Dirichlet integrals and Gaffney-Friedrichs inequalities in convex domains

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Abstract

We study geometrical conditions guaranteeing the validity of the classical Gaffney-Friedrichs estimate

$$\|u\|_{H^{1,2}(\Omega)} \leq C\left(\|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \tag{0.1}$$

granted that the differential form u has a vanishing tangential or normal component on $\partial\Omega$. Our main result is that (0.1) holds provided Ω satisfies a suitable convexity assumption. In the Euclidean setting, a uniform exterior ball condition suffices. As applications, certain regularity results of PDE's and eigenvalue inequalities in non-smooth domains are presented.

1 Introduction

The aim of this paper is to analyze how geometric and analytic properties of the underlying domain affect the global regularity of solutions of PDE's in the context of minimal smoothness assumptions. In order to be more specific, consider a Riemannian manifold $\mathcal{M}$ and let d, δ stand, respectively, for the exterior differentiation operator and its formal adjoint. Also, fix a domain $\Omega \subset \mathcal{M}$ and a differential l-form $u \in L^2(\Omega, \Lambda^l T\mathcal{M})$, $0 \leq l \leq \dim \mathcal{M}$. The issue is to measure the optimal (global) smoothness of u, given that

$$(d+\delta)u \in L^2(\Omega), \qquad \nu \wedge u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

on the usual scale of Sobolev spaces $H^{s,2}(\Omega)$, $s \in \mathbb{R}$. Here, ν is the outward unit conormal on $\partial\Omega$ and $\wedge$ is the exterior product of forms.

According to an old result, due to M. Gaffney [12] and K. Friedrichs [10], (1.1) entails $u \in H^{1,2}(\Omega)$ and

$$\|u\|_{H^{1,2}(\Omega)} \leq C\left(\|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right) \tag{1.2}$$

provided $\partial\Omega \in C^\infty$. The nontrivial aspect of this estimate stems from the fact that $|\nabla u|$ is not (pointwise) majorized by (a multiple of) $|(d+\delta)u|$. Nonetheless, in modern

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terms, (1.2) is simply the natural *a priori* estimate to be expected in this case for the *regular elliptic* BVP $\{d+\delta,\, \nu\wedge\cdot\}$; see, e.g., [27], Proposition 11.2, Vol. 1. The Gaffney-Friedrichs estimate (1.2) is important inasmuch as it reflects the *coerciveness* of the Dirichlet integral

$$D(u) := \iint_{\Omega} \left[|du|^2 + |\delta u|^2\right] d\mathrm{Vol}. \tag{1.3}$$

As such, it has applications to many basic PDE's of mathematical physics like, for example, those arising in electromagnetism and hydrodynamics.

On the other hand, if Ω is an arbitrary *Lipschitz* domain, it has been proved in [19] that (1.1) always implies $u \in H^{1/2,2}(\Omega)$ and

$$\|u\|_{H^{1/2,2}(\Omega)} \leq C\left(\|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right). \tag{1.4}$$

Furthermore, the exponent $1/2$ is sharp in the class of Lipschitz domains.

The question which we address in this paper is that of finding additional geometric conditions on Ω guaranteeing the validity of (1.2) even when $\partial\Omega$ contains irregularities. Another issue of concern is the dependence of the constant C in (1.2) on the metric tensor $g = \sum_{j,k} g_{jk} dx_j \otimes dx_k$. Roughly speaking, we would like to ensure that

$$C \text{ depends on at most } \textit{first order} \text{ derivatives of } g_{jk}\text{'s}. \tag{1.5}$$

One possible line of attack, which avoids the use of pseudodifferential operators (and, hence, can be used in the nonsmooth setting), is via the Bochner-Lichnerowicz-Weitzenbock identity, to the effect that

$$(d+\delta)^2 = d\delta + \delta d = \nabla^*\nabla + \mathcal{R}, \tag{1.6}$$

where ∇ is the Levi-Civita connection and $\mathcal{R}$ is the curvature endomorphism; cf., e.g., [5]. However, we cannot make direct use of (1.6) as $\mathcal{R}$ involves *second order* derivatives of the metric tensor. Our approach is based on deriving an "integrated" version of (1.6) in which one keeps careful track of how the metric is involved. This scenario has connections with the solution of the classical $\bar{\partial}$-problem for several complex variables based on integral identities (cf., e.g., [27] for a general discussion) but the aims are different.

In the class of Lipschitz domains, a satisfactory answer to the question raised two paragraphs above is given by introducing the concept of *l-convexity*, $0 \leq l \leq \dim \mathcal{M}$. For the purpose of this introduction, let us point out that for a smooth domain $\Omega \subset \mathbb{R}^m$, being l-convex amounts to the fact that any collection of $m-l$ principal curvatures (of $\partial\Omega$) has a nonnegative sum. For a general Lipschitz domain $\Omega \subset \mathbb{R}^m$, the requirement is that Ω is approximable by a sequence of smooth, l-convex subdomains. The case when Ω is taken on a manifold is similar in spirit but technically more involved. One of our main results is as follows.

Theorem 1.1. *Let $\Omega \subset \mathcal{M}$ be a Lipschitz domain which is l-convex for some $0 \leq l \leq$ $\dim \mathcal{M}$ and let $u \in L^2(\Omega, \Lambda^l T\mathcal{M})$. Then, granted* (1.1)*, $u \in H^{1,2}(\Omega)$ and* (1.2) *holds for a constant which satisfies* (1.5)*.*

This theorem has several remarkable applications to the regularity of PDE's in Lipschitz domains on manifolds, such as Dirac and Maxwell's equations and Hodge decompositions; see the discussion in § 5.

In the flat-space Euclidean setting, we also consider the case of Lipschitz domains satisfying a *uniform exterior ball condition*. The philosophy of this condition is that boundary irregularities (such as "wedges" and "corners") are directed outward. Formally, it can be also interpreted as a weak bound on the curvature. Our main result in this regard is the following theorem.

Theorem 1.2. *Assume that $\Omega \subset \mathbb{R}^m$ is a bounded Lipschitz domain which satisfies a uniform exterior ball condition. Then for any l-form u, $0 \leq l \leq m$, with components in $L^2(\Omega)$ and so that (1.1) holds, we have that $u \in H^{1,2}(\Omega)$ and (1.2) holds for a constant which depends only on the Lipschitz character of Ω and the exterior ball condition constant.*

Various particular cases of this theorem have been known for some time and are scattered in the literature. Take, for example, the case when u is a 1-form of the type $u := \nabla v$ where v is some scalar-valued function. Then Theorem 1.2 becomes a statement about the regularity of the solution of the Poisson problem for the Laplacian with Dirichlet boundary conditions ([16], [1]). At the other extreme, for certain $(m-1)$-forms u, Theorem 1.2 can be rephrased in terms of the regularity of the solution of the Poisson problem for the Laplacian with *Neumann* boundary conditions (cf. also [3]). Finally, the case of arbitrary vector fields (i.e. when $l = 1$ or, under suitable identifications, when $l = m-1$) and when Ω is (geometrically) convex, has been considered in [6], [14], [25], [8]. Here we provide a unified treatment as well as a significant generalization of such results.

The organization of the paper is as follows. Section 2 contains some preliminary material, whereas Section 3 is reserved for a detailed discussion of l-convexity. Identities for Dirichlet type integrals are considered in Section 4, while Gaffney-Friedrichs estimates in l-convex domains are, subsequently, derived in Section 5. Let us point out that in the second part of §5 we discuss several remarkable consequences of the Gaffney-Friedrichs type estimates alluded to before in the context of eigenvalue estimates for the Hodge-Laplacian and PDE's problems in non-smooth domains. Finally, in Section 6, we conduct a similar study at the level of Lipschitz domains satisfying a uniform exterior ball condition.

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2 Notation, definitions and preliminary results

Let $\mathcal{M}$ be a smooth, compact, boundaryless, orientable Riemannian manifold of real dimension m. For $0 \leq l \leq m$, denote by $\Lambda^l T\mathcal{M}$ the l-th exterior power of the tangent bundle $T\mathcal{M}$; sections in $\Lambda^l T\mathcal{M}$ are l-forms. Fix $\mathcal{A} = (U_\nu)_\nu$, a finite atlas of $\mathcal{M}$. Then the metric tensor g is locally written in each U_ν as $\sum_{j,k} g^\nu_{jk} dx_j \otimes dx_k$. In the sequel, we shall drop the index ν, i.e. write U, g_{jk}, etc., but we shall always understand that any localization process is done with respect to the aforementioned atlas $\mathcal{A}$. Also, denote by $(g^{jk})_{jk}$ the inverse matrix of $(g_{jk})_{jk}$.

Throughout the paper, in order to avoid excessive technicalities, we shall assume that g_{jk} are smooth but we emphasize that, as far as the metric is concerned,

$$\text{all our estimates will only involve } \|g_{jk}\|_{L^\infty},\ \|g^{jk}\|_{L^\infty} \text{ and } \|\nabla g_{jk}\|_{L^\infty} \tag{2.1}$$

(written with respect to $\mathcal{A}$). To indicate the dependence of a constant C on the above quantities in a more explicit fashion, we simply write $C = C(g)$.

As is customary, if $(x_1, \ldots, x_m)$ are local coordinates and if $\wedge$ stands for the usual *exterior product* of forms, we set $dx^I := dx_{i_1} \wedge \ldots \wedge dx_{i_l}$ for any increasing l-tuple $I = (i_1, \ldots, i_l)$. We let $|I|\,(= l)$ stand for the cardinality of I.

The Hermitian structure on fibers of $T\mathcal{M}$ extends naturally to $\oplus_{0\leq l\leq m}\Lambda^l T\mathcal{M}$ by setting

$$\langle dx^I, dx^J\rangle := \det\left((g^{ij})_{i\in I, j\in J}\right) \tag{2.2}$$

if $|I| = |J|$ and zero otherwise.

Let $\{w^1, \ldots, w^m\}$ be an orthonormal basis of $\Lambda^1 T\mathcal{M}$ with Lipschitz coefficients in some open set $U \subseteq \mathcal{M}$, and denote by $\left\{\frac{\partial}{\partial w^1}, \cdots, \frac{\partial}{\partial w^m}\right\}$ the corresponding dual basis of $T\mathcal{M}$ in U. Then, if d is the usual exterior derivative operator, we have that

$$du = \sum_{1\leq j\leq m} \frac{\partial u}{\partial w^j} w^j, \tag{2.3}$$

for any C^1, scalar-valued function u. A useful consequence of this is the principal symbol calculation

$$\sigma\left(\frac{\partial}{\partial w^j}; du\right) = \sqrt{-1}\frac{\partial u}{\partial w^j}, \qquad 1 \leq j \leq m. \tag{2.4}$$

Also, set

$$\alpha^i_{jk} := \langle dw^i, w^j \wedge w^k\rangle, \qquad 1 \leq i, j, k \leq m. \tag{2.5}$$

Note that if U is a coordinate patch and $\{w^j\}_j$ is obtain by applying the usual Gramm-Schmidt orthonormalization process to $dx_1, \ldots, dx_m$, where $(x_1, \ldots, x_m)$ are local coordinates in U, then

$$\|\alpha^i_{jk}\|_{L^\infty} \leq C(\|g\|_{L^\infty}, \|\nabla g\|_{L^\infty}). \tag{2.6}$$

We make (2.6) a standing assumption throughout the paper.

The main, elementary properties of the system of functions $(\alpha^i_{jk})_{ijk}$ are collected in the next lemma.

Lemma 2.1. *With the above notation, the following hold:*

(i) α^i_{jk} *is antisymmetric in* j *and* k*;*

(ii) $dw^i = \sum_{j<k} \alpha^i_{jk} w^j \wedge w^k = \frac{1}{2}\sum_{j,k} \alpha^i_{jk} w^j \wedge w^k$*;*

(iii) $\operatorname{div}\left(\frac{\partial}{\partial w^j}\right) = \sum_l (-1)^l \alpha^l_{lj}$*. In particular,* $\left(\frac{\partial}{\partial w_j}\right)^* = -\frac{\partial}{\partial w_j} - \sum_l (-1)^l \alpha^l_{lj}$*;*

(iv) $\frac{1}{2}\left[\frac{\partial}{\partial w^k}, \frac{\partial}{\partial w^j}\right] = \sum_i \alpha^i_{jk}\frac{\partial}{\partial w^i}$*, where* $[\cdot,\cdot]$ *stands for the usual commutator bracket;*

(v) $\frac{1}{2}\left\langle\left[\frac{\partial}{\partial w^k}, \frac{\partial}{\partial w^j}\right], \frac{\partial}{\partial w^i}\right\rangle = \alpha^i_{jk}$*.*

Proof. The point (i) follows directly from (2.5), as $w^j \wedge w^k = -w^k \wedge w^j$. Next, (ii) is a consequence of (2.5) and the fact that $\{w^j \wedge w^k\}_{1\leq j<k\leq m}$ is an orthonormal basis for $\Lambda^2 T\mathcal{M}|_U$. The first part in (iii) is seen by a direct calculation. Then, the second part,

follows on account of this and the general identity $X^* = -X - \operatorname{div} X$ valid for any $X \in T\mathcal{M}$. Going further, observe that for any C^2, scalar-valued function u we have

$$\begin{aligned} 0 = d^2 u = d\left(\sum_j \frac{\partial u}{\partial w^j} w^j\right) &= d\left(\frac{\partial u}{\partial w_j}\right) \wedge w^j + \frac{\partial u}{\partial w^i} dw^i \\ &= \sum_{j,k}\left(\frac{\partial^2 u}{\partial w^k \partial w^j} - \sum_i \alpha^i_{jk} \frac{\partial u}{\partial w^i}\right) w^k \wedge w^j. \end{aligned} \tag{2.7}$$

Thus, since $\{w^k \wedge w^j\}_{1 \leq k < j \leq m}$ is an orthonormal basis in $\Lambda^2 T\mathcal{M}|_U$ and u is arbitrary, necessarily

$$\frac{\partial^2}{\partial w^k \partial w^j} - \sum_i \alpha^i_{jk} \frac{\partial}{\partial w^i} \quad \text{is symmetric in } j \text{ and } k. \tag{2.8}$$

In particular, based on this and (i), we conclude that

$$\left[\frac{\partial}{\partial w^k}, \frac{\partial}{\partial w^j}\right] = \frac{\partial^2}{\partial w^k \partial w^j} - \frac{\partial^2}{\partial w^j \partial w^k} = 2\sum_i \alpha^i_{jk} \frac{\partial}{\partial w^i}, \tag{2.9}$$

i.e. (iv). Now (v) follows immediately from this. This completes the proof of the lemma. ■

