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$\mathscr{L}_2$-Gain analysis and control of uncertain nonlinear systems with bounded disturbance inputs

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SUMMARY

This paper proposes a convex approach to regional stability and $\mathscr{L}_2$-gain analysis and control synthesis for a class of nonlinear systems subject to bounded disturbance signals, where the system matrices are allowed to be rational functions of the state and uncertain parameters. To derive sufficient conditions for analysing input-to-output properties, we consider polynomial Lyapunov functions of the state and uncertain parameters (assumed to be bounded) and a differential-algebraic representation of the nonlinear system. The analysis conditions are written in terms of linear matrix inequalities determining a bound on the $\mathscr{L}_2$-gain of the input-to-output operator for a class of (bounded) admissible disturbance signals. Through a suitable parametrization involving the Lyapunov and control matrices, we also propose a linear (full-order) output feedback controller with a guaranteed bound on the $\mathscr{L}_2$-gain. Numerical examples are used to illustrate the proposed approach. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The input-to-output analysis of dynamical systems is one of the most important problems in control systems theory [1, 2]. For instance, the linear robust $\mathscr{H}_\infty$ control theory is a well-established research area [3, 4]. The nonlinear counterpart is characterized in terms of the $\mathscr{L}_2$-gain of the input-to-output operator that can be considered as a generalization of the

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$\mathscr{H}_\infty$-norm of linear systems [5]. Unfortunately, the nonlinear $\mathscr{H}_\infty$ problem (with an abuse of notation) needs the solution of Hamilton–Jacob inequalities (HJI or equations—HJE) which are difficult to solve. Some alternative approaches have been developed to solve HJI (or HJE) indirectly by reducing the problem to algebraic inequalities (or equations), but this is only feasible for very simple nonlinear systems [6].

On the other hand, the so-called linear matrix inequality (LMI) approach has been widely used to solve several problems for linear systems such as robust control, gain-scheduling, multi-objective control, and filtering [7, 8]. Since the work [9] that showed a solution to the nonlinear problem using LMIs, researchers have proposed different solutions to the nonlinear robust $\mathscr{H}_\infty$ control problem using a wide range of modelling techniques [10–12]. At the same time, many authors extended the linear parameter varying (LPV) theory to deal with nonlinear systems, resulting in the *quasi-LPV* representation [13]. In other words, a nonlinear system $\dot{x} = f(x)$ is described by the parameter-dependent system $\dot{x} = A(\sigma(x))x$, $\sigma(x) \in \Sigma_x$, where the nonlinearities are viewed as state-dependent parameters concentrated in the vector $\sigma(x)$ that belongs to a known polyhedral set Σ_x. However, the *quasi-LPV* approach may lead to serious conservativeness, since the nonlinearities of the system are considered as free time-varying parameters which are actually determined by the system trajectories [14]. Moreover, there are some shortcomings related to the quasi-LPV form that may lead to the instability of the nonlinear closed-loop system [15], i.e. the closed-loop LPV system is stable but the original nonlinear system may be unstable. In addition, the quasi-LPV approach with parameter-dependent Lyapunov functions needs an estimate of the parameter variation rate (i.e. $\dot{\sigma}(x)$ must belong to a polytopic region) and this assumption has to be verified after the control design [13]. In parallel, for a more restricted class of systems, some researchers have devised a convex formulation for polynomial systems based on the sum of squares (SOS) decomposition [16, 17].

In this paper, we address the nonlinear $\mathscr{H}_\infty$ control problem using the LMI framework. We consider a class of nonlinear systems which are subjected to both parameter uncertainties in the system model and an affine (bounded) disturbance input. In contrast to the *quasi-LPV* representation, the system matrices are allowed to be rational functions of the state and uncertain parameters. More specifically, the nonlinear system is recast in terms of the following representation:

$$\dot{x} = A_1 x + A_2 \pi, \quad 0 = \Pi_1(x)x + \Pi_2(x)\pi$$

where the auxiliary vector $\pi = \pi(x)$ contains all the nonlinearities. These nonlinearities are defined by means of the equality constraint $\Pi_1(x)x + \Pi_2(x)\pi = 0$. It can be shown that the above representation can model the whole class of rational systems well posed inside $\mathscr{X}$. For this particular class of systems, we determine the regional stability and a bound on the $\mathscr{L}_2$-gain of the nonlinear system considering polynomial Lyapunov functions. We then extend these results to the synthesis problem, that is, we design a full-order linear dynamic output feedback controller that minimizes an upper bound on the $\mathscr{L}_2$-gain while guaranteeing that the state stays inside a given region for a given class of disturbance signals. Both problems are solved in terms of LMIs. Numerical examples are used to demonstrate the approach.

The rest of the paper is structured as follows. Section 2 states the problems of concern, Section 3 introduces the key concepts for rewriting the stability conditions in terms of LMIs, Section 4 addresses the problem of solving state-dependent LMIs, Sections 5 and 6 present the main results of this paper, respectively, Section 7 illustrates the approach, and Section 8 concludes the paper.

The notation used in this paper is standard. $\mathbb{R}_+$ is the set of non-negative real numbers, $\mathbb{R}^n$ denotes the set of n-dimensional real vectors, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices, I_n is the $n\times n$ identity matrix, $0_{n\times m}$ is the $n\times m$ matrix of zeros and 0_n is the $n\times n$ matrix of zeros. For a real matrix S, S' denotes its transpose, $S^\perp$ a basis for the null space of S, and $S>0$ means that S is symmetric and positive definite. For block matrices, diag$\{\cdots\}$ is a diagonal block matrix. The time derivative of a function $r(t)$ will be denoted by $\dot{r}(t)$ and the argument (t) is often omitted. The finite-horizon 2-norm of a bounded signal $c(t)$ is defined as $\|c(t)\|_{2,[0,T]} = (\int_0^T c(t)'c(t)\,\mathrm{d}t)^{1/2}$. For two sets $\mathscr{A}\subset\mathbb{R}^{n_a}$ and $\mathscr{B}\subset\mathbb{R}^{n_b}$, the notation $\mathscr{A}\times\mathscr{B}$ represents that $(\mathscr{A}\times\mathscr{B})\subset\mathbb{R}^{(n_a+n_b)}$ is a meta set obtained by the cartesian product. For a known polytope Φ, $\mathscr{V}(\Phi)$ is the set of all vertices of Φ. Matrix and vector dimensions are omitted whenever they can be inferred from the context.

2. PROBLEM STATEMENT

Consider the following time-invariant nonlinear system:

$$\dot{x}(t) = f(x(t),\delta,w(t)),\quad z(t) = g(x(t),\delta,w(t)),\quad x(0)=0 \tag{1}$$

where $x(t)\in\mathbb{R}^{n_x}$ is the state taking values on the set $\mathscr{X}$, $\delta\in\mathbb{R}^{n_\delta}$ the uncertain (constant) parameters taking values on the set Δ, $w(t)\in\mathbb{R}^{n_w}$ the disturbance input taking values on the set $\mathscr{W}$, and $z(t)\in\mathbb{R}^{n_z}$ the performance output taking values on the set $\mathscr{Z}$. Hereafter, the time argument is generally omitted for simplicity of notation.

The main goal of this paper is to analyse the input-to-output properties of system (1) in the nearby of the system equilibrium point for zero initial conditions, i.e. $x(0)=0$. Therefore, we need for any disturbance input $w\in\mathscr{W}$, zero initial conditions, and any interval of time $[0,T]$, $T\geqslant 0$, that there exist uniquely defined signals $x:[0,T]\mapsto\mathscr{X}$ and $z:[0,T]\mapsto\mathscr{Z}$ satisfying (1). The input-to-output properties will be characterized by the following map:

$$s:\mathscr{W}\times\mathscr{Z}\mapsto\mathbb{R} \tag{2}$$

that is assumed to be locally absolutely integrable from 0 to T, i.e.

$$\int_0^T |s(w,z)|\,\mathrm{d}t<\infty\quad \forall T\geqslant 0$$

for all pair (w,z) satisfying (1). The mapping $s(w,z)$ will be referred to as the supply function.

To guarantee the existence and uniqueness of solution, we assume for system (1) that:

A1: $f(x,\delta,w)$ is locally Lipschitz and $g(x,\delta,w)$ is continuous and bounded over $\mathscr{X}\times\Delta\times\mathscr{W}$.

As usual for nonlinear systems, we further assume the following to simplify the analysis.

A2: The origin of the unforced system belongs to $\mathscr{X}$ and it is an equilibrium point for all admissible uncertainty.

Adapting the dissipative theory of dynamical systems to the above scenario, the following definition will be of interest.

