Extracting Information on Implied Volatilities and Discrete Dividends From American Option Prices

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This paper deals with options on assets, such as stocks or indexes, which pay cash dividends. Pricing methods which consider discrete dividends are usually computationally expensive and become infeasible when one considers multiple dividends paid during the option lifetime. This is the case of long-term options and options on indexes. The first purpose of this paper is to assess efficient and accurate numerical procedures which yield consistent prices for both European and American options when the underlying asset pays discrete dividends. The authors then analyze some methodologies to extract information on implied volatilities and dividends from quoted option prices. Implied dividends can also be computed using a modified version of the well-known put-call parity relationship. This technique is straightforward, nevertheless, its use is limited to European options, and when dealing with equities, most traded options are of American type. As an alternative, the numerical inversion of pricing methods, such as efficient interpolated binomial method, can be used. This paper applies different procedures to obtain implied volatilities and dividends of listed stocks of the Italian derivatives market (IDEM).

Keywords: options on stocks, discrete dividends, lattice methods, implied volatilities, implied dividends

Introduction

Stock options are normally unprotected from cash dividends paid on the underlying. Dividend payments during the option life reduce the stock price by an amount proportional to the size of the dividend, and hence, reduce (increase) the value of call (put) options. In the event of extraordinary cash dividends, the options clearing corporation protects the value of options by adjusting the exercise prices. When considering aggregated dividends which are the case when dealing with indexes, one can assume that the uncertainty is balanced out, but stock options can be affected by a single cash dividend. Thus, a change in the latter has a significant impact on the option prices.

A valuation of options on stocks which pay discrete dividends is a rather hard problem, which has drawn a lot of attentions in the financial literature, but there is still some confusion concerning the evaluation approaches. Different methods have been proposed for the pricing of both European and American options on dividend paying stocks, which suggest various model adjustments (for example, subtracting the present value of the dividend from the asset spot price). Nevertheless, all such approximations have some drawbacks and are not so efficient.
E. G. Haug and J. Haug (1998) and Beneder and Vorst (2002) proposed a volatility adjustment, which took into account the timing of the dividend. A more sophisticated volatility adjustment to be used in combination with the escrowed dividend model is proposed by Bos, Gairat, and Shepeleva (2003). A slightly different implementation is suggested by Bos and Vandermark (2002), which adjusts the stock price and the strike at the same time. While De Matos, Dilao, and Ferreira (2006) derived arbitrarily accurate lower and upper bounds for the value of European options on a stock paying a discrete dividend. E. G. Haug, J. Haug, and Lewis \(^1\) (2003) provided an integral representation formula that could be considered as the exact solution to problems of evaluating both European and American call options and European put options. Veiga and Wystup (2009) provided a closed formula for European options in a Black-Scholes (BS) framework. A review of some pricing methods proposed in the literature\(^2\) is provided in Section 2.

In this paper, the authors focus on American-style options, since most listed options on stocks are of this type. As is well-known (Merton, 1973), in the absence of dividends, it is never optimal to exercise an American call before maturity. If a cash dividend payment is expected during the lifetime of the option, it may be optimal to exercise an American call right before the ex-dividend date. For an American put option, it may be optimal to exercise at any time prior to expiration, even in the absence of dividends. Unfortunately, no analytical solutions for both the option price and the exercise strategy are available. Hence, one is generally forced to numerical solutions, such as binomial approaches.

Lattice methods are commonly used for pricing options and one can easily deal with early exercise. In the binomial model (Cox, Ross, & Rubinstein, 1979), the pricing problem is solved by a backward induction along the tree. In particular, for American options, at each node of the lattice, one has to compare the early exercise value with the continuation value. Nevertheless, when considering discrete dividends, the tree is no longer reconnecting in correspondence with ex-dividend nodes. Hence, the method becomes computationally very expensive.