In U, define the Hessian of a C^2 scalar-valued function u relative to the basis $\{\frac{\partial}{\partial w^i}\}_j$ to be the matrix

$$\operatorname{Hess}(u) := (u_{j_k})_{j,k} := \text{ the symmetric part of} \left\{\frac{\partial^2 u}{\partial w^j \partial w^k}\right\}_{j,k}. \tag{2.10}$$

It follows, via Lemma 2.1, that

$$u_{jk} = \frac{\partial^2 u}{\partial w^j \partial w^k} + \sum_i \alpha^i_{jk} \frac{\partial u}{\partial w^i}, \quad \forall j, k. \tag{2.11}$$

If $u \in \Lambda^l T\mathcal{M}$, then $u|_U$ has the form $\sum'_{|I|=l} u_I w^I$, where $\sum'$ indicates that the sum is performed over ordered l-tuples $I = (i_1, \ldots, i_l)$, $1 \leq i_1 < \ldots < i_l \leq m$ and, for each such I, $w^I := w^{i_1} \wedge \ldots \wedge w^{i_l}$. At the level of l-forms, if $u = \sum'_{|I|=l} u_I w^I$ in U, then

$$\begin{aligned} du = \sum_{|I|=l}{}' (du_I \wedge w^I + u_I dw^I) = &\sum_{|L|=l+1}{}' \sum_{|I|=l}{}' \sum_j \epsilon^{jI}_L \frac{\partial u_I}{\partial w_j} w^L \\ &+ \sum_{|L|=l+1}{}' \sum_{|I|=l} \sum_{i \in I} \sum_{j<k} \epsilon^{iL}_{jkI} \alpha^i_{jk} u_I w^L, \end{aligned} \tag{2.12}$$

where

$$\epsilon^I_J := \begin{cases} \det(\delta_{ij})_{i \in I, j \in J}, & \text{if} \quad |I| = |J|, \\ 0, & \text{otherwise}, \end{cases} \tag{2.13}$$

is the generalized Kronecker symbol. If we denote by d' the principal part of d then, from (2.6) and (2.12),

$$|du - d'u| \leq C(\|g\|_{L^\infty}, \|\nabla g\|_{L^\infty})|u|. \tag{2.14}$$

The exterior co-differential operator, δ, is the formal adjoint of d. Based on (2.12), (iii) in Lemma 2.1 and integrations by parts it follows that, in U,

$$\begin{aligned} \delta u = &- \sum_{|K|=l-1}{}' \sum_j \frac{\partial u_{jK}}{\partial w^j} w^K + \sum_{|K|=l-1}{}' \sum_{|L|=l}{}' \sum_{j<k} \sum_{i\in K} \epsilon^{iL}_{jkK} \alpha^i_{jk} u_L w^K \\ &- \sum_{|K|=l-1}{}' \sum_j \sum_i (-1)^i \alpha^i_{ij} u_{jk} w^K. \end{aligned} \tag{2.15}$$

Upon denoting the principal part of δ by δ', (2.15) becomes

$$|\delta u - \delta' u| \leq C(\|g\|_{L^\infty}, \|\nabla g\|_{L^\infty})|u|. \tag{2.16}$$

We will make use of the Hodge star operator, which can be characterized as the unique vector bundle morphism $* : \Lambda^l T\mathcal{M} \to \Lambda^{m-l} T\mathcal{M}$ such that

$$u \wedge (*u) = |u|^2 \, d\mathrm{Vol}, \tag{2.17}$$

where dVol stands for the volume form on $\mathcal{M}$. We define the *interior product* between a 1-form α and an l-form u by setting

$$\alpha \vee u := (-1)^{(l-1)m} * (\alpha \wedge *u). \tag{2.18}$$

Thus, as it is well known, for $\xi \in T^*\mathcal{M} \backslash 0$, the principal symbols of d, δ are, respectively, given by

$$\sigma(d;\xi)u = \sqrt{-1}\,\xi \wedge u \quad \text{and} \quad \sigma(\delta;\xi)u = -\sqrt{-1}\,\xi \vee u. \tag{2.19}$$

Recall that a domain $\Omega \subset \mathcal{M}$ is called *Lipschitz* if $\partial\Omega$ can be described in local coordinates by graphs of Lipschitz functions (mapping $\mathbb{R}^{m-1}$ into $\mathbb{R}$). In particular, if Ω is a Lipschitz subdomain of $\mathcal{M}$ with outward unit conormal ν defined a.e. on $\partial\Omega$, with respect to the surface measure $d\sigma$, and $u \in C^1(\bar{\Omega}, \Lambda^l T\mathcal{M})$, $v \in C^1(\bar{\Omega}, \Lambda^{l+1} T\mathcal{M})$, then

$$\iint_\Omega \langle du, v \rangle \, d\mathrm{Vol} - \iint_\Omega \langle u, \delta v \rangle \, d\mathrm{Vol} = \int_{\partial\Omega} \langle \nu \wedge u, v \rangle \, d\sigma = \int_{\partial\Omega} \langle u, \nu \vee v \rangle \, d\sigma. \tag{2.20}$$

For further reference as well as for the convenience of the reader, some basic, elementary properties of these objects are summarized in the following lemma.

Lemma 2.2. *For arbitrary one-forms α, β, and any l-form u, $(m-l)$-form v, and $(l+1)$-form w, the following are true:*

(1) $* * u = (-1)^{l(m-l)} u$;

(2) $\langle u, *v \rangle = (-1)^{l(m-l)} \langle *u, v \rangle$ *and* $\langle *u, *v \rangle = \langle u, v \rangle$;

(3) $\alpha \wedge (\alpha \wedge u) = 0$ *and* $\alpha \vee (\alpha \vee u) = 0$;

(4) $\alpha \wedge (\beta \vee u) + \beta \vee (\alpha \wedge u) = \langle \alpha, \beta \rangle u$*;*

(5) $\langle \alpha \wedge u, w \rangle = \langle u, \alpha \vee w \rangle$*;*

(6) $*(\alpha \wedge u) = (-1)^l \alpha \vee (*u)$ *and* $*(\alpha \vee u) = (-1)^{l-1} \alpha \wedge (*u)$

Moreover, if α *is normalized such that* $\langle \alpha, \alpha \rangle = 1$*, then also:*

(7) $u = \alpha \wedge (\alpha \vee u) + \alpha \vee (\alpha \wedge u)$*;*

(8) $|\alpha \wedge (\alpha \vee u)| = |\alpha \vee u|$ *and* $|\alpha \vee (\alpha \wedge u)| = |\alpha \wedge u|$*.*

Finally,

(9) $dd = 0$*,* $\delta\delta = 0$*;*

(10) $\delta = (-1)^{m(l+1)=1} * d*$ *and* $*\delta = (-1)^l d*$*,* $\delta * = (-1)^{l+1} * d$ *on* l*-forms.*

We denote by $H^{s,p}$ and $B^{s,p}$, respectively, the usual scales of Sobolev and Besov spaces. In particular, if Ω is a Lipschitz subdomain of $\mathcal{M}$, $H^{s,p}(\Omega, \Lambda^l T\mathcal{M})$ stands for differential l-forms with $H^{s,p}(\Omega)$-coefficients, etc. Denote the outgoing unit conormal on $\partial\Omega$ by ν. Also, let $1 \leq l \leq m$ and $u \in L^p(\Omega, \Lambda^l T\mathcal{M})$ be such that $\delta u \in L^p(\Omega, \Lambda^{l-1} T\mathcal{M})$ for some $1 < p < \infty$. Then we define the distribution $\nu \vee u$ on $\partial\Omega$ by requiring that

$$\langle\langle \nu \vee u, \varphi \rangle\rangle := -\iint_\Omega \langle \delta u, v \rangle \, d\mathrm{Vol} + \iint_\Omega \langle u, dv \rangle \, d\mathrm{Vol}, \tag{2.21}$$

for any $v \in H^{1,q}(\Omega, \Lambda^{l-1} T\mathcal{M})$, $1/p + 1/q = 1$, with $\mathrm{Tr}\, v = \varphi$. Here, $\mathrm{Tr} : C^0(\bar{\Omega}, \Lambda^l T\mathcal{M}) \to C^0(\partial\Omega, \Lambda^l T\mathcal{M})$ is the ordinary trace operator, which further extends from $H^{s,p}(\Omega, \Lambda^l T\mathcal{M})$ into $B^{s-1/p,p}(\partial\Omega, \Lambda^l T\mathcal{M})$ for any $s \in \left(\frac{1}{p}, 1 + \frac{1}{p}\right)$, $1 < p < \infty$. Also, $\langle\langle \cdot, \cdot \rangle\rangle$ is the distributional pairing. Thus, the right side of (2.21) is well defined for $\varphi \in B^{1/p,q}(\partial\Omega, \Lambda^{l-1} T\mathcal{M})$, independently of the choice of such v, so we have

$$\nu \vee u \in B^{-1/p,p}(\partial\Omega, \Lambda^{l-1} T\mathcal{M}) \tag{2.22}$$

with naturally accompanying estimates. Further, if $u \in L^p(\Omega, \Lambda^l T\mathcal{M})$ is such that $du \in L^p(\Omega, \Lambda^{l+1} T\mathcal{M})$ then, similarly, we can define $\nu \wedge u \in B^{-1/p,p}(\partial\Omega, \Lambda^{l+1} T\mathcal{M})$ or, equivalently, set $\nu \wedge u := (-1)^{1+m(l+1)} * (\nu \vee *u)$.

3 Discussion of l-convexity

Let Ω be a C^2 domain in $\mathcal{M}$ and let $\rho \in C^2$ be a defining function for Ω. That is, $\rho > 0$ in $\mathcal{M} \setminus \bar{\Omega}$, $\rho < 0$ in Ω and $d\rho \neq 0$ on $\partial\Omega$. In particular, $d\rho$ is parallel to ν, the outward unit conormal to $\partial\Omega$. In the definition below, recall that ρ_{jk} is defined as in (2.10)–(2.11).

Definition 3.1. *Call the* C^2*-smooth domain* Ω l*-convex,* $0 \leq l \leq m$*, if for any* $x_0 \in \partial\Omega$ *there exist an open neighborhood* U *of* x_0 *and* $\{w^j\}_j$*, orthonormal basis of* $\Lambda^1 T\mathcal{M}$ *in* U *with* C^1*-coefficients so that* (2.6) *is satisfied and, if* ρ *is a defining function for* Ω*, then for any* $x \in \partial\Omega \cap U$

$$\begin{gathered} \sum_{j,k} \rho_{jk} \langle w^j \wedge u, w^k \wedge u \rangle \geq 0 \quad \textit{at} \quad x \quad \textit{whenever} \\ u \in \Lambda^l T_x\mathcal{M} \quad \textit{is such that} \quad \nu(x) \wedge u = 0. \end{gathered} \tag{3.1}$$

If Ω is merely Lipschitz, we call Ω l-convex if there exists a nested sequence of C^2 domains $\{\Omega_\mu\}_\mu$, with $\Omega_\mu \subset\subset \Omega$, $\cup_\mu \Omega_\mu = \Omega$, and such that the following is true. For any $x_0 \in \partial\Omega$, there exists an open set $U \subseteq \mathcal{M}$ with $x_0 \in U$, and $\{w^j\}_j$ orthonormal basis of $\Lambda^1 T\mathcal{M}$ with C^1-coefficients and satisfying (2.6) in U so that, if ρ_μ is a defining function for Ω_μ, then for a.e. $x \in \partial\Omega_\mu \cap U$ the condition (3.1) holds with ρ, ν replaced by ρ_μ and ν_μ, respectively.

Before going any further, several remarks are in order here.

(i) The above definition is actually independent of the particular choice of the defining function ρ. Indeed, a different choice for ρ changes $\sum_{j,k} \rho_{jk}\langle w^j \wedge u, w^k \wedge u\rangle$ only by a positive factor whenever $\nu \wedge u = 0$.

(ii) Due to symmetry considerations, ρ_{jk} can be replaced by $\frac{\partial^2 \rho}{\partial w^j \partial w^k}$ in (3.1).

(iii) For $l = 0$ and $l = m$, the condition (3.1) is trivially satisfied regardless of the domain Ω. Thus, l-convexity is a genuine restriction only for $1 \leq l \leq m-1$.

(iv) When $l = m-1$, the condition (3.1) is equivalent to

$$\begin{aligned} &\sum_{j,k} \rho_{jk}(x)\, t_j t_k \geq 0 \quad \text{whenever the real numbers} \\ &\{t_j\}_j \quad \text{are such that} \quad \sum_{1\leq j\leq m} t_j \frac{\partial \rho}{\partial w_j}(x) = 0. \end{aligned} \tag{3.2}$$

Indeed, in one direction, if $\{t_j\}_j$ are so that $\sum_j t_j \frac{\partial\rho}{\partial w^j}(x) = 0$, set

$$u := \sum_{1\leq j\leq m} (-1)^{j-1} t_j\, w_x^1 \wedge \ldots \wedge w_x^{j-1} \wedge w_x^{j+1} \wedge \ldots \wedge w_x^m \in \Lambda^{m-1} T_x\mathcal{M}. \tag{3.3}$$

If $h := |d\rho|$ then $\nu \wedge u = h(x)^{-1}\left(\sum_j t_j \frac{\partial\rho}{\partial w^j}(x)\right) d\mathrm{Vol} = 0$ and $\langle w_x^j \wedge u, w_x^k \wedge u\rangle = t_j t_k$. Thus, (3.2) is implied by (3.1) when $l = m-1$. The converse implication is similar in spirit and, hence, omitted.

(v) 1-convexity (for a smooth domain) simply means

$$\sum_{j,k} \rho_{jk}\langle w^j \wedge \nu, w^k \wedge \nu\rangle \geq 0 \quad \text{on} \quad \partial\Omega \cap U. \tag{3.4}$$

Since $\nu = h^{-1}\, d\rho = h^{-1}\left(\sum_l \frac{\partial\rho}{\partial w^l} w^l\right)$ where $h := |d\rho|$, a direct calculation shows that (3.4) is further equivalent to

$$\left(\sum_j \rho_{jj}\right)\left(\sum_k \left(\frac{\partial\rho}{\partial w^k}\right)^2\right) - \sum_j \sum_k \rho_{jk} \frac{\partial\rho}{\partial w^j}\frac{\partial\rho}{\partial w^k} \geq 0 \quad \text{on} \quad \partial\Omega \cap U, \tag{3.5}$$

i.e. to

$$\mathrm{Trace}\,[\mathrm{Hess}\,(\rho)] - \langle \mathrm{Hess}\,(\rho)\nu, \nu\rangle \geq 0 \quad \text{on} \quad \partial\Omega \cap U. \tag{3.6}$$

Here, $\mathrm{Hess}\,(\rho)$ (cf. (2.10)) is taken with respect to $\{\frac{\partial}{\partial w^j}\}_j$.