Definition 1 (Dissipativity [1, 18])

A dynamical system as defined in (1) with a supply function $s(w,z)$ is said to be dissipative if there exists a positive definite function $V:\mathscr{X}\mapsto\mathbb{R}_+$ (called the storage function) such that the

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following holds:

$$\int_t^T s(w(\tau), z(\tau))\, \mathrm{d}\tau \geqslant V(x(T)) - V(x(t)) \quad \forall t \in [0, T) \tag{3}$$

for all signals (x, w, z) satisfying (1) and all $T \geqslant 0$.

In particular, we are interested in the following supply function:

$$s(w, z) \triangleq \gamma w' w - \frac{1}{\lambda} z' z \tag{4}$$

where γ and λ are positive scalars. Notice for a suitable choice of λ that the above function can encompass the problems of (i) $\mathscr{L}_2$-gain analysis with $\lambda = \gamma$, and (ii) the regional stability of system (1) with respect to (w.r.t.) $\mathscr{X}$ and $\mathscr{W}$ when $\lambda \to \infty$, as detailed later in this paper.

(P1) The problem to be addressed in this paper is to investigate under what conditions the zero-state dissipation inequality (3) for the supply function as defined in (4) is satisfied using semi-definite programming (SDP).

The following definition of regional stability will be of interest.

Definition 2 (Regional stability)
System (1) is regionally stable with respect to $\mathscr{X}$ and $\mathscr{W}$, if the trajectory $x(t)$ driven by $w(t) \in \mathscr{W}$ belongs to $\mathscr{X}$ for all $\delta \in \Delta$ and $t \geqslant 0$. In such case, the set $\mathscr{W}$ is called an admissible disturbance set.

A connection between the dissipative theory [1] and the Lyapunov stability [19] is provided by the following lemma.

Lemma 1 (Dissipativity and Lyapunov stability)
Consider system (1) with A1 and A2. Let $\mathscr{W}$ be an admissible set in the sense of Definition 2. Let $s : \mathscr{W} \times \mathscr{Z} \mapsto \mathbb{R}$ be a given supply function. Suppose there is a continuously differentiable function $V : \mathscr{X} \times \Delta \mapsto \mathbb{R}_+$ satisfying the following conditions for positive scalars ε_1 and ε_2, and a storage function as defined in (4):

$$\varepsilon_1 x' x \leqslant V(x, \delta) \leqslant \varepsilon_2 x' x \quad \forall (x, \delta) \in \mathscr{X} \times \Delta \tag{5}$$

$$\dot{V}(x, \delta) < s(w, z) \quad \forall (x, \delta, w) \in \mathscr{X} \times \Delta \times \mathscr{W} \tag{6}$$

Then, the trajectory $x(t)$ driven by $w(t) \in \mathscr{W}$ lies in the following bounding set:

$$\mathscr{R}(c^*) \triangleq \{x \in \mathscr{X} : V(x, \delta) \leqslant c^*,\ \delta \in \Delta,\ c^* \in \mathbb{R}_+\} \tag{7}$$

for some

$$c^* = \max_c \{c : \mathscr{R}(c) \subset \mathscr{X}\}$$

Moreover, the dissipation inequality in (3) is satisfied for all signals x, w, z satisfying (1) and all $T \geqslant 0$.

The proof of the above lemma is straightforward from the dissipativity theory [1, Theorem 6], and Lyapunov stability [19, Theorems 3.1 and 3.4].

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3. BASIC RESULTS

This section introduces the key points for obtaining a convex characterization of Lemma 1. Firstly, a differential-algebraic representation of nonlinear systems is presented. Then, the class of Lyapunov functions to be considered throughout this paper is defined. Finally, some comments regarding invariant sets will end this section.

3.1. System model

Let us suppose that the nonlinear system defined in (1) can be recast in the following form:

$$\begin{aligned} \dot{x} &= A_1 x + A_2 \pi + A_3 w \\ z &= E_1 x + E_2 \pi + E_3 w \\ 0 &= \Pi_1(x,\delta)x + \Pi_2(x,\delta)\pi + \Pi_3(x,\delta)w \end{aligned} \tag{8}$$

where $\pi \in \mathbb{R}^{n_\pi}$ is an auxiliary vector; $\Pi_1(x,\delta) \in \mathbb{R}^{m\times n_x}$, $\Pi_2(x,\delta) \in \mathbb{R}^{m\times n_\pi}$ and $\Pi_3(x,\delta) \in \mathbb{R}^{m\times n_w}$ are affine matrix functions of (x,δ); and $A_1, A_2, \ldots, E_3$ are constant matrices with appropriate dimensions. To simplify the notation, we may use the auxiliary matrices and vectors without explicitly mentioning their respective dependence on x and δ. The above representation is hereafter denoted as *differential-algebraic representation* or DAR in short.

The basic methodology for rewriting a nonlinear system in terms of a DAR is to lump all nonlinear and uncertain terms in the vector π, and then define the relationship between their elements by means of the algebraic constraint $\Pi_1 x + \Pi_2 \pi + \Pi_3 w = 0$.

Notice that a possible differential representation of system (8) can be obtained as follows:

$$\begin{aligned} \dot{x} &= (A_1 - A_2(\Pi_2'\Pi_2)^{-1}\Pi_2'\Pi_1)x + (A_3 - A_2(\Pi_2'\Pi_2)^{-1}\Pi_2'\Pi_3)w \\ z &= (E_1 - E_2(\Pi_2'\Pi_2)^{-1}\Pi_2'\Pi_1)x + (E_3 - E_2(\Pi_2'\Pi_2)^{-1}\Pi_2'\Pi_3)w \end{aligned} \tag{9}$$

A straight consequence of the above relations is that matrix $\Pi_2'\Pi_2$ should be nonsingular for all $x \in \mathcal{X}$ and $\delta \in \Delta$. To guarantee that the DAR is well defined or, equivalently, that (9) holds, we further assume the following.

A3: The matrix Π_2 is full-column rank for all $x \in \mathcal{X}$ and $\delta \in \Delta$.

Under the above assumption, the DAR in (8) is regular in the sense of Definition 1 in [20], which follows from Lemma 1 in the same reference.

The following lemma characterizes the class of nonlinear systems which has an equivalent DAR formulation.

Lemma 2 (Rational matrix decomposition)
Let $\sigma \in \Sigma \subset \mathbb{R}^{n_\sigma}$ be a generic parameter. For any rational matrix function $M : \Sigma \mapsto \mathbb{R}^{n_1\times n_2}$ with no singularities at Σ, there exist constant matrices M_1, M_2, and affine matrix functions $\Gamma_1(\sigma), \Gamma_2(\sigma)$ with appropriate dimensions such that

$$M(\sigma) = M_1 - M_2(\Gamma_2'(\sigma)\Gamma_2(\sigma))^{-1}\Gamma_2'(\sigma)\Gamma_1(\sigma)$$

The proof of the above lemma is straightforward from [9, Lemma 2.1]. A natural conclusion from the above result is that the DAR can model the whole class of rational systems with no singularities inside $\mathcal{X} \times \Delta$.

To illustrate the representation, we give the following example.

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Example 1
Consider the following scalar (autonomous) system:

$$\dot{x} = -\frac{x}{1+x^4} + w, \quad z = x \tag{10}$$

which is well posed for all $x \in \mathbb{R}$.

The above system can be rewritten as in (8) by the following definitions:

$$\pi = \begin{bmatrix} \dfrac{x}{1+x^4} \\ \dfrac{x^2}{1+x^4} \\ \dfrac{x^3}{1+x^4} \\ \dfrac{x^4}{1+x^4} \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \end{bmatrix}, \quad \begin{array}{c} \Pi_3 = 0, A_1 = 0, A_3 = 1 \\ A_2 = [-1 \ 0 \ 0 \ 0] \\ E_1 = 1, E_2 = 0, E_3 = 0 \end{array}$$

Observe that Π_2 is non-singular for all $x \in \mathbb{R}$, and so the regularity of the representation is guaranteed.

3.1.1. Related representations of dynamical systems. It turns out that the DAR is not new in control systems theory. For instance, the same modelling technique is used in [20] in the context of uncertain systems.