The authors analyze binomial algorithms for the evaluation of options written on stocks which pay discrete dividends. In Section 2.3 (Binomial Models), the authors consider non-recombining binomial trees, hybrid binomial algorithms for both European and American call options (based on the BS formula for the evaluation of the option after the ex-dividend date and up to maturity), a binomial method which implements the efficient continuous approximation proposed in Bos and Vandermark (2002), and a binomial method based on an interpolation idea of Vellekoop and Nieuwenhuis (2006), in which the recombining feature is maintained. The authors also apply the model based on the interpolation procedure to the case of multiple dividends. Some numerical experiments are performed and the results are compared in Section 2.4 (Numerical Applications).

The first purpose of this paper is to study methods which yield consistent prices for options with discrete dividends. The second aim is to analyze some methodologies to extract information on volatilities and dividends from observable option prices.

In the case of European options, implied dividends can be derived by exploiting the well-known put-call parity relationship. Such a relationship, which can be proved by arbitrage arguments, is independent of a pricing model, and therefore, it can be used to test the market efficiency. Harvey and Whaley (1992) and Brooks (1994) used the parity to predict dividends on standard and poor (S&P) index and single stocks. This

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1 Hence HHL.

2 See also Haug (2007) for a review and references therein.
Extraction information from American option prices

Technique is straightforward. Nevertheless, it does not apply to American options. As an alternative, the numerical inversion of pricing methods discussed in Section 2 (Option Pricing with Discrete Dividends) can be used to derive implied dividends from market data. By equating the observed market prices and the corresponding theoretical option values, one has to solve a problem in two unknowns: the implied volatility and the implied dividend. As a solution, the authors propose to fix the volatility by using a model-free implied volatility and to solve for the dividend. The main interest of this paper is in American options on dividend paying stocks, for which very few empirical contributions have been published. In particular, when considering the Italian market, to the authors’ best knowledge, a similar study has not been carried out.

In Section 3, the authors derive implied volatilities, and Section 4 deals with implied dividends. The authors apply different procedures to obtain implied volatilities and dividends of listed stocks of the Italian derivatives market (IDEM). Section 5 concludes and suggests further researches.

Option Pricing with Discrete Dividends

The authors consider options written on assets (typically stocks) which pay discrete dividends. Usually, the derivative pricing theory assumes that stocks pay known dividends, both in size and timing. Even if this might not be realistic, in this section, the authors assume that they know both the amount of dividends and times of payment. Let the dividend \( D \) be a pure cash amount to be paid at a specified time \( t_D \). Empirically, one observes that at the ex-dividend date, the stock price drops by an amount proportional to the size of the dividend. In order to exclude arbitrage opportunities, in a frictionless market, the jump in the price should be equal to the net dividend. Dividend payments during the option lifetime imply lower call premia and higher put premia.

The underlying price dynamics depends on the timing of the dividend payment. In a continuous time setting, the authors assume that the stock price process \( S \) satisfies the following stochastic differential Equation (1):

\[
dS_t = rS_t dt + \sigma S_t dW_t \quad t \neq t_D
\]

\[
S_{t_D} = S_{t_D} - D
\]

where \( r \) is the risk-free continuously-compound interest rate, \( \sigma > 0 \) is the volatility, \((W_t)_{t \geq 0}\) is a standard Wiener process, and \( S_{t_D} \) and \( S_{t_D} \) denote the stock price levels right before and after the jump respectively at time \( t_D \).

Due to this discontinuity, the solution to Equation (1) is no longer lognormal, but in the form of Equation (2):

\[
S_t = S_0 e^{(r - \sigma^2/2) t + \sigma W_t} - D e^{(r - \sigma^2/2)(t - t_D) + \sigma W_{t-D}} I_{(t \geq t_D)}
\]

where \( I_A \) denotes the indicator function of \( A \).

Valuation of options on stocks which pay discrete dividends is a rather hard problem which has received a lot of attentions in the financial literature. In next sub-sections, the authors will consider some pricing models proposed in the literature for European and American options.