(vi) More generally, since $\nu(x) \wedge u = 0$ is equivalent to the existence of $v \in \Lambda^{l-1}T_x\mathcal{M}$ so that $u = \nu(x) \wedge v$, we see that (3.1) is also equivalent to

$$\sum_{j,k} \rho_{jk}\langle w^j \wedge (\nu \wedge v), w^k \wedge (\nu \wedge v)\rangle \geq 0 \quad \text{on} \quad \partial\Omega \cap U \qquad \text{for any} \quad v \in \Lambda^{l-1}T\mathcal{M}|_{\partial\Omega}. \tag{3.7}$$

Writing $v = \sum'_{|I|=l-1} v_I w^I$, with v_I arbitrary, we get that (3.7) is equivalent to

$$\sum_{j,k} \sum_{|I|=|J|=l-1}{}' \sum_{p,q} \epsilon^{jpI}_{kqJ} \rho_{jk} \frac{\partial \rho}{\partial w^p} \frac{\partial \rho}{\partial w^q} v_I v_J \geq 0, \quad \forall\, (v_I)_I, \quad x \in \partial\Omega \cap U, \tag{3.8}$$

i.e. to

$$\left(\sum_{j,k} \sum_{p,q} \epsilon^{jpI}_{kqJ} \rho_{jk} \nu_p \nu_q \right)_{|I|=l-1, |J|=l-1} \geq 0 \quad \text{at a.e.} \quad x \in \partial\Omega \cap U, \tag{3.9}$$

where $(\nu_p)_p$ are the components of ν in the basis $\{w^p\}_p$.

(vii) We elaborate further on the condition (3.9) in the case when Ω is the domain in $\mathbb{R}^m$ (equipped with the ordinary Euclidean metric) lying above the graph of the C^2 function $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$. In this situation, due to the invariance of (3.9) to translations and rotations, matters can be arranged so that

$$\varphi(0) = 0, \quad \nabla\varphi(0) = 0 \quad \text{and} \quad \partial_j\partial_k\varphi(0) = 0 \quad \text{if} \quad j \neq k. \tag{3.10}$$

Choosing $\rho(x) := \varphi(x') - x_m$, where $x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$, as a defining function for Ω, we see that, at $0 \in \mathbb{R}^m$, the positivity condition (3.9) amounts to

$$\sum_{j \in L} \partial_j^2 \varphi(0) \geq 0, \quad \forall L \subseteq \{1, \ldots, m-1\}, \quad |L| = m - l. \tag{3.11}$$

In particular, if $\Omega \subseteq \mathbb{R}^m$ is geometrically convex then Ω is l-convex for any $l \in \{0, 1, \ldots, m\}$.

Parenthetically, let us point out that the condition (3.11) can be expressed in an invariant form as follows. Consider $\tilde{\nu}$ a C^1 vector field in a neighborhood of $\partial\Omega$ so that $\tilde{\nu}|_{\partial\Omega} = \nu$, the outward unit normal to $\partial\Omega$ (e.g., $d\rho/\|d\rho\|$ will do). Next, introduce the (real, symmetric) $m \times m$ curvature matrix

$$\kappa := \left(\partial_j \tilde{\nu}_i - \sum_{k=1}^{m} \tilde{\nu}_k \tilde{\nu}_j \partial_k \tilde{\nu}_i \right)_{i,j}. \tag{3.12}$$

Clearly, κ has the eigenvalue 0 corresponding to the eigenvector $\tilde{\nu}$. Denote by $(\kappa_j)_j$ the other $m-1$ eigenvalues, the so-called *principal curvatures* of Ω. Then, with this notation, (3.11) becomes

$$\text{the sum of any } (m-l) \text{ principal curvatures is nonnegative.} \tag{3.13}$$

We conclude this section with the following.

Proposition 3.2. *For a Lipschitz domain $\Omega \subseteq \mathcal{M}$,*

$$(m-1)\text{-}convexity \Longrightarrow l\text{-}convexity, \quad \forall\, l \in \{0,1,...,m\}. \tag{3.14}$$

Proof. Clearly, it is enough to prove (3.14) at the level of smooth domains. To this end, let $u = \sum'_{|I|=l} u_I w^I \in \Lambda^l T\mathcal{M}$ be such that

$$\nu \wedge u = 0, \quad \text{on } \partial\Omega. \tag{3.15}$$

Then,

$$\begin{aligned}
\sum_{j,k} \rho_{jk} \langle w^j \wedge u, w^k \wedge u \rangle &= \sum_{j,k} \sum_{M}{}' \sum_{I}{}' \sum_{J}{}' \rho_{jk} \epsilon^{jI}_{M} \epsilon^{kJ}_{M} u_I u_J \\
&= \sum_{M}{}' \sum_{j,k} \rho_{jk} \left(\sum_{I}{}' \epsilon^{jI}_{M} u_I \right) \left(\sum_{I}{}' \epsilon^{kI}_{M} u_I \right) \\
&= \sum_{M}{}' \left(\sum_{j,k} \rho_{jk} t^M_j t^M_k \right),
\end{aligned} \tag{3.16}$$

where $t^M_j := \sum'_I \epsilon^{jI}_M u_I$. Note that, for any M,

$$\sum_j t^M_j \frac{\partial \rho}{\partial w^j} = \sum_I{}' \sum_j \epsilon^{jI}_M \frac{\partial \rho}{\partial w_j} u_I = (d\rho \wedge u)_M = 0, \tag{3.17}$$

by (3.15), since $d\rho$ is parallel to ν. Consequently, by hypothesis and the Remark (iv) above, $\sum_{j,k} \rho_{jk} t^M_j t^M_k \geq 0$ on $\partial\Omega$. Hence $\sum_{j,k} \rho_{jk} \langle w^j \wedge u, w^k \wedge u \rangle \geq 0$ on $\partial\Omega$, as desired.
■

4 Dirichlet integrals

Let $\Omega \subseteq \mathcal{M}$ be a C^2-domain. Our aim is to analyze the family of Dirichlet integrals

$$D_l(u,v) := \iint_\Omega \langle du, dv \rangle \, d\text{Vol} + \iint_\Omega \langle \delta u, \delta v \rangle \, d\text{Vol}, \tag{4.1}$$

where $u, v \in H^{1,2}(\Omega, \Lambda^l T\mathcal{M})$, $0 \leq l \leq m$.

We shall do so under the additional assumption that u and v are supported in some open set $U \subseteq \mathcal{M}$ in which an orthonormal basis with C^1-coefficients $\{w^1, \ldots, w^m\}$ for $\Lambda^1 T\mathcal{M}$ has been selected. Of course, this is not restrictive inasmuch as matters can be reduced to this case via a partition of unity. In connection with $\{w^j\}_j$ we shall employ the notation introduced in § 2.

The main result of this section is the identity contained in the theorem below. To state it, we need some more notation. Specifically, we introduce the boundary differential operators

$$d_\partial := -d\rho \wedge d(d\rho \vee \cdot), \qquad \delta_\partial := -d\rho \vee \delta(d\rho \wedge \cdot), \tag{4.2}$$

for a fixed, defining function ρ so that $|d\rho| = 1$ on $\partial\Omega$. In particular, $d\rho|_{\partial\Omega} = \nu$, the outward unit conormal to $\partial\Omega$. A more detailed account on d_∂, δ_∂ can be found in [19]. Also, for an arbitrary form ω, we write $\omega_{\tan} := d\rho \vee (d\rho \wedge \omega)$, called the tangential component of ω, and $\omega_{\rm nor} := d\rho \wedge (d\rho \vee \omega)$, the normal component of ω. Note that $\omega = \omega_{\tan} + \omega_{\rm nor}$ and $\langle \omega_{\tan}, \omega_{\rm nor} \rangle = 0$. Finally, denote by $d\sigma$ the surface area on $\partial\Omega$.

Theorem 4.1. *Assume that the differential forms $u, v \in H^{1,2}(\bar\Omega \cap U, \Lambda^l T\mathcal{M})$, $0 \le l \le m$, are so that* $\operatorname{supp} u, \operatorname{supp} v \subseteq U$ *and that $u = \sum'_{|I|=l} u_I w^I$, $v = \sum'_{|I|=l} v_I w^I$ in U.*

Then there exist

$$G_l,\ H_l \in \operatorname{End}\left(\Lambda^l T\mathcal{M}|_U\right) \tag{4.3}$$

so that

$$\begin{aligned} D_l(u,v) = & {\sum_{|I|=l}}' \sum_{j=1}^m \iint_\Omega \frac{\partial u_I}{\partial w^j} \frac{\partial v_I}{\partial w^j}\, dVol + \int_{\partial\Omega} \sum_{j,k} \frac{\partial^2 \rho}{\partial w^k \partial w^j} \langle w^j \wedge u_{\rm nor}, w^k \wedge v_{\rm nor} \rangle \, d\sigma \\ & + \int_{\partial\Omega} \sum_{j,k} \frac{\partial^2 \rho}{\partial w^k \partial w^j} \langle w^j \vee u_{\tan}, w^k \vee v_{\tan} \rangle \, d\sigma \\ & + \int_{\partial\Omega} \langle G_l(u_{\tan}), v_{\rm nor} \rangle \, d\sigma + \int_{\partial\Omega} \langle H_l(u_{\rm nor}), v_{\tan} \rangle \, d\sigma \\ & - \int_{\partial\Omega} \langle \nu \vee u, \delta_\partial v_{\tan} \rangle \, d\sigma - \int_{\partial\Omega} \langle \nu \wedge u, d_\partial v_{\rm nor} \rangle \, d\sigma \\ & + R(u,v), \end{aligned} \tag{4.4}$$

where the residual term $R(u,v)$ satisfies

$$|R(u,v)| \le C(g) \left[\|u\|_{H^{1,2}(\Omega)} \|v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|v\|_{H^{1,2}(\Omega)} \right]. \tag{4.5}$$

The departure point in the proof of Theorem 4.1 is the following partial result.

Proposition 4.2. *Assume that $u = \sum'_{|I|=l} u_I w^I$, $v = \sum'_{|I|=l} v_I w^I \in C^1(\bar\Omega \cap U, \Lambda^l T\mathcal{M})$ and that, in addition,*

$$\nu \wedge u = 0, \quad \nu \wedge v = 0 \quad \text{on} \quad \partial\Omega \cap U. \tag{4.6}$$

Then

$$\begin{aligned} \iint_\Omega \langle du, dv \rangle \, dVol + \iint_\Omega \langle \delta u, \delta v \rangle \, dVol = & {\sum_{|I|=l}}' \sum_{j=1}^m \iint_\Omega \frac{\partial u_I}{\partial w^j} \frac{\partial v_I}{\partial w^j} \, dVol \\ & + \sum_{j,k} \int_{\partial\Omega} \frac{\partial^2 \rho}{\partial w^k \partial w^j} \langle w^j \wedge u, w^k \wedge v \rangle \, d\sigma \\ & + R(u,v), \end{aligned} \tag{4.7}$$

where the residual term $R(u,v)$ satisfies

$$|R(u,v)| \le C(g)\left[\|u\|_{H^{1,2}(\Omega)}\|v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v\|_{H^{1,2}(\Omega)}\right]. \tag{4.8}$$

Proof. Recall the operators d' and δ' from § 2. The effect of changing du, dv to $d'u\; d'v$ and $\delta u, \delta v$ to $\delta' u, \delta' v$ in the left side of (4.7) can, by (2.14) and (2.16), be controlled as in (4.8). Thus, we may replace d, δ by d' and δ', respectively. As such, we are interested in

$$\iint_\Omega \langle d'u, d'v\rangle \, d\mathrm{Vol} + \iint_\Omega \langle \delta' u, \delta' u\rangle \, d\mathrm{Vol} = \iint_\Omega \langle [(d')^* d' + (\delta')^* \delta'] u, v\rangle \, d\mathrm{Vol} + \int_{\partial\Omega} \langle d'u, \nu \wedge v\rangle \, d\sigma - \int_{\partial\Omega} \langle \delta' u, \nu \vee v\rangle \, d\sigma. \tag{4.9}$$

Note that the differences $(d')^* - \delta'$ and $(\delta')^* - d'$ are zero-order operators whose coefficients are $\mathcal{O}(\|g\|_{L^\infty} + \|\nabla g\|_{L^\infty})$. Consequently,

$$\iint_\Omega \langle [(d')^* d' + (\delta')^* \delta'] u, v\rangle \, d\mathrm{Vol} = \iint_\Omega \langle (\delta' d' + d'\delta') u, v\rangle \, d\mathrm{Vol} + R_1(u,v), \tag{4.10}$$

where $R_1(u,v)$ obeys the estimate (4.8). Going further, a direct calculation based on the explicit form of d' and δ' shows that if $u = \sum'_{|I|=l} u_I w^I$, then

$$\delta' d' u = -{\sum_{|K|=l}}' \left({\sum_{|I|=l}}' \sum_{j,k} \epsilon^{jI}_{kK} \frac{\partial^2 u_I}{\partial w^k \partial w^j} \right) w^K, \tag{4.11}$$

$$d'\delta' u = -{\sum_{|K|=l}}' \left({\sum_{|L|=l-1}}' \sum_{j,k} \epsilon^{jL}_{K} \frac{\partial^2 u_{kL}}{\partial w^j \partial w^k} \right) w^K. \tag{4.12}$$

In the sequel, we shall simply ignore the zero terms. Now, for the non-zero terms in the RHS of (4.11) which have $j = k$ it follows that, necessarily, $I = K$. Thus, since in this case $\epsilon^{jI}_{kK} = 1$,

$$\text{the "}j = k\text{" part in the RHS of (4.11) is } -{\sum_{|K|=l}}' \left(\sum_{j \notin K} \frac{\partial^2 u_K}{\partial w^j \partial w^j} \right) w^K. \tag{4.13}$$

Also, clearly,

$$\text{the "}j = k\text{" part in the RHS of (4.12) is } -{\sum_{|K|=l}}' \left(\sum_{j \in K} \frac{\partial^2 u_K}{\partial w^j \partial w^j} \right) w^K. \tag{4.14}$$

Next, consider the remaining part in the RHS of (4.11). Write $M \equiv N$ if two arrays M and N coincide as sets. Note that $j \neq k$ in the RHS of (4.11) forces $I \equiv kL$ and $K \equiv jL$ for some L with $|L| = l-1$, in which case $\epsilon^{jI}_{kK} = \epsilon^{jkL}_{kjL} = -1$. Hence,

$$\text{the "}j \neq k\text{" part in the RHS of (4.11) is } \sum_{|L|=l-1}{}' \sum_{k \neq j} \frac{\partial^2 u_{kL}}{\partial w^k \partial w^j} w^{jL}. \tag{4.15}$$

Except for the order of differentiation and the sign, this is precisely the "$j \neq k$" part in the RHS of (4.12). Adding up the two then gives

$$\sum_{|L|=l-1}{}' \sum_{j,k} \left(\left[\frac{\partial}{\partial w_k}, \frac{\partial}{\partial w_j} \right] u_{kL} \right) w^{jL} = \frac{1}{2} \sum_{|L|=l-1}{}' \sum_{j,k} \sum_i \alpha^i_{jk} \frac{\partial u_{kL}}{\partial w^i} w^{jL}, \tag{4.16}$$

by Lemma 2.1. When paired with $v = \sum'_{|I|=l} v_I w^I$, this yields the term

$$\iint_\Omega \frac{1}{2} \sum_{|L|=l-1}{}' \sum_{j,k} \sum_i \alpha^i_{jk} \frac{\partial u_{kL}}{\partial w^i} v_{jL} \, d\mathrm{Vol} = \mathcal{O}\left(\|\alpha^i_{jk}\|_{L^\infty} \|u\|_{H^{1,2}(\Omega)} \|v\|_{L^2(\Omega)} \right). \tag{4.17}$$

By (2.6), this qualifies as residual in the sense of (4.8) and can be absorbed in what we shall call, in the end, $R(u, v)$.

