algorithm without parameter

For the case with unit width and arbitrary height of Grid problem, we can use algorithm $E$ to design an exact algorithm that its time complexity is only measured in the size of the input.

In Algorithm $E$, we maintain two tables $T_{\text{left}}$ and $T_{\text{right}}$ for each stage. At each stage, the core operations are to construct $T_{\text{right}}$, merge $T_{\text{left}}$ and $T_{\text{right}}$, and filter the resulting table. In the case with unit width and arbitrary height, one may observe that the functionalities of these core operations are not affected by the length of the windows representing $T_{\text{left}}$ and $T_{\text{right}}$. For example, we can restrict the window length to be a constant but not be related to the cliques in the interval graph, and the algorithm still works correctly. By fixing the lengths of all windows, a new exact algorithm is obtained. Without loss of generality, we assume that the number of timeslots $\tau$ is even. We enforce all windows to have length 2, i.e. we have $\tau/2$ windows in total. By this setting, the new algorithm runs in $O((\tau/2) \cdot 4^{2n} \cdot n)$ time where $n$ is the number of jobs. This is because the numbers of configurations for the three components in the algorithm are at most $4^n$. Note that the input size $N$ of the problem is $3n \log \tau + n \log h_{\text{max}}$ where $h_{\text{max}}$ is the maximum height over all jobs. Since $\log \tau = O(N)$, the running time becomes $2^{O(N)}$. Thus we have the following theorem.

**Theorem 40.** There is an exact algorithm running in $2^{O(N)}$ time for the Grid problem with unit width and arbitrary height where $N$ is the length of the input.

Our algorithm is highly more efficient than a brute force search. Such naive method would enumerate all possible schedules and check if they are feasible and optimal, which requires $O(\tau^n n)$ time. The running time can be rewritten as $2^{O(Nn)}$ or more clearly, $(2^{O(N)})^n$. The exact algorithm modified from our fixed parameter algorithm indeed crosses out an ‘$n$’ in the exponent.

6.3 Lower bound

This section provides two lower bounds on the running time of the Grid problem under a certain condition.

Jansen et al. [21] derived several lower bounds for scheduling and packing problems which can be used to develop lower bounds for our problem. Their lower bounds assume Exponential Time Hypothesis (ETH) holds, which conjectures that there is a positive real $\epsilon$ such that 3-SAT cannot be decided in time $2^{o(n)}N^{O(1)}$ where $n$ is the number of variables in the formula and $N$ is the length of the input. A lower bound for other problems can be shown by making use of strong reductions, i.e. reductions that increase the parameter at most linearly. Through a sequence of strong reductions, they obtain two lower bounds for PARTITION, $2^{o(n)}N^{O(1)}$ and $2^{o(\sqrt{N})}$ where $n$ is the cardinality of the given set and $N$ is the length of the input.

**Reduction.** We design a strong reduction from PARTITION to the decision version of Grid problem with unit width and arbitrary height. Here is a sketch of the reduction. Recall that PARTITION is a decision problem that decides if a given set $S$ of integers can be partitioned into two disjoint subsets such that the two subsets have equal sum. For each integer $s \in S$, we convert it to a job $J$ with $r(J) = 0$, $d(J) = 2$, $w(J) = 1$ and $h(J) = 2s$. We claim that $S$ is a partition if and only if the set of jobs $J$ can be scheduled with cost at most $2(\sum_{s \in S} s)^{n}$. Note that the specified cost appears when jobs can be put into two timeslots with equal loads. By setting the length of the input as the parameter, we observe that the parameter increases at most linearly from PARTITION to our problem. (Note that a strong reduction from PARTITION to the case with unit height and arbitrary width can be done similarly, and the results also apply on that case.) Furthermore, we can choose the number of jobs as a parameter of the problem. Note that the reduction above does not increase this parameter with respect to the parameter of PARTITION, which is the number of integers.

**Theorem 41.** There is a lower bound of $2^{o(\sqrt{N})}$ and a lower bound of $2^{o(n)}N^{O(1)}$ on the running time for the Grid problem unless ETH fails, where $n$ is the number of jobs and $N$ is the length of the input.
7 Conclusion

We develop the first online algorithm with log-competitive ratio and the first FPT algorithms for non-preemptive smart grid scheduling problem in general case. We also derive matching lower bound for the competitive ratio. Constant competitive online algorithms are presented for several special input instances. To our best knowledge, this paper presents the first constant-ratio online algorithm for the problem in non-trivial special case.

There are quite a few directions in extending the problem setting: different cost functions perhaps to capture varying electricity cost over time of the day; jobs with varying power requests during its execution (it is a constant value in this paper); other objectives like response time. A preliminary result is that we can extend our online algorithm to the case where a job may have varying power requests during its execution, in other words, a job can be viewed as having rectilinear shape instead of being rectangular. In such case, the competitive ratio is increased by a factor which measures the maximum height to the minimum height ratio of a job.

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References