In the context of nonlinear systems, the DAR is closely related to the *linear fractional representation* (LFR) introduced by *El Ghaoui* and co-authors [9, 21]. The approaches differ between each other in the way that the nonlinear systems is rewritten to apply the LMI framework. More precisely, El Ghaoui *et al.* interpret system (1) as an interconnected system given by

$$\dot{x} = Ax + B_w w + B_p p, \quad q = C_q x + D_{qw} w + D_{qp} p, \quad z = C_z x + D_{zw} w + D_{zp} p, \quad p = \Omega(x, \delta) q \tag{11}$$

where p, q are respectively fictitious inputs and outputs, and

$$\Omega(x, \delta) = \operatorname{diag}\{x_1 I_{r_1}, \ldots, x_n I_{r_{n_x}}, \delta_1 I_{\hat{r}_1}, \ldots, \delta_{n_\delta} I_{\hat{r}_{n_\delta}}\} \tag{12}$$

A DAR of a nonlinear system can be obtained from its LFR model by means of the following equalities:

$$\begin{gathered} A_1 = A, \quad A_2 = B_p, \quad A_3 = B_w, \quad E_1 = C_z, \quad E_2 = D_{zp}, \quad E_3 = D_{zw} \\ \Pi_1 = \Omega(x, \delta) C_q, \quad \Pi_2 = \Omega(x, \delta) D_{qp} - I, \quad \Pi_3 = \Omega(x, \delta) D_{qw}, \quad \pi = p \end{gathered} \tag{13}$$

A straightforward consequence from the above relations is that the LFR tools developed in [22, 23] can be applied to obtain a DAR model of the nonlinear system. However, the LFR representation as well as the DAR one are not unique. In other words, the system equivalence can be obtained from a different choices of $A, B_w, \ldots, \Omega(x, \delta)$ in (11) or $A_1, A_2, \ldots, \Pi_3$ and π in (8). As a result, a bad choice of these matrices may lead to very conservative results.

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In the LFR context, the optimal[‡] choice is obtained from the notion of a minimal LFR [22, 24], which is not always easy to find. On the other hand, the non-uniqueness of representations having the form $\dot{x} = A(x)x$ was discussed in [6] for state-dependent algebraic Riccati equations and then extended to the LMI framework in [13]. In this paper, we follow the similar solution proposed by Trofino [25] for DAR models that addresses the problem from a different perspective leading to less conservative results. More details are given later in Section 4.

3.2. Lyapunov function candidate

Consider the following class of Lyapunov functions:

$$V(x,\delta) = x'\mathscr{P}(x,\delta)x, \quad \mathscr{P}(x,\delta) = \begin{bmatrix} \Theta(x,\delta) \\ I_{n_x} \end{bmatrix}' \begin{bmatrix} P_2 & P_1' \\ P_1 & P_0 \end{bmatrix} \begin{bmatrix} \Theta(x,\delta) \\ I_{n_x} \end{bmatrix} \tag{14}$$

where $P_0 = P_0' \in \mathbb{R}^{n_x \times n_x}$, $P_1 \in \mathbb{R}^{n_x \times n_\theta}$, $P_2 = P_2' \in \mathbb{R}^{n_\theta \times n_\theta}$ are constant matrices to be determined; and $\Theta(x,\delta) \in \mathbb{R}^{n_\theta \times n_x}$ is a given rational matrix function of (x,δ) having no singularities for all x and δ of interest.

For simplicity, define the following auxiliary notation:

$$\zeta = \begin{bmatrix} \xi \\ x \end{bmatrix}, \quad \xi = \Theta(x,\delta)x, \quad P = \begin{bmatrix} P_2 & P_1' \\ P_1 & P_0 \end{bmatrix} \tag{15}$$

From the above, the Lyapunov function and its time derivative can be written as $V(x,\delta) = \zeta' P \zeta$ and $\dot{V}(x,\delta) = 2\zeta' P \dot{\zeta}$, respectively, where $\dot{\zeta}$ is as follows:

$$\dot{\zeta} = \begin{bmatrix} \dot{\xi} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \Theta(x,\delta)\dot{x} + \dot{\Theta}(x,\delta)x \\ \dot{x} \end{bmatrix} \tag{16}$$

In light of (8) and (15), notice that $\dot{\xi}$ is a rational function of (x,δ,π,w). Hence, the following DAR of $\dot{\xi}$ will always exist:

$$\begin{aligned} \dot{\xi} &= J_1 x + J_2 \phi + J_3 w \\ 0 &= \Phi_1(x,\delta)x + \Phi_2(x,\delta)\phi + \Phi_3(x,\delta)w \end{aligned} \tag{17}$$

where $J_1, \ldots, \Phi_3$ and ϕ are defined similarly to the ones in (8).

From the above definitions, the time derivative of the Lyapunov function is as follows:

$$\dot{V}(x,\delta) = 2\zeta' P \begin{bmatrix} J_1 & 0 & J_2 & J_3 \\ A_1 & A_2 & 0 & A_3 \end{bmatrix} \begin{bmatrix} x \\ \pi \\ \phi \\ w \end{bmatrix} \tag{18}$$

Notice that matrix $\Theta(x,\delta)$ plays an important rule on the conservativeness of the approach, since it defines the complexity of the Lyapunov function. In other words, a more complex $\Theta(x,\delta)$ leads to less conservative results at the cost of extra computations.

[‡]The term optimal is used for expressing the less conservative representation.

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Remark 1
The representation in (17) is similar to the *complete square matricial representation* (CSMR) of homogeneous forms proposed in [26] to represent polynomial forms arising from the stability analysis of polynomial systems. The main differences between these representations are: (i) the approach in [27] considers a quadratic equality constraint to define the augmented vector; and (ii) the Lyapunov function is constrained to be a homogeneous form. In contrast, the proposed approach allows the use of rational or polynomial Lyapunov functions.

3.2.1. A particular representation of $\dot{\xi}$. From the authors' experience, a good compromise between conservativeness and computational effort is achieved by choosing $\Theta(x,\delta)$ as an affine matrix function of (x,δ).

The above claim is based on two main points: (i) a systematic form of constructing the DAR in (17) is possible to obtain when $\Theta(x,\delta)$ is affine; and (ii) the most general Lyapunov function will be a polynomial form of degree 4 in x and 2 for δ with the following definition:

$$\Theta(x,\delta) = [x_1 I_{n_x} \ \cdots \ x_{n_x} I_{n_x} \ \delta_1 I_{n_x} \ \cdots \ \delta_{n_\delta} I_{n_x}]'$$

In light of the above arguments, the proposed solution is from now on particularized to the case where $\Theta(x,\delta)$ is an affine function of (x,δ).

Thus, the matrix $\Theta(x,\delta)$ can be represented in the following form:

$$\Theta(x,\delta) = \sum_{i=1}^{n_x} R_i x_i + \sum_{i=1}^{n_\delta} S_i \delta_i + U \tag{19}$$

where R_i (for $i=1,\ldots,n_x$), S_i (for $i=1,\ldots,n_\delta$) and U are constant matrices with the same dimensions of $\Theta(x,\delta)$, and x_i,δ_i are, respectively, the elements of x and δ.

In view of (16) and (19), the following $\dot{V}(x,\delta)$ is obtained:

$$\dot{V}(x,\delta) = 2\zeta' P \begin{bmatrix} \Theta + \sum_{i=1}^{n_x} R_i x r_i \\ I_{n_x} \end{bmatrix} \dot{x} = 2\zeta' P\tilde{\Theta}\dot{x}, \quad \tilde{\Theta} = \begin{bmatrix} \Theta\left(x, \delta + \sum_{i=1}^{n_x} R_i x r_i\right) \\ I_{n_x} \end{bmatrix} \tag{20}$$

where r_i denotes the ith row of the identity matrix I_{n_x}.

Taking into account (8), the above results in

$$\dot{V}(x,\delta) = 2\zeta' P\tilde{\Theta}(A_1 N_1 \zeta + A_2 \pi + A_3 w) \tag{21}$$

where N_1 is a matrix such that $N_1\zeta = x$, e.g. $N_1 = [0_{n_x \times n_\theta} \ I_{n_x}]$.

3.3. Estimating the bounding sets

The estimation of invariant sets from quadratic Lyapunov functions is a standard problem in the LMI framework, see e.g. [8, Section 5.2].

In this paper, a normalized estimate $\mathscr{R}$ as defined below is considered for simplicity.

$$\mathscr{R} \triangleq \frac{\mathscr{R}(c^*)}{c^*} = \{x : x'\mathscr{P}(x,\delta)x \leqslant 1, \ \delta \in \Delta\} \tag{22}$$

where the Lyapunov matrix is redefined as $\mathscr{P}(x,\delta) = \mathscr{P}(x,\delta)/c^*$. Therefore, a sufficient condition to guarantee that $\mathscr{R} \subset \mathscr{X}$ is derived in the following.