At this stage, from (4.11)-(4.17), the only non-residual terms produced by the volume integral in the right side of (4.10) are

$$\begin{aligned} -\iint_\Omega \sum_{|K|=l}{}' \sum_{j=1}^m \frac{\partial^2 u_K}{\partial w^j \partial w^j} v_K \, d\mathrm{Vol} &= \iint_\Omega \sum_{|K|=l}{}' \sum_{j=1}^m \frac{\partial u_K}{\partial w^j} \frac{\partial v_K}{\partial w^j} \, d\mathrm{Vol} \\ &\quad + \iint_\Omega \sum_{|K|=l}{}' \sum_{j=1}^m \frac{\partial u_K}{\partial w^j} \left(\mathrm{div} \, \frac{\partial}{\partial w^j} \right) v_K \, d\mathrm{Vol} \\ &\quad - \int_{\partial\Omega} \sum_{|K|=l}{}' \sum_{j=1}^m \frac{\partial u_K}{\partial w_j} v_K \frac{\partial \rho}{\partial w^j} \, d\sigma \\ &=: A + B + C. \end{aligned} \tag{4.18}$$

We shall keep A, which is part of the final identity. Also, by Lemma 2.1, B is residual in the sense of (4.8) and can be omitted. On the other hand, C is still subject to cancellations with surface integrals in (4.9).

In order to make this connection more transparent, we digress for a moment and observe that

$$\nu \wedge u = 0 \Leftrightarrow \sum_{|I|=l}{}' \sum_{j=1}^m \epsilon^{jI}_M \frac{\partial \rho}{\partial w^j} u_I = 0, \quad \forall\, |M| = l+1. \tag{4.19}$$

Now, generally speaking, if $(a_j)_j$, $(b_k)_k$ are scalar functions so that

$$\sum_j a_j \frac{\partial \rho}{\partial w^j} = \sum_k b_k \frac{\partial \rho}{\partial w^k} = 0 \quad \text{on } \partial\Omega, \tag{4.20}$$

then $\left(\sum_j a_j \frac{\partial}{\partial w^j}\right)\left(\sum_k b_k \frac{\partial \rho}{\partial w^k}\right) = 0$ on $\partial\Omega$; that is,

$$\sum_j \sum_k \left(a_j b_k \frac{\partial^2 \rho}{\partial w^j \partial w^k} + a_j \frac{\partial b_k}{\partial w^j} \frac{\partial \rho}{\partial w^k} \right) = 0 \quad \text{on} \quad \partial\Omega. \tag{4.21}$$

For fixed, arbitrary M with $|M| = l+1$, set $a_j := \sum'_{|I|=l} \epsilon^{jI}_M u_I$, $b_k := \sum'_{|I|=l} \epsilon^{kJ}_M u_J$ and notice that (4.20) is satisfied in this case, thanks to (4.19) and (4.6). Writing the identity (4.21) for these a_j's, b_k's and then adding in M, it follows that

$$-\sum_{|M|=l+1}{}' \sum_{I,J}{}' \sum_{j,k} \epsilon^{jI}_M \epsilon^{kJ}_M v_I \frac{\partial u_J}{\partial w^j} \frac{\partial \rho}{\partial w^k} = \sum_M{}' \sum_{I,J}{}' \sum_{j,k} \frac{\partial^2 \rho}{\partial w^j \partial w^k} \epsilon^{jI}_M \epsilon^{kJ}_M v_I u_J. \tag{4.22}$$

In fact, further inspection reveals that the right side of (4.22) can be organized as

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial w^k \partial w^j} \langle w^j \wedge u, w^k \wedge u \rangle. \tag{4.23}$$

Returning to the mainstream discussion, in the light of the observation made in the previous paragraph, there remains to show that surface integrands in (4.9) amount precisely to the left side of (4.22). To this end, we split the sum in the left-hand side of (4.22) according to whether j equals k or not. When $j = k$ then, necessarily, $I = J$ and, after a slight change in notation, this part becomes

$$-\sum_{|K|=l}{}' \sum_{j \notin K} \frac{\partial \rho}{\partial w^j} \frac{\partial u_I}{\partial w^j} v_K. \tag{4.24}$$

This is covered by the "$j \notin K$" part of the term C in (4.18).

Also, for non-zero terms with $j \neq k$ in the LHS of (4.22) one has, necessarily, $I \equiv kL$ and $J \equiv jL$ for some L with $|L| = l-1$. Since $\sum'_M \epsilon^{jkL}_M \epsilon^{kjL}_M = -1$ if $j, k \notin L$, the corresponding part reads

$$\sum_{|L|=l-1}{}' \sum_{\substack{j \neq k \\ j,k \notin L}} \frac{\partial \rho}{\partial w^k} \frac{\partial u_{jL}}{\partial w^j} v_{kL}. \tag{4.25}$$

In turn, this is covered by the "$j \neq k$" piece from

$$-\langle \delta' u, \nu \vee v \rangle = -\langle \nu \wedge \delta' u, v \rangle = \sum_{|L|=l-1}{}' \sum_{j,k \notin L} \frac{\partial \rho}{\partial w^k} \frac{\partial u_{jL}}{\partial w^j} v_{kL}. \tag{4.26}$$

The remaining piece in (4.26), corresponding to $j = k$, will then cancel the "$j \in K$" part of C in (4.18). Since $\int_{\partial\Omega} \langle d' u, \nu \wedge v \rangle \, d\sigma = 0$, by hypothesis, all terms are accounted for and (4.7) follows. ■

We are now in a position to present the

Proof of Theorem 4.1. The first order of business is to assume that the l-forms u and v are smooth on $\bar{\Omega} \cap U$ and to analyze the effect of dropping the hypothesis (4.6) on the identity (4.7). To this end, an inspection of the previous proof reveals that (4.6) has only been used to transform

$$\int_{\partial\Omega}\langle d'u,\nu\wedge v\rangle\,d\sigma-\int_{\partial\Omega}\langle\delta'u,\nu\vee v\rangle\,d\sigma-{\sum_{|K|=l}}'\sum_{j=1}^{m}\int_{\partial\Omega}\frac{\partial\rho}{\partial w^j}\frac{\partial u_K}{\partial w^j}v_K\,d\sigma \tag{4.27}$$

in the boundary integral in (4.7). In order to explain what happens in the general case under discussion, let us introduce the *ad hoc* notation

$$\frac{\partial\omega}{\partial n}:={\sum_{|I|=l}}'\sum_{j=1}^{m}\frac{\partial\rho}{\partial w^j}\frac{\partial\omega_I}{\partial w^j}w^I,\quad\text{if}\quad\omega={\sum_{|I|=l}}'\omega_Iw^I\quad\text{in}\quad U. \tag{4.28}$$

Decomposing u,v in normal and tangential components in (4.27) yields twelve integrals, four of which are zero by simple orthogonality considerations (cf. Lemma 2.2). We group four of the remaining eight as follows:

$$I:=-\int_{\partial\Omega}\langle\delta'u_{\rm nor},\nu\vee v_{\rm nor}\rangle\,d\sigma-\int_{\partial\Omega}\left\langle\frac{\partial u_{\rm nor}}{\partial n},v_{\rm nor}\right\rangle\,d\sigma, \tag{4.29}$$

$$II:=\int_{\partial\Omega}\langle d'u_{\rm tan},\nu\wedge v_{\rm tan}\rangle\,d\sigma-\int_{\partial\Omega}\left\langle\frac{\partial u_{\rm tan}}{\partial n},v_{\rm tan}\right\rangle\,d\sigma, \tag{4.30}$$

and treat the rest, i.e.

$$\begin{array}{ll}-\displaystyle\int_{\partial\Omega}\left\langle\frac{\partial u_{\rm tan}}{\partial n},v_{\rm tan}\right\rangle\,d\sigma, & -\displaystyle\int_{\partial\Omega}\left\langle\frac{\partial u_{\rm nor}}{\partial n},v_{\rm tan}\right\rangle\,d\sigma,\\ \displaystyle\int_{\partial\Omega}\langle d'u_{\rm nor},\nu\wedge v_{\rm tan}\rangle\,d\sigma, & -\displaystyle\int_{\partial\Omega}\langle\delta'u_{\rm tan},\nu\vee v_{\rm nor}\rangle\,d\sigma,\end{array} \tag{4.31}$$

separately, one at a time.

As far as I is concerned, the point is that now $\nu\wedge u_{\rm nor}=\nu\wedge v_{\rm nor}=0$, i.e. (4.6) is satisfied for $u_{\rm nor}$ and $v_{\rm nor}$. Thus, by our previous calculation, I in (4.29) can be brought in the form

$$I=\int_{\partial\Omega}\sum_{j,k}\frac{\partial^2\rho}{\partial w^k\partial w^j}\langle w^j\wedge u_{\rm nor},w^k\wedge v_{\rm nor}\rangle\,d\sigma. \tag{4.32}$$

The next observation is that (4.30) corresponds to (4.29) written for $*u$ and $*v$ in place of u,v. Indeed, the Hodge $*$-isomorphism intertwines $(\dots)_{\rm tan}$ with $(\dots)_{\rm nor}$, $\wedge$ with $\vee$, commutes with $\frac{\partial}{\partial n}$, satisfies $\delta'*=(-1)^{l+1}*d$ on l-forms, and is an isometry. Hence, much as before, II in (4.30) can be rewritten as

$$II=\int_{\partial\Omega}\sum_{j,k}\frac{\partial^2\rho}{\partial w^k\partial w^j}\langle w^j\vee u_{\rm tan},w^k\vee v_{\rm tan}\rangle\,d\sigma. \tag{4.33}$$

Next we turn our attention to the four integrals in (4.31). In order to treat the first one, we need to study the commutator between $\frac{\partial}{\partial n}$ and the interior multiplication by $d\rho$. Concretely, if $\omega=\sum_I'\omega_Iw^I$, then

$$
\begin{aligned}
\left[\frac{\partial}{\partial n}, d\rho\vee\cdot\right]\omega =& \frac{\partial}{\partial n}(d\rho\vee\omega) - d\rho\vee\frac{\partial\omega}{\partial n}\\
=& \Big(\sum\sum\sum \frac{\partial\omega_{jL}}{\partial w^k}\frac{\partial\rho}{\partial w^k}\frac{\partial\rho}{\partial w^j}w^L + \sum\sum\sum \omega_{jL}\frac{\partial^2\rho}{\partial w^k\partial w^j}\frac{\partial\rho}{\partial w^k}w^L\Big)\\
&-\sum\sum\sum \frac{\partial\omega_{jL}}{\partial w^k}\frac{\partial\rho}{\partial w^k}\frac{\partial\rho}{\partial w^j}w^L\\
=& \sum\sum\sum \omega_{jL}\frac{\partial^2\rho}{\partial w^k\partial w^j}\frac{\partial\rho}{\partial w^k}w^L =: N'\omega.
\end{aligned}
\tag{4.34}
$$

The conclusion is that $N' = \left[\frac{\partial}{\partial n}, d\rho\vee\cdot\right]$ is a zero order operator. With this piece of notation, the first integral in (4.31) is

$$
\begin{aligned}
-\int_{\partial\Omega}\left\langle \frac{\partial}{\partial n}\big(d\rho\vee(d\rho\wedge u)\big), v_{\rm nor}\right\rangle d\sigma =& -\int_{\partial\Omega}\left\langle d\rho\vee\big(\frac{\partial}{\partial n}(d\rho\wedge u)\big), v_{\rm nor}\right\rangle d\sigma\\
&-\int_{\partial\Omega}\langle N'(d\rho\wedge u), v_{\rm nor}\rangle\, d\sigma\\
=& -\int_{\partial\Omega}\langle N'(d\rho\wedge u_{\rm tan}), v_{\rm nor}\rangle\, d\sigma,
\end{aligned}
\tag{4.35}
$$

since $d\rho\wedge u_{\rm nor} = 0$. If we set $N'' := (-1)^{m(l-1)} * N' *$ then, based on the properties of the Hodge $*$-isomorphism and the above calculation we see that the second integral in (4.31) is

$$
-\int_{\partial\Omega}\langle N''(d\rho\vee u_{\rm nor}), v_{\rm tan}\rangle\, d\sigma. \tag{4.36}
$$

Observe next that $d(u_{\rm nor}) = d(d\rho\wedge(d\rho\vee u)) = -d\rho\wedge d(d\rho\vee u)$ so that

$$
\begin{aligned}
\int_{\partial\Omega}\langle du_{\rm nor}, \nu\wedge v_{\rm tan}\rangle\, d\sigma =& -\int_{\partial\Omega}\langle d\rho\wedge d(d\rho\vee u), d\rho\wedge v_{\rm tan}\rangle\, d\sigma\\
=& -\int_{\partial\Omega}\langle d(d\rho\vee u), v_{\rm tan}\rangle\, d\sigma\\
=& -\int_{\partial\Omega}\langle \nu\vee u, \delta_\partial v_{\rm tan}\rangle\, d\sigma,
\end{aligned}
\tag{4.37}
$$

where we recall, from (4.2), that $\delta_\partial = -\nu\vee\delta(\nu\wedge\cdot)$ on $\partial\Omega$ and the last equality follows by repeated integrations by parts.