European Options

The simplest evaluation approach to price options on stocks consists of adjusting the well-known generalized Black-Scholes-Merton (BSM) formula by replacing the stock price \( S_0 \) with the stock price minus the present value of the dividend (see Equation (3)).

\[
S_0 - D e^{-rt_D}
\]
This is called the escrowed dividend model and typically implies too low absolute price volatility $\sigma S$ in the period before $t_D$. Since the initial stock price is lowered under the true observed price, this approach typically undervalues call options. Such a mispricing is larger, the later the dividend is paid during the option’s lifetime. In order to overcome such a problem, several corrections of volatility have been suggested in literature. The following adjustment is popular among practitioners:

$$\sigma_2 = \frac{\sigma S}{S - De^{-r t_D}}$$ (4)

Such an adjustment increases the volatility relative to the basic escrow-divided process, but yields too high volatility, if the dividend is paid early in the lifetime of the contract. In this case, the approach typically leads to an overpricing of call options and can allow for arbitrage opportunities.

Another volatility adjustment takes into account the timing of the dividend (E. G. Haug & J. Haug, 1998; Beneder & Vorst, 2002). The idea behind the approximation is to leave the volatility unchanged before the dividend payment and to apply the volatility $\sigma_2$ after the dividend payment. This method performs particularly poor in the presence of multiple dividends. A more sophisticated volatility adjustment to be used in combination with the escrowed dividend model is proposed by Bos et al. (2003). This approximation is quite accurate in most cases. Nevertheless, for very large dividends or in the case of multiple dividends, the method can lead to a significant mispricing. A slightly different implementation (Bos & Vandermark, 2002) adjusts the stock price and the strike at the same time. The dividends are divided into two parts, “near” and “far”, which are used for the adjustments to the spot and the strike price respectively. This approach seems to work better than the approximation mentioned above. De Matos et al. (2006) arbitrarily derived accurate lower and upper bounds for the value of European options on stocks paying a discrete dividend.

A different approach is proposed by Haug et al. (2003). The authors derive an integral representation formula for the fair price of a European call option on a dividend paying stock. The basic idea is that after the dividend payment, option pricing reduces to a simple BS formula for a non-dividend paying stock. Before $t_D$, one considers the discounted expected value of the BS formula adjusted for the dividend payment. In the geometric setup of Brownian motion, the HHL formula is given as follows:

$$C_{HHL}(S_0, D, t_D) = e^{-r t_D} \int_0^\infty c_{e} (S_x - D, t_D) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$ (5)

where:

$$d = \frac{\ln (D/S_0) - (r - \sigma^2/2) t_D}{\sigma \sqrt{t_D}}$$ (6)

$$S_x = S_0 e^{(r - \sigma^2/2) t_D + \sigma \sqrt{t_D} x}$$ (7)

and $c_{e}(S_x - D, t_D)$ is simply the BS formula with time to maturity $T - t_D$. The integral Representation (5) can be considered as the exact solution to the problem of valuing a European call option written on stock with a discrete and known dividend. Let the authors observe that the well-known put-call parity relationship allows one to calculate immediately the theoretical price of a European put option with a discrete dividend. Formula (5) can be extended to the multiple dividend case, but at a much higher computational expense.

American-Style Options

Most traded options on stocks are of American style. The effect of dividends on American option prices is different than for European options: Dividend payments have an impact not only on the option price, but also on the optimal exercise strategy. As is well-known, it is never optimal to exercise an American call option on non-dividend paying stocks before maturity. As a result, the American call has the same value as its European
counterpart. In the presence of dividends, it may be optimal to exercise the American call and put before maturity. In general, early exercise is optimal when it leads to an alternative income stream, i.e., dividends from the stock in case of a call and interests on cash in case of a put option. In the case of discrete dividends, the call option may be optimally and instantaneously exercised prior to the ex-dividend date $t_D$. Note that after the dividend date $t_D$, the option is a standard European call, which can be priced using the BS formula\(^3\). While for an American put, it may be optimal to exercise at every time until maturity.