Assume that the state domain, $\mathscr{X}$, is a given polytope with known vertices. So, it can be represented in terms of a collection of hyperplanes as follows:

$$\mathscr{X} = \{x : a_k' x \leqslant 1,\ k \in \mathscr{K}\}, \quad \mathscr{K} = \{1, 2, \ldots, n_f\} \tag{23}$$

where $a_k \in \mathbb{R}^{n_x}$ are given constant vectors associated with the n_f faces of $\mathscr{X}$. It turns out that $\mathscr{X}$ can be equivalently represented by $\mathscr{V}(\mathscr{X})$.

From the above definition, the condition $\mathscr{R} \subset \mathscr{X}$ can be cast as follows:

$$2 - a_k' x - x' a_k \geqslant 0 \quad \forall x : x' \mathscr{P}(x,\delta) x - 1 \leqslant 0 \quad \forall (x,\delta,k) \in \mathscr{X} \times \Delta \times \mathscr{K}$$

Applying the $\mathscr{S}$-procedure (see, e.g. [8, Section 2.6]) to the above yields

$$\begin{bmatrix} 1 \\ x \end{bmatrix}' \begin{bmatrix} (2\mu - 1) & -\mu a_k' \\ -\mu a_k & \mathscr{P}(x,\delta) \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geqslant 0 \quad \forall (x,\delta,k) \in \mathscr{X} \times \Delta \times \mathscr{K}$$

where $\mu > 0$ is a free scalar introduced by the $\mathscr{S}$-procedure.

In view of (14) and (15), the above can be recast as follows:

$$\begin{bmatrix} 1 \\ \zeta \end{bmatrix}' \begin{bmatrix} (2\mu - 1) & -\mu a_k' N_1 \\ -\mu N_1' a_k & P \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \end{bmatrix} \geqslant 0 \tag{24}$$

If the above is satisfied in addition to (5) and (6), then $\mathscr{R}$ as defined in (22) is an invariant set for all $w \in \mathscr{W}$, where $\mathscr{W}$ is an admissible disturbance set.

4. STATE- AND PARAMETER-DEPENDENT LMIs

We start this section recalling the Finsler lemma [28, 29].

Lemma 3 (Finsler's Lemma, [29])

Let $x \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n \times n}$ (symmetric), and $\Psi \in \mathbb{R}^{m \times n}$ be such that $\text{rank}(\Psi) \leqslant n$. The following statements are equivalent:

(i) $x'Qx > 0,\ \forall \Psi x = 0,\ x \neq 0$;
(ii) $(\Psi^{\perp})' Q \Psi^{\perp} > 0$;
(iii) $\exists \mu \in \mathbb{R} : Q - \mu \Psi' \Psi > 0$;
(iv) $\exists L \in \mathbb{R}^{n \times m} : Q + L\Psi + \Psi' L' > 0$.

To obtain a convex characterization of Lemma 1 (i.e. to solve problem P1), the Finsler's lemma is applied to the results stated in Section 3. This procedure leads to a set of convex conditions in terms of state- and parameter-dependent LMIs (or in short SPDLMIs). A discussion on this issue is developed in the rest of this section.

4.1. *Handling equality constraints*

The approach used in this paper involves representing the nonlinear system in an augmented space composed by the state vector x and the algebraic vector π. The relationship between these two vectors is defined by means of an equality constraint. Besides, the Lyapunov function is formulated in terms of an auxiliary variable ζ that is dependent on x by the following

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constraint:

$$\Psi_1\zeta = 0, \quad \Psi_1 = [I_{n_0} \;\; -\Theta] \tag{25}$$

As a consequence, the conditions of Lemma 1 are associated with a set of equality constraints. For instance, the positiveness of $V(x,\delta)$ can be checked as follows:

$$\zeta' P\zeta > 0 \quad \forall\zeta : \Psi_1\zeta = 0, \quad (x,\delta) \in \mathscr{X} \times \Delta \tag{26}$$

where $\zeta = \zeta(x,\delta)$ and $\Psi_1 = \Psi_1(x,\delta)$.

One can check the above by guaranteeing that P is positive definite, which is generally conservative. The solution considered in this paper is to apply Lemma 3 yielding the following convex condition to test if $V(x,\delta) > 0$:

$$P + L\Psi_1 + \Psi_1' L' > 0 \quad \forall (x,\delta) \in \mathscr{V}(\mathscr{X} \times \Delta) \tag{27}$$

where P is a symmetric matrix and L a free multiplier to be determined. Notice that we have assumed that $\mathscr{X}$ and Δ are given polytopes to solve the problem.

To prove that (27) implies (26) is quite simple. That is, pre- and post-multiplying (27) by ζ' and ζ leads to (26) and the results holds for all $(x,\delta) \in \mathscr{X} \times \Delta$ from convexity.

4.2. Conservativeness of parameterized LMIs

From the later discussion, the conditions of Lemma 1 can be written as a set of parameterized inequalities taking the form

$$\sigma' \Gamma(\sigma)\sigma > 0 \quad \forall \sigma \in \Sigma \tag{28}$$

where $\sigma \in \mathbb{R}^{n_\sigma}$ represents a generic parameter belonging to a polytope Σ with known vertices, and $\Gamma(\cdot) = \Gamma(\cdot)'$ is an affine matrix function of σ. For instance, condition (27) can be cast as above with $\Gamma(\sigma) = P + L\Psi_1(\sigma) + \Psi_1'(\sigma)L'$.

Obviously, to check (28) the test below

$$\Gamma(\sigma) > 0 \quad \forall \sigma \in \mathscr{V}(\Sigma) \tag{29}$$

may be conservative, since the above implies $\rho'\Gamma(\sigma)\rho > 0$ for all $\sigma \in \Sigma$, $\rho \in \mathbb{R}^{n_\sigma}$. Trofino [25, 30] introduced the idea of linear annihilators to decrease the conservativeness of testing parameterized LMIs.

The main advantage of the approach in [25, 30] is that the annihilator is built over the entire parameter-dependent LMIs contrasting with related techniques in which the annihilator is only associated with either the system representation or the Lyapunov function time-derivative representation as detailed later in this section. In this way, this approach is considered throughout the paper, and the idea is as stated below.

A matrix $\mathscr{N}(\sigma)$ is a linear annihilator of σ if it is a linear function of σ and $\mathscr{N}(\sigma)\sigma = 0$. For instance, consider the following matrix:

$$\mathscr{N}(\sigma) = \begin{bmatrix} \sigma_2 & -\sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_3 & -\sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \sigma_{n_\sigma} & -\sigma_{(n_\sigma - 1)} \end{bmatrix} \tag{30}$$

where σ_i are ith elements of σ and $\mathscr{N}(\sigma) \in \mathbb{R}^{(n_\sigma - 1)\times n_\sigma}$.

To add the constraint $\mathcal{N}(\sigma)\sigma = 0$ to (28), the Finsler lemma can be applied leading to the following parameter-dependent LMI:

$$\Gamma(\sigma) + L\mathcal{N}(\sigma) + \mathcal{N}(\sigma)'L' > 0 \quad \forall \sigma \in \mathcal{V}(\Sigma) \tag{31}$$

where L is a constant multiplier added to the problem. Notice that the above implies the original condition by pre- and post-multiplying it by σ' and σ, respectively.

4.2.1. Related work. One of the first attempts on addressing the problem of the conservativeness associated to state-dependent conditions are the works of Huang and co-authors [6, 13]. They considered the problem in the following form.

Consider the following class of nonlinear systems:

$$\dot{x} = f(x) = A(x)x, \quad x \in \mathcal{X}$$

where $A(x)$ is a nonlinear matrix function of x smooth at origin with appropriate dimensions.

The choice of $A(x)$ is a potential source of conservativeness, since there is a wide range of techniques to address the problem leading to different representations that are not completely equivalent. To analyse the conservativeness of linear-like representations, the following lemma is of interest.

Lemma 4 (Parametrization of linear-like representations, [13])
Let $\dot{x} = f(x)$ be a nonlinear system, where $f(x): \mathbb{R}^n \mapsto \mathcal{X}$ is such that the conditions for existence and uniqueness are guaranteed for all $x \in \mathcal{X}$ and $f(0) = 0$. Suppose there exists a matrix function $A_0(x): \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ such that $f(x) = A_0(x)x$. Then, all possible $A(x)$ satisfying $f(x) = A(x)x$ can be parameterized as follows:

$$A(x) = A_0(x) + \Upsilon(x)\mathcal{N}(x) \tag{32}$$

where $\mathcal{N}(x)$ is an annihilator of x.