Therefore, denoting the zero order (error) operator $d - d'$ by E', it follows that the third integral in (4.31) can be written as

$$
\begin{aligned}
&-\int_{\partial\Omega}\langle \nu\vee u, \delta_\partial v_{\rm tan}\rangle\, d\sigma - \int_{\partial\Omega}\langle E'u_{\rm nor}, \nu\wedge v_{\rm tan}\rangle\, d\sigma\\
&= -\int_{\partial\Omega}\langle \nu\vee u, \delta_\partial v_{\rm tan}\rangle\, d\sigma - \int_{\partial\Omega}\langle d\rho\vee E'u_{\rm nor}, v_{\rm tan}\rangle\, d\sigma.
\end{aligned}
\tag{4.38}
$$

Similarly, if $E'' := \delta - \delta'$, then the last integral in (4.31) gives

$$-\int_{\partial\Omega} \langle \nu \wedge u, d_\partial v_{\rm nor} \rangle \, d\sigma + \int_{\partial\Omega} \langle d\rho \wedge E'' u_{\rm tan}, v_{\rm nor} \rangle \, d\sigma, \tag{4.39}$$

where $d_\partial = -\nu \wedge d(\nu \vee \cdot)$ on $\partial\Omega$; cf (4.2).

Combining all the above, the identity (4.4) follows in the case when u, v are of class C^1 in $\bar{\Omega} \cap U$ if we set

$$G_l := d\rho \wedge E'' - N'(d\rho \wedge \cdot), \qquad H_l := -d\rho \vee E' - N''(d\rho \vee \cdot). \tag{4.40}$$

Finally, we need to show that the C^1 regularity assumption on u, v can be relaxed to $u, v \in H^{1,2}(\Omega, \Lambda^l T\mathcal{M})$. Indeed, this follows from what we have proved so far and a density argument. The only thing left to check is that

$$\int_{\partial\Omega} \langle \nu \vee u, \delta_\partial v_{\rm tan} \rangle \, d\sigma \quad \text{and} \quad \int_{\partial\Omega} \langle \nu \wedge u, d_\partial v_{\rm nor} \rangle \, d\sigma$$

have a proper interpretation when $u, v \in H^{1,2}(\Omega)$ only. Nonetheless, by the trace theorem, $u|_{\partial\Omega}$, $v|_{\partial\Omega}$ have $H^{1/2,2}(\partial\Omega)$ coefficients, whereas δ_∂, d_∂, on tangential and normal forms, respectively, involve only tangential derivative operators. This latter fact is seen from (4.2) and a symbol calculation:

$$\begin{aligned} \sigma(d_\partial; d\rho) &= -\sqrt{-1}\, d\rho \wedge (d\rho \wedge (d\rho \vee \cdot)) = 0, \\ \sigma(\delta_\partial; d\rho) &= \sqrt{-1}\, d\rho \vee (d\rho \vee (d\rho \wedge \cdot)) = 0. \end{aligned} \tag{4.41}$$

Thus, we can interpret the two integrals under discussion in the sense of the natural pairing between $H^{1/2,2}(\partial\Omega)$ and $(H^{1/2,2}(\partial\Omega))^*$. This finishes the proof of Theorem 4.1. ■

Corollary 4.3. *For any $u = \sum'_{|I|=l} u_I w^I \in H^{1,2}(U \cap \Omega, \Lambda^l T\mathcal{M})$ with $\operatorname{supp} u \subseteq \bar{\Omega} \cap U$ and satisfying $\nu \wedge u = 0$ on $\partial\Omega \cap U$, there holds*

$$\begin{aligned} \iint_\Omega [|du|^2 + |\delta u|^2] \, dVol = & \sum_{|I|=l}{}' \sum_{j=1}^m \iint_\Omega \left| \frac{\partial u_I}{\partial w^j} \right|^2 dVol \\ & + \int_{\partial\Omega} \sum_{j,k} \rho_{jk} \langle w^j \wedge u, w^k \wedge u \rangle \, d\sigma + R(u), \end{aligned} \tag{4.42}$$

where

$$|R(u)| \le C(\|g\|_{L^\infty}, \|\nabla g\|_{L^\infty}) \, \|u\|_{H^{1,2}(\Omega)} \|u\|_{L^2(\Omega)}. \tag{4.43}$$

Proof. This follows from Theorem 4.1 by taking $u = v$. Note that, in this case, the boundary term in (4.7) can be "symmetrized", i.e. we may replace $\left\{ \frac{\partial^2 \rho}{\partial w^k \partial w^j} \right\}_{j,k}$ by its symmetric part. This justifies the form of the boundary integrand in (4.42). ■

5 Gaffney-Friedrichs inequalities and applications

In this section we shall prove a quantitative version of the identity (4.7) which amounts to the Gaffney-Friedrichs estimate for l-convex domains. The main result in this respect is Theorem 5.1 below. Several applications to PDE's and eigenvalue estimates are discussed in Corollaries 5.7–5.11.

Theorem 5.1. *Let Ω be a Lipschitz, l-convex domain in $\mathcal{M}$, $0 \leq l \leq m$, and let $u \in L^2(\Omega, \Lambda^l T\mathcal{M})$ be so that $du \in L^2(\Omega, \Lambda^{l+1} T\mathcal{M})$, $\delta u \in L^2(\Omega, \Lambda^{l-1} T\mathcal{M})$ and $\nu \wedge u = 0$ on $\partial\Omega$. Then $u \in H^{1,2}(\Omega, \Lambda^l T\mathcal{M})$ and*

$$\|u\|_{H^{1,2}(\Omega)} \leq C(\|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \tag{5.1}$$

for some $C > 0$ which depends only on $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$ and $\|\nabla g\|_{L^\infty}$.

From this and Hodge star-duality, the following consequence is easily derived.

Corollary 5.2. *Assume that $\Omega \subseteq \mathcal{M}$ is a $(m-l)$-convex Lipschitz domain, $0 \leq l \leq m$, and $v \in L^2(\Omega, \Lambda^l T\mathcal{M})$ is such that $dv \in L^2(\Omega, \Lambda^{l+1} T\mathcal{M})$, $\delta v \in L^2(\Omega, \Lambda^{l-1} T\mathcal{M})$ and $\nu \vee v = 0$ on $\partial\Omega$. Then $v \in H^{1,2}(\Omega, \Lambda^l T\mathcal{M})$ and*

$$\|v\|_{H^{1,2}(\Omega)} \leq C(\|dv\|_{L^2(\Omega)} + \|\delta v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}) \tag{5.2}$$

for some positive C which depends exclusively on $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$ and $\|\nabla g\|_{L^\infty}$.

Proof of Theorem 5.1. Let $\{\Omega_\mu\}_\mu$ be a sequence of l-convex, C^2-domains such that $\cup_\mu \Omega_\mu = \Omega$ and denote by $\rho_\mu \in C^2$ a defining function for Ω_μ. Since it suffices to prove a local version of (5.1), there is no loss of generality in assuming that $\operatorname{supp} u \subseteq U \cap \bar{\Omega}$, for some small open subset U of $\mathcal{M}$ in which some orthonormal basis $\{w^j\}_j$ for $\Lambda^2 T\mathcal{M}|_U$ satisfying (2.6) has been fixed so that

$$\sum_{j,k} (\rho_\mu)_{jk} \langle w^j \wedge \omega, w^k \wedge \omega \rangle \geq 0 \quad \text{at} \quad x \in \partial\Omega_\mu \cap U \\ \text{whenever } \omega \in \Lambda^l T_x\mathcal{M} \text{ is such that } \nu_\mu(x) \wedge \omega = 0. \tag{5.3}$$

Fix some strictly positive number λ and, for each μ, let u_μ be the unique solution of the boundary problem

$$\begin{cases} u_\mu \in L^2(\Omega_\mu, \Lambda^l T\mathcal{M}),\ du_\mu \in L^2(\Omega_\mu, \Lambda^{l+1} T\mathcal{M}), \\ (\Delta - \lambda) u_\mu = 0 \quad \text{in} \quad \Omega_\mu, \\ \delta u_\mu = 0 \quad \text{in} \quad \Omega_\mu, \\ \nu_\mu \wedge u_\mu = \nu_\mu \wedge (u|_{\Omega_\mu}) \in H^{-1/2,2}(\partial\Omega_\mu, \Lambda^l T\mathcal{M}). \end{cases} \tag{5.4}$$

Here, and elsewhere, $\Delta := -d\delta - \delta d$ stands for the usual Hodge Laplacian and the boundary condition is understood in the sense explained in the last part of § 2. It is known (cf., e.g., [18]) that (5.4) is uniquely solvable and, for any μ,

$$\|u_\mu\|_{L^2(\Omega_\mu)} + \|du_\mu\|_{L^2(\Omega_\mu)} \leq C(\lambda)(\|u\|_{L^2(\Omega_\mu)} + \|du\|_{L^2(\Omega_\mu)}). \tag{5.5}$$

Let us also observe that, since our hypothesis imply $u \in H^{1,2}_{\text{loc}}(\Omega, \Lambda^l T\mathcal{M})$ and since Ω_μ is smooth, we also have $u_\mu \in H^{1,2}(\bar{\Omega}_\mu, \Lambda^l T\mathcal{M})$. This follows from, e.g., the regularity of the L^2-Hodge decomposition in smooth domains (on which the solution to (5.4) is based); cf. [27] Vol. I, [26]. In particular, if $\eta \in C^\infty_{\text{comp}}(U)$ is such that $\eta \equiv 1$ near x_0, then

$$v_\mu := (u|_{\Omega_\mu} - u_\mu)\eta \in H^{1,2}(\Omega_\mu, \Lambda^l T\mathcal{M}), \tag{5.6}$$

satisfies $\operatorname{supp} v_\mu \subseteq \bar{\Omega} \cap U$,

$$\|v_\mu\|_{L^2(\Omega_\mu)} + \|dv_\mu\|_{L^2(\Omega_\mu)} + \|\delta v_\mu\|_{L^2(\Omega_\mu)} \leq C(\lambda)[\|u\|_{L^2(\Omega)} + \|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)}], \tag{5.7}$$

and

$$\nu_\mu \wedge v_\mu = 0 \quad \text{on} \quad \partial\Omega_\mu, \quad \forall\, \mu. \tag{5.8}$$

From Corollary 4.3 it follows that, if $v_\mu = \sum'_{|I|=l} v_I^\mu w^I$, then

$$\begin{aligned} \iint_{\Omega_\mu} [|dv_\mu|^2 + |\delta v_\mu|^2]\, d\text{Vol} = & \sum_{|I|=l}{}' \sum_{j=1}^{m} \iint_{\Omega_\mu} \left| \frac{\partial v_I^\mu}{\partial w^j} \right|^2 d\text{Vol} \\ & + \sum_{j,k} \int_{\partial\Omega_\mu} (\rho_\mu)_{jk} \langle w^j \wedge v_\mu, w^k \wedge v_\mu \rangle \, d\sigma_\mu \\ & + R(v_\mu), \end{aligned} \tag{5.9}$$

where $d\sigma_\mu$ is the surface measure on $\partial\Omega_\mu$ and

$$|R(v_\mu)| \leq C(g) \|v_\mu\|_{H^{1,2}(\Omega_\mu)} \|v_\mu\|_{L^2(\Omega_\mu)}. \tag{5.10}$$

This, (5.9), (5.8), (5.3) and (5.7) imply that

$$\begin{aligned} \|v_\mu\|_{H^{1,2}(\Omega_\mu)} & \leq C(g,\lambda)(\|dv_\mu\|_{L^2(\Omega_\mu)} + \|\delta v_\mu\|_{L^2(\Omega_\mu)} + \|v_\mu\|_{L^2(\Omega_\mu)}) \\ & \leq C(g,\lambda)(\|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \end{aligned} \tag{5.11}$$

uniformly in μ, where C depends only on the metric and λ. It follows that there exists $v \in H^{1,2}(\Omega, \Lambda^l T\mathcal{M})$ so that

$$\|v\|_{H^{1,2}(\Omega)} \leq C(g,\lambda) \left(\|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right) \tag{5.12}$$

and

$$v_\mu \to v \text{ weakly in } H^{1,2} \text{ on compact subsets of } \Omega, \text{ as } \mu \to \infty. \tag{5.13}$$

Next, we claim that

$$u_\mu \to 0 \text{ weakly in } L^2 \text{ on compact subsets of } \Omega, \text{ as } \mu \to \infty. \tag{5.14}$$

Accepting (5.14) for a moment, it follows from this, (5.13) and (5.6) that $u = v$ in Ω near x_0. In view of (5.12) and since x_0 was arbitrary, this finishes the proof of the theorem (modulo that of (5.14)).

Turning our attention to (5.14), we first observe that, from (5.5), there exists $\omega \in L^2(\Omega, \Lambda^l T\mathcal{M})$ with $d\omega \in L^2(\Omega, \Lambda^{l+1} T\mathcal{M})$ and such that

$$\begin{aligned} &\iint_{\Omega_\mu} \langle u_\mu - \omega, \varphi \rangle \, d\mathrm{Vol} \to 0 \quad \text{as} \quad \mu \to \infty, \\ &\iint_{\Omega_\mu} \langle du_\mu - d\omega, \psi \rangle \, d\mathrm{Vol} \to 0 \quad \text{as} \quad \mu \to \infty, \end{aligned} \tag{5.15}$$

for any $\varphi \in L^2(\Omega, \Lambda^l T\mathcal{M})$, $\psi \in L^2(\Omega, \Lambda^{l+1} T\mathcal{M})$. In particular, from (5.4),

$$(\Delta - \lambda)\omega = 0 \quad \text{and} \quad \delta\omega = 0 \quad \text{in} \quad \Omega. \tag{5.16}$$

Next, we argue that also

$$\nu \wedge \omega = 0. \tag{5.17}$$

This is going to imply that ω solves the boundary value problem

$$\begin{cases} \omega \in L^2(\Omega, \Lambda^l T\mathcal{M}),\ d\omega \in L^2(\Omega, \Lambda^{l+1} T\mathcal{M}), \\ (\Delta - \lambda)\omega = 0 \quad \text{in} \quad \Omega, \\ \delta\omega = 0 \quad \text{in} \quad \Omega, \\ \nu \wedge \omega = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.18}$$

Given that $\lambda > 0$, it follows (cf., e.g., [18]) that $\omega = 0$, i.e. (5.14) holds.