The first approximation to the value of an American call on a dividend paying stock has been suggested by Black (1975). This is basically the escrowed dividend method, where the stock price in the BS formula is replaced by the stock price minus the present value of the dividend. In order to account for early exercise, one also computes an option value just before the dividend payment, without subtracting the dividend. The maximum of these values is considered as the theoretical value of the option.

A model which is often used and implemented in a lot of commercial software was proposed, simplified, and corrected by Roll (1977), Geske (1979b, 1981), and Whaley (1981) respectively. Henceforth, the authors refer to the Roll-Geske-Whaley (RGW) model. These scholars construct a portfolio of three European call options, which replicates an American call and accounts for the possibility of early exercise right before the ex-dividend date. The portfolio consists of two long positions with exercise prices $X$ and $S^* + D$ and maturities $T$ and $t_D$ respectively. The third option is a short call on the first of the two long calls with exercise price $S^* + D - X$ and maturity $t_D$. The stock price $S^*$ makes the holder of the option indifferent between early exercise at time $t_D$ and continuing with the option. Formally, the authors have:

$$C(S^*, T - t_D, X) = S^* + D - X$$  \hspace{0.5cm} (8)

Equation (8) can be solved, if the ex-dividend date is known. The two long positions follow from the BS formula, while Geske (1979a) provided an analytical solution for the compound option.

For more than 20 years, the RGW model has been considered as a brilliant solution in closed form to the problem of evaluating American call options on equities that pay a discrete dividend. Although some authoritative scholars still consider the RGW formula as the exact solution, the model does not yield good results in many cases of practical interest. Moreover, it is possible to find situations in which the use of the RGW formula allows for arbitrage. Whaley (2006), in his recent monograph, presented an example showing the limitations of the RGW model. The data reported by Whaley for the American call option in the presence of a discrete dividend are: $S_0 = X = 50$, $T = 90$ days, $\sigma = 0.36$, $D = 2$, $t_D = 75$ days, and $r = 0.05$. With these data, the critical price is $S^* = 49.060$ and the value of the American call computed in the RGW model is $C_{RGW} = 3.445$, while the correct value is $C_{HHL} = 3.57041$. The BS price of a European call option with the same features but with maturity the day before ex-dividend date ($t = 74$ days) is 3.47193 and therefore higher than the price obtained in the RGW model.

Haug et al. (2003) derived an integral representation formula for the fair price of American call options in the presence of a single dividend $D$ paid at time $t_D$. Since early exercise is only optimal instantaneously prior to the ex-dividend date, in order to obtain the exact solution for an American call option with a discrete dividend, one can merely replace Relation (5) with:

$$C_{HHL}(S_0, D, t_D) = e^{-r t_D} \int_0^{S_0} \max \{S_x - X, c_e(S_x - D, t_D)\} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx,$$  \hspace{0.5cm} (9)

For American-style put options, early exercise may be optimal at any time prior to expiration, even in the

\(^3\) The authors have implemented this idea in a hybrid BS-binomial model.
absence of dividends. So, in this case, one is generally forced to a numerical solution, such as lattice approaches which are discussed in next section.

Binomial Models

The evaluation of options using binomial methods is particularly easy to implement and efficient at standard conditions, but it becomes difficult to manage in the case in which the underlying asset pays one or more discrete dividends, due to the fact that the number of nodes grows considerably and entails huge calculations. In the absence of dividends or when dividends are assumed proportional to the stock price, the binomial tree reconnects in the sense that the price after an up-down movement coincides with the price after a down-up movement. As a result, the number of nodes at each step grows linearly.