The above lemma can be applied to decrease the conservativeness of choosing linear-like representations of nonlinear systems. To illustrate this point, consider the following bilinear system:

$$\dot{x}_1 = -\frac{x_1 + x_1x_2}{2}, \quad \dot{x}_2 = \frac{x_1^2 - x_2}{2}$$

The above system can be represented in a linear-like form as follows:

$$\dot{x} = A_0(x)x, \quad A_0(x) = \begin{bmatrix} -\dfrac{1+x_2}{2} & 0 \\ \dfrac{x_1+x_2}{2} & -\dfrac{1+x_1}{2} \end{bmatrix}$$

For a Lyapunov matrix $P = I_2$, it is not possible to prove that $A_0(x)' + A_0(x) < 0$ for all $x \neq 0$. However, if $A(x) = A_0(x) + \Upsilon\mathcal{N}(x)$, where

$$\Upsilon = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} \quad \text{and} \quad \mathcal{N}(x) = [x_2 \quad -x_1]$$

then $A(x)' + A(x) = -I_2$, and so the system is globally stable.

In the context of the SOS approach, Chesi *et al.* [27] have proposed a similar technique to represent $\dot{V}(x)$—namely, the CSMR. More precisely, the technique in [27] models a rational system by means of an LFR in which the state is viewed as a time-varying parameter $\theta(t)$ (i.e. $\Omega(x,\delta)=\Omega(\theta(t))$) and considers an homogeneous polynomial Lyapunov function of degree $2m$. A quadratic form of degree $2m$ is defined as $v_{2m}(x)=x^{\{m\}'}(W+L(\alpha))x^{\{m\}}$, where $x^{\{m\}}$ is a vector containing only monomials of degree m in x and $L(\alpha)$ is a linear parametrization of the set of quadratic annihilators of $x^{\{m\}}$, i.e. $\mathscr{L}=\{L=L' : x^{\{m\}'}Lx^{\{m\}}=0\}$.

For a Lyapunov function $V(x)=x^{\{m\}'}P_m x^{\{m\}}$, the time derivative is as follows:

$$\dot{V}(x)=2x^{\{m\}'}P_m\dot{x}^{\{m\}}=2x^{\{m\}'}P_m\frac{\partial x^{\{m\}}}{\partial x}\dot{x}=2x^{\{m\}'}P_m\frac{\partial x^{\{m\}}}{\partial x}(Ax+B_p\Omega(\theta(t))q) \tag{33}$$

$$=2x^{\{m\}'}P_m\bar{A}x^{\{m\}}+2x^{\{m\}'}P_m\bar{B}_p(q\otimes x^{\{m-1\}}) \tag{34}$$

where $\bar{A}$ and $\bar{B}_p$ are suitable matrices. The representation in (34) is the CSMR of $\dot{V}(x)$. The above technique considers quadratic annihilators to $x^{\{m\}}$ and $q\otimes x^{\{m-1\}}$. However, the coupling between the system nonlinearities (represented by the time-varying parameter $\theta(t)$) and the state x is not taken into account.

5. INPUT-TO-OUTPUT ANALYSIS

We start this section by giving the following theorem that proposes a convex solution to problem (P1) in terms of SPDLMIs.

Theorem 1

Consider system (1) with A1 and A2, its DAR in (8) satisfying A3 and the following notation:

$$\Psi_2=\begin{bmatrix} x & -N_1 \\ 0 & \Psi_1 \end{bmatrix},\quad \Psi_3=\begin{bmatrix} \Pi_1 N_1 & \Pi_2 & \Pi_3 \\ \mathscr{N}(x)N_1 & 0 & 0 \end{bmatrix}$$

Let $\mathscr{X}$ and Δ be given polytopes. Let Θ be a given affine matrix function of (x,δ). Suppose the matrices $P=P'$, L, M_k $(k\in\mathscr{K})$, W, and positive scalars μ, γ and λ are a solution to (27) and the following LMIs that are constructed at $\mathscr{V}(\mathscr{X}\times\Delta)$:

$$\begin{bmatrix} (2\mu-1) & -\mu a_k' N_1 \\ -\mu N_1' a_k & P \end{bmatrix}+M_k\Psi_2+\Psi_2' M_k'>0,\quad k\in\mathscr{K} \tag{35}$$

$$\begin{bmatrix} N_1'A_1'\tilde{\Theta}'P+P\tilde{\Theta}A_1N_1 & P\tilde{\Theta}A_2 & P\tilde{\Theta}A_3 & N_1'E_1' \\ A_2'\tilde{\Theta}'P & 0 & 0 & E_2' \\ A_3'\tilde{\Theta}'P & 0 & -\gamma I_{n_w} & E_3' \\ E_1N_1 & E_2 & E_3 & -\lambda I_{n_z} \end{bmatrix}+W[\Psi_3\ \ 0]+\begin{bmatrix}\Psi_3' \\ 0\end{bmatrix}W'<0 \tag{36}$$

Then, the trajectory $x(t)$ driven by an admissible disturbance signal lies in the set $\mathscr{R}$ as defined in (22) for all $\delta\in\Delta$. Moreover, the dissipation inequality in (3) is satisfied.

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Proof
Suppose (27), (35) and (36) are satisfied for all $(x,\delta)\in\mathscr{V}(\mathscr{X}\times\Delta)$. Then, by convexity, they are also satisfied for all $x\in\mathscr{X}$ and $\delta\in\Delta$.

Pre- and post-multiplying the LMI (27) by ζ' and ζ, respectively, yields

$$x'\mathscr{P}(x,\delta)x>0 \quad \forall(x,\delta)\in\mathscr{X}\times\Delta \tag{37}$$

since $\Psi_1\zeta=0$ and $N_1\zeta=x$.

Define two scalars as follows: $\varepsilon_1=\inf_{(x,\delta)\in\mathscr{X}\times\Delta}\operatorname{eig}(\mathscr{P}(x,\delta))$ and $\varepsilon_2=\sup_{(x,\delta)\in\mathscr{X}\times\Delta}\operatorname{eig}(\mathscr{P}(x,\delta))$, where $\operatorname{eig}(\mathscr{P}(x,\delta))$ stands for the eigenvalues of $\mathscr{P}(x,\delta)$. In view of (37) and the above, the following holds:

$$\varepsilon_1 x'x\leqslant x'\mathscr{P}(x,\delta)x\leqslant\varepsilon_2 x'x \quad \forall(x,\delta)\in\mathscr{X}\times\Delta \tag{38}$$

Consider the LMI (36). Pre- and post-multiplying it by $[\zeta'\ \pi'\ w'\ v']$ and its transpose, respectively, and applying the Schur complement leads to

$$\dot{V}(x,\delta)+\frac{z'z}{\lambda}-\gamma w'w<0 \quad \forall(x,\delta)\in\mathscr{X}\times\Delta \tag{39}$$

since $\Psi_3[\zeta'\ \pi'\ w']'=0$, and $\dot{V}(x,\delta)$ is as defined in (21).

Similarly, pre- and post-multiplying (35) by $[1\ \zeta']$ and its transpose, respectively, implies (24). In light of (38) and (39), $\mathscr{R}=\{x: V(x,\delta)\leqslant 1\}$ is an invariant set for all $w(t)\in\mathscr{W}$, where $\mathscr{W}$ is admissible. The rest of the proof follows from Lemma 1. □

Remark 2
Notice that the conditions of Theorem 1 are satisfied then the regularity of the DAR is guaranteed. This is implied by: (i) partitioning W in (36) accordingly to the matrix $[\Psi_3\ 0]$, (ii) applying the Schur complement to the second row and column block in the left-hand side matrix of (36), and (iii) the fact that these operations results in $W_{21}\Pi_2+\Pi_2'W_{21}'<0$. In other words, Π_2 is full-column rank and the DAR is regular in the sense of [20, Definition 1].

The following two results are straight applications of Theorem 1.