Returning to (5.17), let $\varphi \in H^{1/2,2}(\partial\Omega, \Lambda^{l+1} T\mathcal{M})$ be arbitrary and $\tilde{\varphi} \in H^{1,2}(\Omega, \Lambda^{l+1} T\mathcal{M})$ be an extension (in the Sobolev trace sense) of φ. We have

$$\begin{aligned} \langle\!\langle \nu \wedge \omega, \varphi \rangle\!\rangle &= \iint_\Omega \langle d\omega, \tilde{\varphi} \rangle \, d\mathrm{Vol} - \iint_\Omega \langle \omega, \delta\tilde{\varphi} \rangle \, d\mathrm{Vol} \\ &= \lim_{\mu \to \infty} \left[\iint_{\Omega_\mu} \langle du_\mu, \tilde{\varphi} \rangle \, d\mathrm{Vol} - \iint_{\Omega_\mu} \langle u_\mu, \delta\tilde{\varphi} \rangle \, d\mathrm{Vol} \right] \\ &= \lim_{\mu \to \infty} \left[\iint_{\Omega_\mu} \langle du, \tilde{\varphi} \rangle \, d\mathrm{Vol} - \iint_{\Omega_\mu} \langle u, d\tilde{\varphi} \rangle \, d\mathrm{Vol} \right] \\ &= \iint_\Omega \langle du, \tilde{\varphi} \rangle \, d\mathrm{Vol} - \iint_\Omega \langle u, \delta\tilde{\varphi} \rangle \, d\mathrm{Vol} = 0 \end{aligned} \tag{5.19}$$

since, by hypothesis, $\nu \wedge u = 0$ on $\partial\Omega$. This shows that (5.17) holds and concludes the proof of Theorem 5.1. ■

Remark. By means of counterexamples it is easy to show that (5.1) fails for general Lipschitz domains. Indeed, fix $\omega \in (0, 2\pi)$ and consider $\Omega_\omega \subseteq \mathbb{R}^2$ given in polar coordinates by $\{0 < r < 1,\ -\omega/2 < \theta < \omega/2\}$. Also, let $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R})$ be such that $\varphi(r) \equiv 0$ for $|r| \geq \frac{1}{2}$ and $\varphi(r) \equiv 1$ for $|r| \leq \frac{1}{4}$. Consider the 1-form $u := d\left[\varphi(r) r^{\pi/\omega} \cos\left(\frac{\pi\theta}{\omega}\right)\right]$ and observe that $du = 0$ and $|\nu \wedge u| = \left|\nabla_{\tan}\left[\varphi(r) r^{\pi/\omega} \cos\left(\pi\theta/\omega\right)\right]\right| = 0$ on $\partial\Omega_\omega$. Also, $u \sim r^{\frac{\pi}{\omega}-1}$ near the origin so that $u \in L^2(\Omega_\omega)$. Similarly, $\delta u = \Delta\left[\varphi(r) r^{\pi/\omega} \cos\left(\frac{\pi\theta}{\omega}\right)\right] \in L^\infty(\Omega_\omega)$. Finally, $\nabla u \sim r^{\frac{\pi}{\omega}-2}$ near the origin so that

$$u \in H^{1,2}(\Omega_\omega) \Leftrightarrow 0 < \omega \leq \pi \Leftrightarrow \Omega_\omega \text{ is convex.} \tag{5.20}$$

In passing, let us observe that $u \in H^{1/2,2}(\Omega_\omega)$ for *any* $0 < \omega < 2\pi$.

This counterexample can be lifted to higher dimensions by adding dummy variables. □

Next, we are going to discuss the flat, Euclidean version of Theorem 5.1. The main point of this analysis is to produce the best constant in the estimate (5.1), i.e. $C = 1$.

To set the stage, we need two preliminary lemmas. In the first one, we single out a global version of (5.9) corresponding to the flat, Euclidean case. Preparatory to stating this result we discuss some notation. Let Ω be a bounded C^2 domain in $\mathbb{R}^m$ and denote by $(\varphi_\alpha)_\alpha$ a family of C^2 functions used to describe $\partial\Omega$. In general, we let D_φ denote the domain above the graph of a C^2 function $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$, used to describe $\partial\Omega$ locally. Also, fix a finite family of smooth, compactly supported functions $(\xi_\alpha)_\alpha$ such that $\sum_\alpha \xi_\alpha^2 = 1$ and so that $\operatorname{supp} \xi_\alpha \cap \partial\Omega \subset \partial D_{\varphi_\alpha}$.

Lemma 5.3. *Assume that Ω is a bounded C^2 domain in $\mathbb{R}^m$ with outward unit normal ν, and that $v \in H^{1,2}(\Omega, \Lambda^l\mathbb{R}^m)$ with $\nu \wedge v = 0$ on $\partial\Omega$ can be written as $v = \sum'_{|I|=l} v_I dx^I$ in Ω. Then*

$$\begin{aligned} \iint_\Omega [|dv|^2 + |\delta v|^2]\, dx = & \sum_{|I|=l}{}' \sum_{j=1}^m \iint_\Omega \left| \frac{\partial v_I}{\partial x_j} \right|^2 dx \\ & + \sum_\alpha \sum_{1 \leq j,k \leq m-1} \int_{\partial\Omega} \xi_\alpha^2 \frac{\partial^2 \varphi_\alpha}{\partial x_j \partial x_k} \langle dx_j \wedge v, dx_k \wedge v \rangle \, d\sigma. \end{aligned} \tag{5.21}$$

Proof. Let us suppose first that $\operatorname{supp} v \subseteq \bar{\Omega} \cap D_\varphi$, where the graph of the C^2 function $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$ gives $\partial\Omega$ locally. Then the analogue of (5.9) is

$$\begin{aligned} \iint_\Omega [|dv|^2 + |\delta v|^2]\, dx = & \sum_{|I|=l}{}' \sum_{j=1}^m \iint_\Omega \left| \frac{\partial v_I}{\partial x_j} \right|^2 dx \\ & + \sum_{1 \leq j,k \leq m-1} \int_{\partial\Omega} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \langle dx_j \wedge v, dx_k \wedge v \rangle \, d\sigma. \end{aligned} \tag{5.22}$$

Consider next the general case, i.e. when the 'smallness' condition on the support of v is removed. To this end, recall the partition of unity introduced before the statement of the lemma. Writing $v\xi_\alpha$ in place of v in (5.22), expanding out the integrands and summing in α gives (5.21) after some algebra (based on Lemma 2.2). We leave the details to the reader. ■

Our second lemma is a density result.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^m$ be a bounded Lipschitz domain with outward unit normal ν and let $u \in L^p(\Omega, \Lambda^l\mathbb{R}^m)$, $1 < p < \infty$, be so that $du \in L^p(\Omega, \Lambda^{l+1}\mathbb{R}^m)$ and $\nu \wedge u = 0$ on $\partial\Omega$.*

Then there exists a sequence $w_\mu \in C^\infty_{\text{comp}}(\Omega, \Lambda^l\mathbb{R}^m)$ such that $w_\mu \longrightarrow u$, $dw_\mu \longrightarrow du$ in $L^p(\Omega, \Lambda^l\mathbb{R}^m)$ and $L^p(\Omega, \Lambda^{l+1}\mathbb{R}^m)$, respectively, as $\mu \to \infty$.

Proof. There is no loss of generality in assuming that the support of u is 'small', i.e. $\operatorname{supp} u \subseteq \bar{\Omega} \cap D_\varphi$, where $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$ is a Lipschitz function used to locally define $\partial\Omega$. Letting tilde denote extension by zero outside Ω, it follows that $\tilde{u} \in L^p_{\text{comp}}(\mathbb{R}^m, \Lambda^l \mathbb{R}^m)$, and $d\tilde{u} \in L^p(\mathbb{R}^m, \Lambda^{l+1}\mathbb{R}^m)$.

Consider next a (circular, vertical, open) cone $\Gamma \subset \mathbb{R}^m$ with vertex at 0 and such that $\Gamma + \partial D_\varphi \subset D_\varphi$. That this is possible is guaranteed by the fact that φ is Lipschitz. Take $\theta \in C^\infty_{\text{comp}}(\Gamma)$ with $\int \theta = 1$ and set $\theta_\mu(x) := \mu^m \theta(\mu x)$, $\mu > 0$, $x \in \mathbb{R}^m$. Then $w_\mu := \tilde{u} * \theta_\mu|_\Omega$ does the job. ■

We are now ready to state and prove the following.

Theorem 5.5. *Fix $0 \le l \le m$ and consider an l-convex subdomain Ω of $\mathbb{R}^m$ whose outward unit normal is denoted by ν. Also, assume that $u = \sum'_{|I|=l} u_I dx^I$ satisfies $u \in L^2(\Omega, \Lambda^l\mathbb{R}^m)$, $du \in L^2(\Omega, \Lambda^{l+1}\mathbb{R}^m)$, $\delta u \in L^2(\Omega, \Lambda^{l-1}\mathbb{R}^m)$ and $\nu \wedge u = 0$ on $\partial\Omega$. Then $u \in H^{1,2}(\Omega, \Lambda^l\mathbb{R}^m)$ and*

$$\sum_{|I|=l}{}' \sum_{j=1}^m \iint_\Omega \left| \frac{\partial u_I}{\partial x_j} \right|^2 dx \le \iint_\Omega [|du|^2 + |\delta u|^2]\, dx. \tag{5.23}$$

The same inequality is valid if Ω is $(m-l)$-convex and the boundary condition is changed to $\nu \vee u = 0$ on $\partial\Omega$.

In particular, if Ω is geometrically convex then (5.23) is valid for any l with either type of boundary condition.

Proof. The arguments closely parallel those in the proof of Theorem 5.1 with a couple of notable changes on which we now elaborate. First, granted the global identity in Lemma 5.3, there is no need to truncate v_μ in (5.6) and we simply consider this time $v_\mu := u|_{\Omega_\mu} - u_\mu$.

Second, we need

$$\sum_{|I|=l}{}' \sum_{j=1}^m \iint_\Omega \left| \frac{\partial (v_\mu)_I}{\partial x_j} \right|^2 dx \le \|du\|^2_{L^2(\Omega)} + \|\delta u\|^2_{L^2(\Omega)} + o(1), \text{ as } \mu \to \infty, \tag{5.24}$$

in place of (5.11). Given Lemma 5.3, this can be derived in the same manner as before, provided that (5.5) is sharpened to

$$\|u_\mu\|_{L^2(\Omega_\mu)} + \|du_\mu\|_{L^2(\Omega_\mu)} = o(1) \qquad \text{as } \mu \to \infty. \tag{5.25}$$

In order to prove (5.25) we observe that the boundary datum in the boundary-value problem solved by u_μ can be written in the form $\nu_\mu \wedge (u|_{\Omega_\mu}) = \nu_\mu \wedge (u|_{\Omega_\mu} - w_\mu)$, where $w_\mu \in C^\infty_{\text{comp}}(\Omega_\mu, \Lambda^l\mathbb{R}^m)$ is a sequence approximating u as in Lemma 5.4. In particular, the analogue of (5.5) becomes

$$\|u_\mu\|_{L^2(\Omega_\mu)} + \|du_\mu\|_{L^2(\Omega_\mu)} \le C(\lambda)(\|u - w_\mu\|_{L^2(\Omega)} + \|du - dw_\mu\|_{L^2(\Omega)}). \tag{5.26}$$

This, of course, suffices to conclude that (5.25) holds and, hence, finishes the proof of the theorem. ■

Given its potential for applications, below we single out the case $m = 3$ of the previous theorem. As such, our result is an extension of a theorem in [13] where a similar estimate is proved for smooth, convex domains and vector fields with components in $H^{1,2}$.

Corollary 5.6. *Let $\Omega \subset \mathbb{R}^3$ be a 1-convex domain and consider a vector field $\vec{u} = (u_1, u_2, u_3)$ with $L^2(\Omega)$ components such that (the components of) $\operatorname{curl}\vec{u}$ and $\operatorname{div}\vec{u}$ are in $L^2(\Omega)$ and $\nu \times \vec{u} = 0$ on $\partial\Omega$. Then $u_j \in H^{1,2}(\Omega)$, $j = 1, 2, 3$, and*

$$\sum_j \iint_\Omega |\nabla u_j|^2\, dx \leq \iint_\Omega |\operatorname{curl}\vec{u}|^2\, dx + \iint_\Omega |\operatorname{div}\vec{u}|^2\, dx. \tag{5.27}$$

The same conclusion is valid when Ω is 2-convex and the boundary condition is changed to $\nu \cdot \vec{u} = 0$.

In particular, when Ω is geometrically convex then (5.27) *holds with either boundary condition imposed on $\vec{u}$.*

In the second half of this section we record several consequences of Theorem 5.1 and Theorem 5.5 which have independent interest. First, recall the *Hodge-Dirac operator* $D := d + d^*$, where d is here the maximally closed unbounded operator defined by the exterior differential on $L^2(\Omega, \oplus_{l=0}^m \Lambda^l T\mathcal{M})$ and d^* its (functional analytic) adjoint. As is well known, $-D^2 = \Delta$, the Hodge-Laplacian.

Corollary 5.7. *If $\Omega \subset \mathcal{M}$ is a Lipschitz domain which is $(m-1)$-convex, then*

$$\mathit{Domain}\,(D) \subseteq H^{1,2}(\Omega, \oplus_{l=0}^m \Lambda^l T\mathcal{M}). \tag{5.28}$$

Proof. In view of the fact that

$$u \in \text{Domain}\,(D) \Leftrightarrow u, du, \delta u \in L^2(\Omega, \oplus_{l=0}^m \Lambda^l T\mathcal{M}) \text{ and } \nu \vee u = 0, \tag{5.29}$$

this follows directly from Corollary 5.2. ■

Recall next the Maxwell system:

$$\begin{cases} E \in L^2(\Omega, \Lambda^l T\mathcal{M}),\ H \in L^2(\Omega, \Lambda^{l+1} T\mathcal{M}), \\ dE - \sqrt{-1}kH = \mathcal{K} \in L^2(\Omega, \Lambda^{l+1} T\mathcal{M}), \\ \delta H + \sqrt{-1}kE = \mathcal{J} \in L^2(\Omega, \Lambda^l T\mathcal{M}), \\ \nu \wedge E = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{5.30}$$

where $k \in \mathbb{C}$ is the so-called wave number and $l \in \{0, 1, \ldots, m\}$.

Corollary 5.8. *Let Ω be a l-convex Lipschitz domain of $\mathcal{M}$ and $k \in \mathbb{C}\backslash\{0\}$. Then, for any solution (E, H) of the Maxwell system* (5.30)*, we have*

$$E \in H^{1,2}(\Omega, \Lambda^l T\mathcal{M}) \Leftrightarrow \delta\mathcal{J} \in L^2(\Omega, \Lambda^{l-1} T\mathcal{M}) \tag{5.31}$$

plus natural estimates.

Proof. The left-to-right implication is clear from (5.30), whereas the opposite one follows from Theorem 5.1 upon noticing that $\delta E = -\sqrt{-1}k^{-1}\delta\mathcal{J}$. ■

As expected, Theorem 5.1 has also implications for the regularity of the classical Hodge decomposition of forms in l-convex Lipschitz domains. Here we record one such instance.