If a dividend of amount \( D \) is paid during the life of the option at each node after the ex-dividend date, a new binomial tree has to be considered, with the result that the total number of nodes increases to the point that it is practically impossible to consider trees with an adequate number of stages. To avoid such a drawback, it is often assumed that the underlying dynamics is characterized by a dividend yield which is discrete and proportional to the stock price. Formally:

\[
\begin{align*}
    S_0 u^j d^{i-j} & \quad \text{if } j = 0, 1, \ldots, i < n_D \\
    S_0 (1-q) u^j d^{i-j} & \quad \text{if } j = 0, 1, \ldots, i \geq n_D
\end{align*}
\]

where \( i = 0, 1, \ldots, n, n_D \) defines the ex-dividend time-step. The first law in Equation (10) applies to the period preceding the ex-dividend date and the second applies the period after the dividend date. \( S_0 \) denotes the initial price, \( q \) is the dividend yield, and \( u \) and \( d \) are the upward and downward coefficients respectively which are defined by:

\[
u = e^{\sigma \sqrt{T/n}}, \quad d = 1/u
\]

The hypothesis of a proportional dividend yield can be accepted as an approximation of dividends paid in the long term, but it is not acceptable in a short period of time during which the stock pays a dividend in cash and its amount is often known in advance or estimated with appropriate accuracy.

If the underlying asset is assumed to pay a discrete dividend \( D \) at time \( t_D < T \), the dividend amount is subtracted at all nodes at step \( n_D \). Due to this discrete shift in the tree, as already noticed, the lattice is no longer recombining beyond time \( t_D \) and the binomial method becomes computationally expensive, since at each node at time \( t_D \), a separate binomial tree has to be evaluated until maturity (see Figure 1). Also in the
presence of multiple dividends, this approach remains theoretically sound, but it is computationally intensive with a few dividend dates, whereas it becomes infeasible with more dividends.

Schroder (1988) described how to implement discrete dividends in a recombining tree. The approach is based on the escrowed dividend process idea, but the method leads to significant pricing errors.

The problem of the enormous growth in the number of nodes that occurs in such a case can be simplified, if it is assumed that the price has a stochastic component \( \hat{S} \) given by:

\[
\hat{S} = \begin{cases} 
S - De^{-r(iD - i\Delta t)} & \text{if } i \leq n_D \\
S & \text{if } i > n_D
\end{cases}
\]  

(12)

and a deterministic component represented by the discounted value of the dividend or of dividends that will be paid in the future. Note that the stochastic component gives rise to a reconnecting tree. Moreover, one can build a new tree (which is still reconnecting) by adding the present value of future dividends to the price of the stochastic component in correspondence with each node. Hence, the tree reconnects and the number of nodes in each period \( i \) is equal to that at step \( i + 1 \).

The recombining technique described above can be improved through a procedure which preserves the structure of the tree until the ex-dividend time and will force the recombination after the dividend payment. For example, one can force the binomial tree to recombine as extreme nodes by taking immediately after the payment of a dividend:

\[
S_{nD+1,0} = (S_{nD,0} - D)d, \quad S_{nD,nD} = (S_{nD,nD} - D)u
\]  

(13)

and by calculating the arithmetic average of the values that are not recombining. This technique has the characteristic of being simple from the computational point of view, but will lead to mispricing.

Alternatively, one can use a technique that the authors call “stretch” which calculates the extreme nodes as in the previous case. In such a way, one forces the reconnection at the intermediate nodes by choosing the upward coefficients as follows:

\[
u(i, j) = e^{\lambda \sqrt{\Delta t/n}}
\]  

(14)

where \( \lambda \) is chosen, in order to make the prices equal after an up-and-down movement. This technique requires a greater amount of computations, since at each stage, both the coefficients and the corresponding probabilities change, and as the technique is based on the arithmetic average, it entails pricing errors.

A method which performs very efficiently and can be applied to both European and American call and put options is a binomial method which maintains the recombining feature and is based on an interpolation idea proposed by Vellekoop and Nieuwenhuis (2006).