Corollary 1 (Regional stability analysis)
Consider Theorem 1 and let $\lambda\to\infty$. Suppose the matrices $P=P'$, L, M_k $(k\in\mathscr{K})$, W, and the scalars μ and γ are a solution to the following optimization problem:

$$\min_{(x,\delta)\in\mathscr{V}(\mathscr{X}\times\Delta)}\gamma \text{ subject to (27), (35) and (36)}$$

Then, system (1) is regionally stable in $\mathscr{R}$ w.r.t. the following class of disturbance signals:

$$\mathscr{W}=\left\{w:\int_0^\infty w'w\,\mathrm{d}t\leqslant\frac{1}{\gamma}\right\} \tag{40}$$

Proof
As $\lambda\to\infty$, Theorem 1 implies $\int_0^\infty\gamma w'w\,\mathrm{d}t>1$. As a consequence, the set defined in (40) is an estimate of the class of admissible disturbance signals. □

Remark 3
To make Corollary 1 numerically tractable, we have to modify (36) by taking into account that $\lambda\to\infty$. From the Schur complement, one can drop the fourth row and column in the left-hand

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side matrix of (36) leading to the following modified LMI:

$$\begin{bmatrix} N_1'A_1'\tilde{\Theta}'P + P\tilde{\Theta}A_1N_1 & P\tilde{\Theta}A_2 & P\tilde{\Theta}A_3 \\ A_2'\tilde{\Theta}'P & 0 & 0 \\ A_3'\tilde{\Theta}'P & 0 & -\gamma I_{n_w} \end{bmatrix} + W\Psi_3 + \Psi_3'W' < 0$$

Corollary 2 ($\mathscr{L}_2$-gain estimate)
Consider Theorem 1 and let $\lambda = \gamma$. Suppose the matrices $P = P'$, L, M_k ($k \in \mathscr{K}$), W, and the scalars μ and γ are a solution to the following optimization problem:

$$\min_{(x,\delta)\in\mathscr{V}(\mathscr{X}\times\Delta)} \gamma \text{ subject to (27), (35) and (36)}$$

Then, the $\mathscr{L}_2$-gain of system (1), $||\mathscr{G}_{wz}||_\infty$, is bounded by γ. More precisely, the following is satisfied:

$$||\mathscr{G}_{wz}||_\infty \triangleq \sup_{0\neq w\in\mathscr{W}} \frac{||z||_2}{||w||_2} < \gamma \quad \forall (x,\delta) \in \mathscr{X} \times \Delta \tag{41}$$

for an admissible $\mathscr{W}$.

6. OUTPUT FEEDBACK CONTROL

Consider the following class of input affine systems:

$$\dot{x} = f(x,\delta,w) + Bu, z = g(x,\delta,w) + Fu, y = h(x,\delta,w), \quad x(0) = 0 \tag{42}$$

where $y \in \mathbb{R}^{n_y}$ is the measurement signal, $u \in \mathbb{R}^{n_u}$ the control input and B, F are constant matrices with appropriate dimensions.

The problem of concern in this section is to design an output feedback controller such that the dissipation inequality (3) is satisfied in the closed-loop.

To this end, consider the following (robust, linear and dynamic) output feedback controller:

$$u = C_c x_c, \quad \dot{x}_c = A_c x_c + B_c y, \quad x_c(0) = 0 \tag{43}$$

where $x_c \subset \mathbb{R}^{n_x}$ is the control state, and A_c, B_c, C_c are constant matrices with appropriate dimensions to be determined.

In light of (42) and (43), the closed-loop system takes the following augmented representation:

$$\dot{x}_a = f_a(x_a,\delta,w), \quad z = g_a(x_a,\delta,w), \quad x_a(0) = 0 \tag{44}$$

where $x_a \in \mathbb{R}^{2n_x}$ is the augmented state, and

$$x_a = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad f_a(x_a,\delta,w) = \begin{bmatrix} f(x,\delta,w) + BC_c x_c \\ B_c h(x,\delta,w) + A_c x_c \end{bmatrix}, \quad g_a(x,\delta,w) = g(x,\delta,w) + FC_c x_c$$

To obtain a convex characterization of the synthesis problem, assume that the measurement signal y can take the following representation:

$$y = C_1 x + C_2 \pi + C_3 w \tag{45}$$

where C_1, C_2, C_3 are constant matrices with appropriate dimensions accordingly to the DAR in (8).

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In view of (8) and the above, the closed-loop system can be recast as follows:

$$\dot{x}_a = A_{1a}x_a + A_{2a}\pi + A_{3a}w, \quad z = E_{1a}x_a + E_2\pi + E_3 w \tag{46}$$

where

$$A_{1a} = \begin{bmatrix} A_1 & BC_c \\ B_cC_1 & A_c \end{bmatrix}, \quad A_{2a} = \begin{bmatrix} A_2 \\ B_cC_2 \end{bmatrix}, \quad A_{3a} = \begin{bmatrix} A_3 \\ B_cC_3 \end{bmatrix}, \quad E_{1a} = [E_1 \;\; FC_c]$$

Accordingly, the Lyapunov function candidate is redefined as below:

$$V(x_a, \delta) = x_a' \mathscr{P}_a(x, \delta) x_a, \quad \mathscr{P}_a(x, \delta) = \begin{bmatrix} \mathscr{P}(x, \delta) & P_3 \\ P_3' & P_4 \end{bmatrix} \tag{47}$$

where $\mathscr{P}(x, \delta)$ is as defined in (14), $P_3 \in \mathbb{R}^{n_x \times n_x}$ and $P_4 = P_4' \in \mathbb{R}^{n_x \times n_x}$ are matrices to be determined.

For convenience, define the following auxiliary notation:

$$\zeta_a = \begin{bmatrix} \zeta \\ x_c \end{bmatrix}, \quad P_a = \begin{bmatrix} P & N_1' P_3 \\ P_3' N_1 & P_4 \end{bmatrix} \tag{48}$$

In light of (48), $V(x_a, \delta) = \zeta_a' P_a \zeta_a$ and the time derivative of the Lyapunov function is as follows:

$$\dot{V}(x_a, \delta) = 2\zeta_a' P_a \dot{\zeta}_a = 2\zeta_a' P_a \tilde{\Theta}_a (A_{1a} N_2 \zeta_a + A_{2a}\pi + A_{3a} w) \tag{49}$$

where

$$\tilde{\Theta}_a = \begin{bmatrix} \tilde{\Theta} \\ I_{n_x} \end{bmatrix}, \quad N_2 = [0_{2n_x \times n_\theta} \;\; I_{2n_x}]$$

The straightforward application of Theorem 1 for control design leads to bilinear matrix inequalities (BMIs) [31] that are hard to solve. Even for linear systems, a convex characterization of the robust output feedback problem is not yet available.[§]

Recently, de Oliveira *et al.* [33] have proposed a convex characterization to the robust output feedback problem for linear systems using arguments from the separation principle. The design problem is performed in two steps: (i) a state-feedback law, $u = C_c x$, is synthesized *a priori*, and (ii) the control matrices A_c and B_c are designed by means of a convex optimization problem for a given gain matrix C_c.

In this paper, we follow a similar idea. To this end, we assume that a state feedback $u = C_c x$ is available such that the dissipation inequality is satisfied, and we recall the parametrization introduce in [34, 35] for filter design. Basically, the Lyapunov inequalities are pre- and post-multiplied by matrices of the type

$$G_i = \mathrm{diag}\{I_{n_i}, P_3 P_4^{-1}, I_{m_i}\}, \quad i = 1, 2, 3$$

and an appropriate parametrization on the Lyapunov and control matrices are performed.

The above procedure leads to the following result.

[§]For a parameter-dependent control law, Scherer [32] has proposed an elegant solution to the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ output feedback problems.

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Theorem 2

Consider system (42) with A1 and A2, and its DAR form satisfying A3 with (45). Consider the notation of Theorem 1. Let Θ be a given affine matrix function of (x, δ). Let $C_p \in \mathbb{R}^{n_u \times n_x}$ be a given constant matrix. Let $\mathscr{X}$ and Δ be given polytopes. Further, define the following notation:

$$\Psi_{1a} = [\Psi_1 \;\; 0], \quad \Psi_{2a} = [\Psi_2 \;\; 0], \quad \Psi_{3a} = \begin{bmatrix} \mathscr{N}(x)N_1 & 0 & 0 & 0 & 0 \\ \Pi_1 N_1 & 0 & \Pi_2 & \Pi_3 & 0 \end{bmatrix}$$