Corollary 5.9. *Let Ω be a Lipschitz domain in $\mathcal{M}$ which is simultaneously $(l-1)$-convex and $(m-l-1)$-convex for some $1 \leq l \leq m-1$. Then for any $u \in L^2(\Omega, \Lambda^l T\mathcal{M})$ there exist $\alpha \in H^{1,2}(\Omega, \Lambda^{l-1} T\mathcal{M})$, $\beta \in H^{1,2}(\Omega, \Lambda^{l+1} T\mathcal{M})$ and $\gamma \in L^2(\Omega, \Lambda^l T\mathcal{M})$ such that*

$$u = d\alpha + \delta\beta + \gamma, \quad \delta\alpha = 0, \quad d\beta = 0, \tag{5.32}$$

$$\nu \wedge \alpha = 0, \quad \nu \vee \beta = 0, \quad d\gamma = 0, \quad \delta\gamma = 0. \tag{5.33}$$

Moreover, natural estimates are valid.

Proof. With the weaker conclusion that $\alpha, \beta \in L^2(\Omega)$ and $d\alpha, \delta\beta \in L^2(\Omega)$, the decomposition (5.32)-(5.33) is well known in Lipschitz domains (cf. [23],[19]). The novelty here is the membership of α and β to $H^{1,2}$. Nonetheless, this follows *a posteriori* from Theorem 5.1 and Corollary 5.2. ■

We conclude this section with a brief discussion of eigenvalue estimates for the Hodge-Laplacian $\Delta := -d\delta - \delta d$ with natural boundary conditions. More specifically, for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^m$ and $\lambda \geq 0$ we consider

$$(\mathrm{BVP}_l) \begin{cases} u \in L^2(\Omega, \Lambda^l \mathbb{R}^m), \ du \in L^2(\Omega, \Lambda^{l+1} \mathbb{R}^m), \\ (\Delta - \lambda)u = 0 \quad \text{in} \quad \Omega, \\ \delta u = 0 \quad \text{in} \quad \Omega, \\ \nu \wedge u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{5.34}$$

as well as

$$(\widetilde{\mathrm{BVP}}_l) \begin{cases} w \in L^2(\Omega, \Lambda^l \mathbb{R}^m), \ \delta w \in L^2(\Omega, \Lambda^{l-1} \mathbb{R}^m), \\ (\Delta - \lambda)w = 0 \quad \text{in} \quad \Omega, \\ dw = 0 \quad \text{in} \quad \Omega, \\ \nu \vee w = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.35}$$

It is illuminating to point out that (BVP_l) reduces precisely to the Dirichlet eigenvalue problem for the ordinary Laplacian on functions when $l = 0$. Also, the (BVP_l) and $(\widetilde{\mathrm{BVP}}_{m-l})$ correspond to each other under the Hodge star isomorphism.

Denote by $b_l(\Omega)$ the l-th Betti number of Ω and by $\mu_1(\Omega)$ the first (nonzero) Neumann eigenvalue of the scalar Laplacian in Ω.

Corollary 5.10. *Let $\Omega \subset \mathbb{R}^m$ be a Lipschitz domain which is $(m-l-1)$-convex for some $0 \leq l \leq m$ and such that $b_{m-l}(\Omega) = b_{m-l-1}(\Omega) = 0$. Then there exist two sequences of eigenvalues*

$$0 < \lambda_1^l(\Omega) \leq \lambda_2^l(\Omega) \leq \ldots \to \infty, \tag{5.36}$$

$$0 < \tilde{\lambda}_1^l(\Omega) \leq \tilde{\lambda}_2^l(\Omega) \leq \ldots \to \infty,$$

of finite multiplicity for (BVP$_l$) *and* ($\widetilde{\text{BVP}}_l$)*, respectively. In addition, they satisfy* $\lambda_j^l(\Omega) = \tilde{\lambda}_j^{m-l}(\Omega)$ *and*

$$\lambda_1^l(\Omega) \geq \mu_1(\Omega). \tag{5.37}$$

In particular, (5.37) *holds for any* l *if* Ω *is geometrically convex.*

Before presenting the proof we would like to point out that in the case of vector fields in smooth, convex subdomains of the three dimensional Euclidean space, the estimate (5.37) has been established in [4].

Proof. The first part of the conclusion does not utilize the convexity assumption and is essentially well-known; the interested reader may also consult [19] for an extension to Lipschitz domains in Riemannian manifolds. Thus, we shall concentrate on (5.37).

To this end, let $\lambda > 0$ be an arbitrary eigenvalue of (BVP$_l$) and let $u \neq 0$ be a corresponding eigenform. Then

$$\lambda\|u\|^2_{L^2(\Omega)} = \iint_\Omega \langle u, \Delta u\rangle = \iint_\Omega \langle u, \delta du\rangle = \|du\|^2_{L^2(\Omega)}. \tag{5.38}$$

On the other hand, Hodge theory plus the topological assumptions we make on Ω guarantee the existence of a $(l+1)$-form w such that

$$w \in L^2(\Omega, \Lambda^{l+1}\mathbb{R}^m), \quad dw = 0 \text{ in } \Omega, \quad \delta w = u \text{ in } \Omega, \text{ and } \nu \wedge w = 0 \text{ on } \partial\Omega. \tag{5.39}$$

In particular, $\|u\|^2_{L^2(\Omega)} = \iint_\Omega |\delta w|^2 = \iint_\Omega \langle du, w\rangle \leq \|du\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}$, thanks to an integration by parts and Cauchy-Schwarz's inequality. Utilizing this inequality back in (5.38) and invoking Theorem 5.5 gives

$$\begin{aligned} \lambda &\geq \frac{\|u\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}} = \frac{\|dw\|^2_{L^2(\Omega)} + \|\delta w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}} \\ &\geq \left({\sum_{|I|=l}}' \sum_{j=1}^m \iint_\Omega \left|\frac{\partial w_I}{\partial x_j}\right|^2 dx \right) \Big/ \|w\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.40}$$

In order to continue, assume that $w = \sum'_{|I|=l+1} w_I dx^I$ in Ω and introduce the constant coefficient form $\tilde{w} := \sum'_{|I|=l+1} \tilde{w}_I dx^I$, where $\tilde{w}_I := \iint_\Omega w_I\, dx$. Finally, set $v := w - \tilde{w}$. It follows that v_I can replace w_I in the numerator of the last expression in (5.40) whereas, so we claim,

$$\|v\|_{L^2(\Omega)} \geq \|w\|_{L^2(\Omega)}. \tag{5.41}$$

Indeed, one simply needs to check that

$$\exists\, \omega \in L^2(\Omega, \Lambda^{l+2}\mathbb{R}^m) \text{ such that } \delta\omega = \tilde{w} \text{ in } \Omega. \tag{5.42}$$

Granted (5.42) and the easily checked fact that w is orthogonal to the image of δ, the estimate becomes a simple consequence of the Pythagorean theorem. However, due to the current topological assumption on Ω, (5.42) is itself a corollary of the obvious fact that $\delta\tilde{w} = 0$. This proves (5.41).

Summarizing, at this point we may continue (5.40) with

$$\lambda \geq \Big({\sum_{|I|=l}}' \sum_{j=1}^{m} \iint_{\Omega} \Big| \frac{\partial v_I}{\partial x_j} \Big|^2 \, dx \Big) \Big/ \Big({\sum_{|I|=l}}' \iint_{\Omega} |v_I|^2 \, dx \Big)$$
$$\geq \min_I \Big(\iint_{\Omega} |\nabla v_I|^2 \, dx \Big) \Big/ \Big(\iint_{\Omega} |v_I|^2 \, dx \Big). \tag{5.43}$$

Now (5.37) is a direct consequence of the inequality (5.43) and the well-known estimate $\|\nabla f\|^2_{L^2(\Omega)}/\|f\|^2_{L^2(\Omega)} \geq \mu_1(\Omega)$, valid for any scalar function $f \in H^{1,2}(\Omega)$ with $\iint_{\Omega} f = 0$. ■

Corollary 5.11. *Let Ω, l, be as in Corollary 5.10. Then the first positive eigenvalue $k_1^l(\Omega)$ of the Maxwell system* (5.30) *satisfies*

$$k_1^l(\Omega) \geq (\mu_1(\Omega))^{1/2}. \tag{5.44}$$

In particular, if Ω is geometrically convex, then for any l the first positive Maxwell eigenvalue satisfies the estimate

$$k_1^l(\Omega) \geq \pi \, (\mathrm{diam}\,(\Omega))^{-1}. \tag{5.45}$$

Proof. The estimate (5.44) follows from Corollary 5.10 by observing that for each solution (E, H) of the *homogeneous* Maxwell system (5.30), the electric component E solves (5.34) for the wave number $k := \sqrt{\lambda}$.

Finally, (5.45) is a direct consequence of (5.44) and a sharp result due to Payne and Weinberger to the effect that $\mu_1(\Omega) \geq \pi^2(\mathrm{diam}\,(\Omega))^{-2}$ if Ω is convex; see [22] (cf. also Remark 2, p. 206 in [17] for a related comment). ■

Parenthetically, let us observe that, in the case of a convex domain Ω, (5.37) contains as a special case the estimate $\lambda_1(\Omega) \geq \mu_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of the (scalar) Dirichlet Laplacian in Ω. Results of this type go back to [24] and the reader is also referred to [21], [17], [9] for a more detailed account. Note that our proof works for convex domains which are not necessarily smooth.

6 Domains satisfying an exterior ball condition

In this section we shall work in $\mathbb{R}^m$ with the standard Euclidean metric. The aim is to state and prove an analogue of Theorem 5.1 for Lipschitz domains satisfying a uniform exterior ball condition. This is done in Theorem 6.5 whereas applications are discussed in Corollaries 6.6-6.7.

Definition 6.1. *We say that a Lipschitz domain $\Omega \subseteq \mathbb{R}^m$ satisfies a uniform exterior ball condition (abbreviated UEBC henceforth) if there exists $R > 0$ with the following property. For each $x \in \partial\Omega$, there exists $p = p(x) \in \mathbb{R}^m$ so that*

$$\overline{B(p,R)} \backslash \{x\} \subseteq \mathbb{R}^m \backslash \Omega \quad \textit{and} \quad x \in \partial B(p,R). \tag{6.1}$$

The supremum over all R's satisfying the above condition is called the EBC constant of the domain.

Obviously, any convex or $C^{1,1}$ domain satisfies a UEBC. We debut with a simple but useful observation.

Lemma 6.2. *Assume that Ω is a Lipschitz domain, and that $x \in \partial\Omega$, $R > 0$, $p \in \mathbb{R}^m$ are so that* (6.1) *is satisfied. Also, suppose that $\partial\Omega$ has a tangent plane π_x at x. Then π_x is tangent to $B(p, R)$ too.*

Proof. There is no loss of generality to assume that $x = 0 \in \mathbb{R}^m$ and that Ω is the domain lying about the graph of a Lipschitz function $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$ with $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. It follows that for any $\epsilon > 0$,

$$|\varphi(x')| \leq \epsilon|x'|, \quad \text{if} \quad |x'| \quad \text{is sufficiently small.} \tag{6.2}$$

In particular, introducing the cone

$$C_{\epsilon,h} := \{(x', x_m) \in \mathbb{R}^m;\, 0 < x_m < \epsilon|x'|,\, |x'| \leq h\} \tag{6.3}$$

it follows that

$$\forall\, \epsilon > 0,\, \exists\, h > 0 \text{ such that } C_{\epsilon,h} \subseteq \Omega. \tag{6.4}$$

Consequently,

$$0 \in \partial B(p, R) \quad \text{and} \quad \forall\, \epsilon > 0,\, \exists\, h > 0 \text{ such that } C_{\epsilon,h} \cap B(p, R) = \emptyset. \tag{6.5}$$

This condition readily implies that $\{x_m = 0\}$ is tangent to $B(p, R)$. ■

Lemma 6.3. *Let Ω be the domain above the graph of a Lipschitz function $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$. Consider the following two assertions:*

- *(i) Ω satisfies a UEBC;*
- *(ii) there exists $C_0 > 0$ such that for a.e. $a \in \mathbb{R}^{m-1}$ and $\forall\, v \in \mathbb{R}^{m-1}$, $|v| \leq C_0$, there holds*

$$2\varphi(a) - \varphi(a + v) - \varphi(a - v) \leq C_0|v|^2. \tag{6.6}$$

Then (i) $\Rightarrow$ (ii).

Moreover, the converse implication is also true if $\varphi \in C^3$. In this latter case, (i) and (ii) are also equivalent to

- *(iii) there exists $C_1 > 0$ such that*

$$-\langle Hess_\varphi(a)v, v\rangle \leq C_1|v|^2, \qquad \forall\, v \in \mathbb{R}^{m-1}, \tag{6.7}$$

uniformly in $a \in \mathbb{R}^{m-1}$; here $Hess_\varphi(a) := \left(\frac{\partial^2\varphi}{\partial x_j \partial x_k}(a)\right)_{j,k}$ is the Hessian of φ at a.

Proof. The LHS of (6.6) is invariant to subtracting affine functions from φ; therefore, we may assume that

$$a = 0, \quad \varphi(0) = 0, \quad \nabla\varphi(0) = 0, \tag{6.8}$$

in which case (6.6) becomes

$$\exists\, C_0 \text{ such that } -\varphi(v) - \varphi(-v) \le C_0|v|^2, \quad \forall\, v \in \mathbb{R}^{m-1}, \quad |v| \le C_0. \tag{6.9}$$

Since Ω satisfies a UEBC, by Lemma 6.2, there exists $R > 0$ such that $B(-Re_m, R)\backslash\{0\} \subseteq \mathbb{R}^m\backslash\Omega$, where $e_m := (0, 0, ..., 0, 1) \in \mathbb{R}^m$. With $\Phi(v) := \sqrt{R^2 - |v|^2} - R$, this entails

$$\begin{aligned} \varphi(v) \ge \Phi(v) &= \Phi(0) + \langle v, \nabla\Phi(0)\rangle + \mathcal{O}(|v|^2) \\ &= \mathcal{O}(|v|^2) \quad \text{as} \quad |v| \to 0. \end{aligned} \tag{6.10}$$

Hence, $-\varphi(v) \le c|v|^2$ if $|v|$ is sufficiently small. Replacing v by $-v$ and adding up yields (6.9). This concludes the proof of the implication $(i) \Rightarrow (ii)$.