Suppose the matrices $P = P'$, A_p, B_p, $Q = Q'$, L, M_k (for $k \in \mathscr{K}$) and W, and positive scalars μ, γ and λ are a solution to the following LMIs which are constructed at $\mathscr{V}(\mathscr{X} \times \Delta)$:

$$\begin{bmatrix} P & N_1' Q \\ Q N_1 & Q \end{bmatrix} + L\Psi_{1a} + \Psi_{1a}' L' > 0 \tag{50}$$

$$\begin{bmatrix} (2\mu - 1) & -\mu a_k' N_1 & 0 \\ -\mu N_1' a_k & P & N_1' Q \\ 0 & Q N_1 & Q \end{bmatrix} + M_k \Psi_{2a} + \Psi_{2a}' M_k' > 0, \quad k \in \mathscr{K} \tag{51}$$

$$\begin{bmatrix} \psi_{11} & \psi_{21}' & \psi_{31}' & \psi_{41}' & N_1' E_1' \\ \psi_{21} & \psi_{22} & \psi_{32}' & \psi_{42}' & C_p' F' \\ \psi_{31} & \psi_{32} & 0 & 0 & E_2' \\ \psi_{41} & \psi_{42} & 0 & -\gamma I_{n_w} & E_3' \\ E_1 N_1 & F C_p & E_2 & E_3 & -\lambda I_{n_z} \end{bmatrix} + W\Psi_{3a} + \Psi_{3a}' W' < 0 \tag{52}$$

where the blocks ψ_{ij} are as follows:

$$\psi_{11} = N_1' A_1' \tilde{\Theta}' P + P\tilde{\Theta} A_1 N_1 + N_1' C_1' B_p' N_1 + N_1' B_p C_1 N_1$$

$$\psi_{21} = Q N_1 \tilde{\Theta} A_1 N_1 + B_p C_1 N_1 + C_p' B' \tilde{\Theta}' P + A_p' N_1$$

$$\psi_{22} = C_p' B' \tilde{\Theta}' Q + Q N_1 \tilde{\Theta} B C_p + A_p' + A_p$$

$$\psi_{31} = A_2' \tilde{\Theta}' P + C_2' B_p' N_1$$

$$\psi_{32} = A_2' \tilde{\Theta}' N_1' Q + C_2' B_p'$$

$$\psi_{41} = A_3' \tilde{\Theta}' P + C_3' B_p' N_1$$

$$\psi_{42} = A_3' \tilde{\Theta}' N_1' Q + C_3' B_p'$$

Then, the dissipation inequality is satisfied for system (42) with (43), where the control matrices are as below:

$$A_c = A_p Q^{-1}, \quad B_c = B_p, \quad C_c = C_p Q^{-1} \tag{53}$$

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Moreover, $V(x, \delta, x_c)$ as defined in (47) with $P_3 = I_{n_x}$ is a Lyapunov function for $(x, x_c) = 0$, and the closed-loop system is regionally stable w.r.t.

$$\mathscr{R} = \{(x, 0) : V(x, \delta, 0) \leqslant 1\} \tag{54}$$

and $\mathscr{W}$, where $\mathscr{W}$ is admissible.

Proof

Suppose the LMIs of Theorem 2 are satisfied for all $(x, \delta) \in \mathscr{V}(\mathscr{X} \times \Delta)$, then by convexity they are also satisfied for all $(x, \delta) \in \mathscr{X} \times \Delta$. Let $P_3 = I_{n_x}$ and $P_4 = Q^{-1}$. From (53), we get the following parametrization:

$$C_p = C_c P_4^{-1}, \quad B_p = B_c, \quad A_p = A_c P_4^{-1} \tag{55}$$

Also, define the following matrices:

$$G_1 = \text{diag}\{I_{n_\theta + n_x}, P_3 P_4^{-1}\}, \quad G_2 = \text{diag}\{I_{1+n_\theta+n_x}, P_3 P_4^{-1}\}, \quad G_3 = \text{diag}\{I_{n_\theta+n_x}, P_3 P_4^{-1}, I_{n_w+n_z}\}$$

Consider the LMI (50). Notice from the above that (50) can be cast as given below:

$$G_1 \left(\begin{bmatrix} P & N_1' P_3 \\ P_3' N_1 & P_4 \end{bmatrix} + \tilde{L} \Psi_{1a} + \Psi_{1a}' \tilde{L}' \right) G_1' > 0 \tag{56}$$

where L is redefined as $\tilde{L}$, accordingly.

Similarly, the LMI in (51) can be written as follows:

$$G_2 \left(\begin{bmatrix} (2\mu - 1) & -\mu a_k' N_1 & 0 \\ -\mu N_1' a_k & P & N_1' P_3 \\ 0 & P_3' N_1 & P_4 \end{bmatrix} + \tilde{M}_k \Psi_{2a} + \Psi_{2a}' \tilde{M}_k' \right) G_2' > 0, \quad k \in \mathscr{K} \tag{57}$$

where the matrices M_k are redefined as $\tilde{M}_k$, accordingly.

Notice that the LMI (52) can also take the following form:

$$G_3 \left(\begin{bmatrix} \phi_{11} & \phi_{21}' & \phi_{31}' & \phi_{41}' & N_1' E_1' \\ \phi_{21} & \phi_{22} & \phi_{32}' & \phi_{42}' & C_c' F' \\ \phi_{31} & \phi_{32} & 0 & 0 & E_2' \\ \phi_{41} & \phi_{42} & 0 & -\gamma I_{n_w} & E_3' \\ E_1 N_1 & F C_c & E_2 & E_3 & -\lambda I_{n_z} \end{bmatrix} + \tilde{W} \Psi_{3a} + \Psi_{3a}' \tilde{W}' \right) G_3' < 0 \tag{58}$$

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where W is redefined as $\tilde{W}$, accordingly, and the blocks ϕ_{ij} are as follows:

$$\phi_{11} = N_1'A_1'\tilde{\Theta}'P + P\tilde{\Theta}A_1N_1 + N_1'C_1'B_c'P_3'N_1 + N_1'P_3B_cC_1N_1$$

$$\phi_{21} = P_3'N_1\tilde{\Theta}A_1N_1 + P_4B_cC_1N_1 + C_c'B'\tilde{\Theta}'P + A_c'P_3'N_1$$

$$\phi_{22} = C_c'B'\tilde{\Theta}'P_3 + P_3'N_1\tilde{\Theta}BC_c + A_c'P_4 + P_4A_c$$

$$\phi_{31} = A_2'\tilde{\Theta}'P + C_2'B_c'P_3'N_1$$

$$\phi_{32} = A_2'\tilde{\Theta}'N_1'P_3 + C_2'B_c'P_4$$

$$\phi_{41} = A_3'\tilde{\Theta}'P + C_3'B_c'P_3'N_1$$

$$\phi_{42} = A_3'\tilde{\Theta}'N_1'P_3 + C_3'B_c'P_4$$

From the Schur complement and the pre- and post-multiplication by $[\zeta' \; x_c' \; \pi' \; w']$ and its transpose, respectively, we get for (58) the following:

$$\dot{V}(x_a, \delta) + \frac{z'z}{\lambda} - \gamma w'w < 0 \tag{59}$$

where $\dot{V}(x_a, \delta)$ is as defined in (49).

In view of (56), (57) and (59), the rest of the proof follows from Theorem 1. □

Remark 4

Matrix P_3 can be viewed as a similarity transformation on the control matrices, since the original parametrization is as follows:

$$A_c = P_3^{-1}A_pQ^{-1}P_3, \quad B_c = P_3^{-1}B_p, \quad C_c = C_pQ^{-1}P_3$$

Thus, P_3 is any non-singular matrix and it can be set as I_{n_x} without loss of generality. From this fact, the Lyapunov function as defined in (47), i.e. $V(x, \delta, x_c) = x'\mathscr{P}(x, \delta)x + 2x'P_3x_c + x_c'P_4x_c$, is not conservative at all, since: (i) P_3 is free, and (ii) a quadratic function is considered to the (linear) control space.

Remark 5

To make the conditions of Theorem 2 convex, we have assumed that a parameterized matrix C_p is given. To determine this matrix, one can consider any (parameterized) robust state-feedback law such that the system is locally stable. When a local solution is not available, we propose the following procedure: (i) consider a linearized system as state below:

$$\dot{\rho} = \hat{A}(\delta)\rho + \hat{B}(\delta)w + Bu, \quad \hat{z} = \hat{C}(\delta)\rho + \hat{D}(\delta)w + Fu \tag{60}$$

where

$$\hat{A}(\delta) = \left.\frac{\partial f(x, \delta, w)}{\partial x}\right|_{x=0, w=0}, \quad \hat{B}(\delta) = \left.\frac{\partial f(x, \delta, w)}{\partial w}\right|_{x=0, w=0}, \quad \hat{C}(\delta) = \left.\frac{\partial g(x, \delta, w)}{\partial x}\right|_{x=0, w=0}$$

and

$$\hat{D}(\delta) = \left.\frac{\partial g(x, \delta, w)}{\partial w}\right|_{x=0, w=0}$$

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and (ii) determine C_p as follows:

$$\min_{X, C_p,\ \delta \in \mathscr{V}(\Delta)} \eta : X > 0, \quad \begin{bmatrix} \hat{A}(\delta)X + X\hat{A}(\delta)' + BC_p + C_p'B' & \hat{B}(\delta) & X\hat{C}(\delta)' + C_p'F' \\ \hat{B}(\delta)' & -\eta I_{n_w} & \hat{D}(\delta)' \\ \hat{C}(\delta)X + FC_p & \hat{D}(\delta) & -\eta I_{n_z} \end{bmatrix} < 0$$

The following two results are straight applications of Theorem 2.

Corollary 3 (Regional stabilization)
Consider Theorem 2 and let $\lambda \to \infty$. Suppose matrices $P = P'$, L, M_k $(k \in \mathscr{K})$, Q, and W, and positive scalars μ and γ are a solution to the following optimization problem:

$$\min_{(x,\delta) \in \mathscr{V}(\mathscr{X} \times \Delta)} \gamma \text{ subject to (50), (51) and (52)}$$

Then, the closed-loop system (42) is regionally stable in $\mathscr{R}$ as defined in (54) w.r.t. to the class of disturbance signals defined in (40).

Remark 6
For Corollary 3 to be numerically tractable, we have to modify (52) by taking into account that $\lambda \to \infty$. To this end, the LMI (52) should be as follows:

$$\begin{bmatrix} \psi_{11} & \psi_{21}' & \psi_{31}' & \psi_{41}' \\ \psi_{21} & \psi_{22} & \psi_{32}' & \psi_{42}' \\ \psi_{31} & \psi_{32} & 0 & 0 \\ \psi_{41} & \psi_{42} & 0 & -\gamma I_{n_w} \end{bmatrix} + W\hat{\Psi}_{3a} + \hat{\Psi}_{3a}'W' < 0$$

where

$$\hat{\Psi}_{3a} = \begin{bmatrix} \mathscr{N}(x)N_1 & 0 & 0 & 0 \\ \Pi_1 N_1 & 0 & \Pi_2 & \Pi_3 \end{bmatrix}$$

Corollary 4 (Nonlinear $\mathscr{H}_\infty$ control)
Consider Theorem 2 and let $\lambda = \gamma$. Suppose the matrices $P = P'$, L, M_k $(k \in \mathscr{K})$, Q, and W, and positive scalars μ and γ are a solution to the following optimization problem:

$$\min_{(x,\delta) \in \mathscr{V}(\mathscr{X} \times \Delta), w \in \mathscr{W}} \gamma \text{ subject to (50) (51) and (52)}$$

Then, the $\mathscr{L}_2$-gain of the closed-loop system (42) is bounded by γ, where $\mathscr{W}$ is admissible.

7. NUMERICAL EXAMPLES

To illustrate the proposed approach, we give the following examples.

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Example 2

Consider a 2DOF camera with an image-based control law, as described in [36], whose closed-loop dynamics is as stated below:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{a_1}{x_1 + a_2} \begin{bmatrix} -a_2 & -\dfrac{8a_2^2 x_2}{(1+x_2^2)^2} \\ 0 & -\dfrac{a_2(1-x_2^2)}{(1+x_2^2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ 0 \end{bmatrix} w, \quad z = [0 \;\; 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{61}$$

where $a_1 = 0.1$ and $a_2 = 1.5$. From physical reasoning, the state vector is constrained to the following regions: $x_1 \in [-1.4, \alpha]$ and $x_2 \in [-0.6, 0.6]$. The problem to be addressed in this example is to estimate an upper bound on $||\mathscr{G}_{wz}||_\infty$ for some α (as large as possible).

To this end, consider the following DAR form for system (61):

$$\dot{x} = A_2\pi + A_3 w, \quad z = C_1 x, \quad 0 = \Pi_1 x + \Pi_2 \pi$$

where $A_3 = [1 \;\; 0]'$, $C_1 = [0 \;\; 1]$, and

$$\pi = \left[\frac{x_1}{x_1+a_2} \; \frac{x_2}{(x_1+a_2)(1+x_2^2)} \; \frac{x_2^3}{(x_1+a_2)(1+x_2^2)} \; \frac{x_2^2}{(x_1+a_2)(1+x_2^2)^2} \; \frac{x_2^2}{(x_1+a_2)(1+x_2^2)} \right.$$
$$\left. \frac{x_2}{1+x_2^2} \; \frac{x_2^2}{1+x_2^2} \; \frac{x_2^3}{(x_1+a_2)(1+x_2^2)^2} \right]'.$$

$$A_2 = \begin{bmatrix} -a_1a_2 & 0 & 0 & -8a_1a_2^2 & 0 & 0 & 0 & 0 \\ 0 & -a_1a_2 & a_1a_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} 1 & 0_{1\times4} & 0 & 0_{1\times2} \\ 0 & 0_{1\times4} & 1 & 0_{1\times2} \end{bmatrix}'$$

$$\Pi_2 = \begin{bmatrix} -(x_1+a_2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (x_1+a_2) & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & x_2 \\ 0 & x_2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & -1 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Notice that the regularity of system (61) is guaranteed if $x_1 > -1.5$ which implies the regularity of the above DAR.

To define the Lyapunov function candidate, we choose $\Theta = [x_1 I_2 \;\; x_2 I_2]'$. For comparison purposes, the LFR approach is also applied to the same example considering a quadratic Lyapunov function [9] and a homogeneous one [27] with

$$x^{\{m\}} = [x_1^2 \;\; x_1x_2 \;\; x_2^2]'$$

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Table I. Upper bounds on $||\mathcal{G}_{wz}||_\infty$.

	Methodology		
Estimates	LFR quadratic	LFR polynomial	Proposed approach
γ	0.05	0.02	0.01
α	1.40	4.00	5.00

The results are summarized in Table I, where the proposed approach obtained the less conservative result. The conservativeness of the approach in [27] is possibly due to the modelling of the nonlinearities as time-varying parameters and the polynomial vector $x^{\{m\}}$ is a homogeneous form.

Example 3
Consider the following Van der Pol equation:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + (0.8 + 0.2\delta)(1 - x_1^2)x_2 + u + w_1, \quad y = x_1 + w_2 \tag{62}$$

where $x = [x_1 \ x_2]'$ is the state vector belonging to $\mathcal{X} = \{x : |x_i| < \alpha\}$, and α is a given positive scalar. The time-invariant parameter δ is unknown, but bounded to $[-1, 1]$. The problem of interest in this example is to determine an estimate of the class of admissible disturbance inputs (as large as possible) such that the system is regionally stable in closed-loop.

To this end, consider the following DAR form for system (62):

$$\dot{x} = A_1 x + A_2 \pi + A_3 w + Bu, \quad 0 = \Pi_1 x + \Pi_2 \pi, \quad y = C_1 x + C_3 w \tag{63}$$

where the system matrices and vectors are given by $w = [w_1 \ w_2]'$ and the following:

$$\pi = [x_1 x_2 \ \ \delta x_2 \ \ x_1^2 x_2 \ \ \delta x_1^2 x_2]', \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0.8 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.2 & -0.8 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = [1 \ \ 0], \quad C_3 = [0 \ \ 1]$$

$$\Pi_1 = \begin{bmatrix} x_2 & 0 \\ 0 & \delta \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ x_1 & 0 & -1 & 0 \\ 0 & 0 & \delta & -1 \end{bmatrix}$$

Applying the procedure on Remark 5, we get the following parameterized matrix C_p:

$$C_p = [-307.42177 \ \ -324.81903]$$

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To define the Lyapunov function candidate, let Θ be as follows:

$$\Theta = \begin{bmatrix} x_1 I_2 \\ [0 \;\; x_2] \\ \delta I_2 \end{bmatrix}$$

From Corollary 3 and Remark 6, we get $\gamma = 0.2$ for $\alpha = 1$, and the following control matrices:

$$A_c = \begin{bmatrix} -2.4991553 & 3.265075 \\ -0.1328802 & -1.2749113 \end{bmatrix}, \quad B_c = \begin{bmatrix} -0.0416680 \\ 0.0001435 \end{bmatrix}$$

and $C_c = [-163.93216 \;\; 125.77996]$.

For comparison purposes, we apply the quasi-LPV technique to the robust controllers proposed in [33, 37], respectively. It turns out that both approaches do not give feasible solutions for the same limits on x and δ.

8. CONCLUDING REMARKS

This paper has proposed a convex approach to the input-to-output analysis and output feedback control for a class of uncertain nonlinear systems subject to bounded disturbances. The proposed conditions are cast in terms of (state- and parameter-dependent) LMIs that assure: (i) the regional stability of the system for bounded disturbance signals and (ii) an upper bound on the system (induced) $\mathscr{L}_2$-gain. The stability results are extended for designing a robust output feedback control law that assures the regional stability of the closed loop with a guaranteed input-to-output performance.

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