For the converse implication, assuming $\varphi \in C^3$ and expanding φ in a Taylor series at a, the LHS of (6.6) becomes

$$-\langle \mathrm{Hess}_\varphi(a)v, v\rangle + \mathcal{O}(|v|^3) \quad \text{as} \quad |v| \to 0. \tag{6.11}$$

Since, by (6.6), the expression in (6.11) is $\le C_0|v|^2$ for $|v| \le C_0$, it follows that (6.7) holds with, e.g., $C_1 := 2C_0$. Thus, $(ii) \Rightarrow (iii)$ if $\phi \in C^3$.

Next, we shall prove that (iii) implies (6.1) when $x = 0$ for the choice $R := \frac{1}{4c_0}$, $p := -Re_m$. Recall that we are assuming (6.8) which can always be arranged by performing a translation and a rotation. To this end, it suffices to show that the points on the graph of φ which are sufficiently close to the origin lie outside of the ball $B(p, R)$, i.e. $(\varphi(x') + R)^2 + |x'|^2 > R^2$ for small $|x'|$ or, equivalently,

$$\varphi(x')^2 + 2R\,\varphi(x') + |x'|^2 > 0, \qquad \text{for small } |x'|. \tag{6.12}$$

Expanding φ in a Taylor series at 0 gives

$$\varphi(x') = \tfrac{1}{2}\big\langle \mathrm{Hess}_\varphi(0)x', x'\big\rangle + \mathcal{O}(|x'|^3), \quad \text{as } |x'| \to 0, \tag{6.13}$$

so that, by (6.7), $2R\,\varphi(x') \ge -\frac{1}{2}|x'|^2$ provided $|x'|$ is sufficiently small (in a uniform fashion). This clearly implies (6.12) and, hence, finishes the proof of the lemma. ■

Lemma 6.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^m$ which satisfies a UEBC. Then there exists a sequence of C^∞ domains $\{\Omega_\epsilon\}_{\epsilon>0}$ such that:*

(i) $\bar{\Omega}_\epsilon \subseteq \Omega, \quad \cup_{1>\epsilon>0}\Omega_\epsilon = \Omega$;

(ii) there exist $C > 0$ and a vector field θ so that, for each $1 > \epsilon > 0$,

$$\langle \theta, n_\epsilon\rangle \ge C \qquad \text{a.e. on } \partial\Omega_\epsilon, \tag{6.14}$$

where n_ϵ is the outward unit normal to $\partial\Omega_\epsilon$;

(iii) Ω_ϵ *satisfies a UEBC with a constant independent of* $\epsilon \in (0,1)$*.*

Proof. The problem localizes and, hence, we may assume that $\Omega = \{(x', x_m);\, \varphi(x') < x_m\}$, for some Lipschitz function $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$ which also satisfies $\|\varphi\|_{L^\infty} < +\infty$. Let $\Phi \in C^0_{\rm comp}(\mathbb{R}^{m-1})$, $1 \geq \Phi \geq 0$, $\Phi \equiv 1$ near 0, $\int_{\mathbb{R}^{m-1}} \Phi\, dx' = 1$ and define $\Phi_\epsilon(x') := \epsilon^{-(m-1)}\Phi(x'/\epsilon)$.

For a large, positive constant C (to be determined shortly), set

$$\varphi_\epsilon(x') := \epsilon\, C + \int_{\mathbb{R}^{m-1}} \Phi_\epsilon(y')\varphi(x'-y')\, dy', \quad x' \in \mathbb{R}^{m-1} \tag{6.15}$$

and introduce

$$\Omega_\epsilon := \{(x', x_m);\, \varphi_\epsilon(x') < x_m\}. \tag{6.16}$$

Clearly, $\partial\Omega_\epsilon \in C^\infty$. To ensure that the sequence $\{\Omega_\epsilon\}_\epsilon$ has all the other desired properties, we choose the constant C in (6.15) so that

$$\frac{d}{d\epsilon}[\varphi_\epsilon(x')] > 0 \quad \text{for} \quad 1 > \epsilon > 0 \;\text{ and }\; x' \in \mathbb{R}^{m-1}. \tag{6.17}$$

To see that this can be done, note that differentiating the RHS of (6.15) with respect to ϵ gives

$$C + \epsilon^{-m} \int_{\mathbb{R}^{m-1}} \langle \nabla\Phi\big(y'/\epsilon\big), y'\rangle \varphi(x'-y')\, dy'. \tag{6.18}$$

Now, $\nabla\Phi(y'/\epsilon) \neq 0 \Leftrightarrow |y'| \approx \epsilon$. Making the change of variables $y' = \epsilon y''$ forces $|y''| \approx 1$ and, hence,

$$\left| \int_{|y''|\approx 1} \langle \nabla\Phi(y''), y''\rangle \varphi(x' - \epsilon y'')\, dy'' \right| \tag{6.19}$$

can be bounded in terms of Φ and $\|\varphi\|_{L^\infty}$ uniformly for $\epsilon \in (0,1)$ and $x' \in \mathbb{R}^{m-1}$. Thus, for C large enough the expression in (6.18) is > 0 uniformly for $\epsilon \in (0,1)$, $x' \in \mathbb{R}^{m-1}$, as wanted. Going further, (6.17) ensures that $\varphi_\epsilon(x') \downarrow \varphi(x)$ as $\epsilon \downarrow 0$ which translates into $\Omega_\epsilon \subseteq \Omega$, $\cup_{\epsilon\in(0,1)}\Omega_\epsilon = \Omega$. The condition (6.14) is satisfied if we choose, e.g., $\theta \equiv e_m$.

There remains to check *(iii)*. The crucial point is that the condition (6.6) in Lemma 6.3 is invariant under the change $\varphi \mapsto \varphi_\epsilon$, with φ_ϵ as in (6.15). Now, since by hypothesis Ω satisfies the UEBC, it follows that φ satisfies (6.6). Hence, φ_ϵ is smooth and satisfies (6.6) and, finally, Ω_ϵ satisfies a UEBC, by Lemma 6.3. The proof of the lemma is finished. ∎

Now we are ready to present the main result of this section.

Theorem 6.5. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^m$ *which satisfies a UEBC. Also, let* $u \in L^2(\Omega, \Lambda^l\mathbb{R}^m)$ *be so that* $du \in L^2(\Omega, \Lambda^{l+1}\mathbb{R}^m)$ *and* $\delta u \in L^2(\Omega, \Lambda^{l-1}\mathbb{R}^m)$ *for some* $l \in \{0, \dots, m\}$*. Finally, assume that either* $\nu \wedge u = 0$ *on* $\partial\Omega$ *or* $\nu \vee u = 0$ *on* $\partial\Omega$*, where* ν *is the unit conormal to* $\partial\Omega$*.*

Then $u \in H^{1,2}(\Omega, \Lambda^l\mathbb{R}^m)$ *and*

$$\|u\|_{H^{1,2}(\Omega)} \leq C(\|du\|_{L^2(\Omega)} + \|\delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \tag{6.20}$$

for some $C > 0$ *which depends exclusively on* Ω*.*

Proof. The problem localizes and we can assume that $\operatorname{supp} u$ is contained in a small open neighborhood U of a boundary point $a \in \partial\Omega$. Furthermore assume that $\partial\Omega \cap U$ lies on the graph of a Lipschitz function $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$.

Let $(\Omega_\epsilon)_{0<\epsilon<1}$ be the sequence of smooth domains approximating Ω which has been constructed in Lemma 6.3. For each ϵ, set $\rho_\epsilon(x) := \varphi_\epsilon(x') - x_m$ (where φ_ϵ is as in (6.15)) for x near a and extend ρ_ϵ to $\mathbb{R}^n$ so that it becomes a defining function for Ω_ϵ. Note that $(\rho_\epsilon)_{jk} = \frac{\partial^2 \rho_\epsilon}{\partial x_j \partial x_k}$ if $1 \leq j, k \leq m-1$, and 0 otherwise.

Up to a certain point we are going to parallel the proof of Theorem 5.1 with the role of smooth approximating domains played by $(\Omega_\epsilon)_\epsilon$. Specifically, for some fixed $\lambda > 0$ we let u_ϵ solve the boundary problem

$$\begin{cases} u_\epsilon \in L^2(\Omega_\epsilon, \Lambda^l \mathbb{R}^m),\ du_\epsilon \in L^2(\Omega_\epsilon, \Lambda^{l+1}\mathbb{R}^m), \\ (\Delta - \lambda) u_\epsilon = 0 \quad \text{in} \quad \Omega_\epsilon, \\ \delta u_\epsilon = 0 \quad \text{in} \quad \Omega_\epsilon, \\ \nu_\epsilon \wedge u_\epsilon = \nu_\epsilon \wedge (u|_{\Omega_\epsilon}) \in H^{-\frac{1}{2},2}(\partial\Omega_\epsilon, \Lambda^l \mathbb{R}^m), \end{cases} \tag{6.21}$$

and set

$$v_\epsilon := (u|_{\Omega_\epsilon} - u_\epsilon)\eta \in H^{1,2}(\Omega_\epsilon, \Lambda^l \mathbb{R}^m). \tag{6.22}$$

where $\eta \in C^\infty_{\mathrm{comp}}(U)$, $\eta \equiv 1$ near a. Writing $v_\epsilon = \sum'_{|I|=l} v_I^\epsilon dx^I$, the identity (5.22) gives

$$\begin{aligned} \iint_{\Omega_\epsilon} [|dv_\epsilon|^2 + |\delta v_\epsilon|^2]\, dx = & \sum_{|I|=l}{}' \sum_{j=1}^{m} \iint_{\Omega_\epsilon} \left| \frac{\partial v_I^\epsilon}{\partial x_j} \right|^2 dx \\ & + \sum_{1 \leq j,k \leq m-1} \int_{\partial\Omega_\epsilon} \frac{\partial^2 \varphi_\epsilon}{\partial x_j \partial x_k} \langle dx_j \wedge v_\epsilon, dx_k \wedge v_\epsilon \rangle \, d\sigma_\epsilon. \end{aligned} \tag{6.23}$$

Note that the boundary integrand can be written in the form

$$\sum_{|M|=l+1}{}' \langle \mathrm{Hess}_{\varphi_\epsilon}(x) v_\epsilon^M(x), v_\epsilon^M(x) \rangle \tag{6.24}$$

where $v_\epsilon^M := (v_{\epsilon,j}^M)_j$ and, further, $v_{\epsilon,j}^M := \sum'_{|I|=l} \epsilon_M^{jI} v_I^\epsilon$, for each M with $|M| = l+1$. This and (6.7) (written for φ_ϵ in place of φ), then give

$$\begin{aligned} \|v_\epsilon\|^2_{H^{1,2}(\Omega_\epsilon)} \leq \|dv_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \|\delta v_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \|v_\epsilon\|^2_{L^2(\Omega_\epsilon)} \\ + C \int_{\partial\Omega_\epsilon} |v_\epsilon|^2 \, d\sigma_\epsilon. \end{aligned} \tag{6.25}$$

The estimate

$$C \int_{\partial\Omega_\epsilon} |v_\epsilon|^2 \, d\sigma_\epsilon \leq \tfrac{1}{2} \|v_\epsilon\|^2_{H^{1,2}(\Omega_\epsilon)} + C' \|v_\epsilon\|^2_{L^2(\Omega_\epsilon)} \tag{6.26}$$

is standard (cf., e.g., [14]) and, granted (6.14), the constant C' can be chosen to depend only on the Lipschitz character of Ω. In particular, it is independent of ϵ. Combining (6.25) and (6.26) we have proved that

$$\|v_\epsilon\|_{H^{1,2}(\Omega_\epsilon)} \leq C(\|dv_\epsilon\|_{L^2(\Omega_\epsilon)} + \|\delta v_\epsilon\|_{L^2(\Omega_\epsilon)} + \|v_\epsilon\|_{L^2(\Omega_\epsilon)}) \tag{6.27}$$

for some constant $C = C(\Omega) > 0$ independent of ϵ.

This plays the role of the estimate (5.11). With this at hand, the proof proceeds analogously to that of Theorem 5.1. Thus, when $\nu \wedge u = 0$, the conclusion follows. When $\nu \vee u = 0$, matters can be reduced to the previous case using Hodge duality. The proof of Theorem 6.5 is therefore complete. ■

In closing, we would like to single out two immediate consequences of Theorem 6.5 which have independent interest.

Corollary 6.6. *Let Ω be a connected, bounded Lipschitz domain in $\mathbb{R}^m$ satisfying a uniform exterior ball condition. Then for any $f \in L^2(\Omega)$ with $\iint_\Omega f\,dx = 0$ the Poisson equation for the Laplacian with homogeneous Neumann boundary conditions*

$$\begin{cases} \Delta u = f & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6.28}$$

has a solution in $H^{2,2}(\Omega)$. The solution is unique (modulo constants) and satisfies

$$\|\nabla u\|_{H^{1,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{6.29}$$

where C depends only on the Lipschitz character of Ω and the EBC constant.

Proof. The only new things are the membership of u to $H^{2,2}(\Omega)$ and the estimate (6.29). They both follow from Theorem 6.5 applied to the 1-form $\omega := du$. ■

Remark. When Ω is convex, L^p-extensions of this result are given in [3]. Building on earlier work in [15], the Poisson problem (6.28) for *general* Lipschitz domains has been solved in [7], [20].

Of course, we can also specialize Theorem 6.5 to obtain a similar result for the Poisson problem with homogeneous Dirichlet boundary conditions. As such, we recover the main result in [1] (cf. also [2], [11] for L^p extensions when Ω is convex). □

Corollary 6.7. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ which satisfies a uniform exterior ball condition and denote by n the outward unit normal to $\partial\Omega$. Then any vector field $\vec{u} \in L^2(\Omega, \mathbb{R}^3)$ such that*

$$\operatorname{curl} \vec{u} \in L^2(\Omega, \mathbb{R}^3), \quad \operatorname{div} \vec{u} \in L^2(\Omega) \tag{6.30}$$

and

$$\textit{either} \quad n \times \vec{u} = 0 \quad \textit{on} \quad \partial\Omega \quad \textit{or} \quad n \cdot \vec{u} = 0 \quad \textit{on} \quad \partial\Omega, \tag{6.31}$$

belongs to $H^{1,2}(\Omega, \mathbb{R}^3)$ and

$$\|\vec{u}\|_{H^{1,2}(\Omega,\mathbb{R}^3)} \leq C(\|\operatorname{curl} \vec{u}\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{div} \vec{u}\|_{L^2(\Omega)} + \|\vec{u}\|_{L^2(\Omega,\mathbb{R}^3)}), \tag{6.32}$$

where C depends only on the EBC constant and the Lipschitz character of Ω.

Proof. This corresponds to translating the case $l = 1$ of Theorem 6.5 in the language of vector fields. ■

Remark. When Ω is convex, this particular result is well-known cf., e.g., [25], [14], [6], [8]. □

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