For an American option, the method can be described as follows: a standard binomial tree is constructed without considering the payment of the dividend (with \( S_{ij} = S_0u^i d^{j-i} \), \( u = e^{\sigma \sqrt{\Delta t/n}} \), and \( d = 1/u \)), then it is evaluated by backward induction from maturity until the dividend payment; at the node corresponding to an ex-dividend date (at step \( n_D \)), the continuation value \( V_{nD} \) is approximated by the interpolation of suitable option values; then it continues backward along the tree. The authors prove the convergence of the method to the continuous time model.

In order to approximate the continuation value \( V_{nD} \), the authors have applied the following linear interpolation:\footnote{One can also use other interpolation schemes, for example splines. Moreover, the interpolation idea can be applied to other numerical schemes, such as finite difference methods.}
for \( j = 0, 1, ..., n, S_{D,k} \leq S_{D,j} \leq S_{D,k+1} \). The authors have implemented a very efficient method which combines this interpolation procedure and the binomial algorithm for the evaluation of American options proposed by Basso, Nardon, and Pianca (2004). The algorithm exploits two devices: (1) The symmetry of the tree, which implies that all the asset prices defined in the lattice at any stage belong to the set \( \{S_{0}t^{j} : j = -n, -n+1, ..., 0, ..., n-1, n\} \); and (2) The fact that in the nodes of the early exercise region, the option value equaling to the intrinsic value, does not need to be recomputed when exploring the tree backwards.

The method can also be easily implemented in the case of multiple dividends (which are not necessarily of the same amount).

Negative prices may arise in some cases, in particular when dividends are high. As a solution, one can impose an absorbing barrier at zero, when the dividend is higher than the underlying price (Dividends are not fully paid due to limited liability).

**Numerical Applications**

In this section, the authors report the results of some numerical examples related to European calls and American calls and puts. In Table 1, the authors compare the prices provided by the HHL exact formula for the European call, with those obtained with the 2,000-step non-recombining binomial method and the binomial method based on Interpolation (15). The second method is accurate and very fast: For a European call, the non-recombining binomial method requires a couple of seconds, while the calculations with a 2000-step interpolated binomial method are immediate. Outcomes of the recombining methods based on the approximation proposed by Bos and Vandermark (2002), the average value, and the stretching procedure are also reported. In particular, the last two methods provide results that entail large errors.

**Table 1**

**European Calls With Dividend** \( D = 5 \) \((S_{0} = 100, T = 1, r = 0.05, \text{and} \sigma = 0.2)\)

<table>
<thead>
<tr>
<th>( t_{D} )</th>
<th>( X )</th>
<th>HHL</th>
<th>Non-rec. bin. ((n = 2,000))</th>
<th>Interp. bin. ((n = 2,000))</th>
<th>Bos-Vand.</th>
<th>Av. value</th>
<th>Stretch</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>7.6444</td>
<td>7.6446</td>
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<td>7.6456</td>
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<tr>
<td></td>
<td>130</td>
<td>0.9997</td>
<td>0.9994</td>
<td>1.0000</td>
<td>0.9956</td>
<td>1.1349</td>
<td>1.0540</td>
</tr>
<tr>
<td>0.5</td>
<td>70</td>
<td>28.8120</td>
<td>28.8120</td>
<td>28.8121</td>
<td>28.8192</td>
<td>28.9291</td>
<td>28.7682</td>
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<tr>
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<td>100</td>
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<td>7.7742</td>
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<td>1.0497</td>
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<td>1.0455</td>
<td>1.1371</td>
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</table>

**Table 2** shows the results related to American options. For the American call, the authors have compared the results yielded by HHL integral representation, a 5,000-step non-recombining hybrid binomial method (in which the authors have used the BS formula, in order to calculate the continuation value at the ex-dividend node), and the 10,000-step binomial method based on the interpolation Procedure (15). For the American put, the authors have compared the 2,000-step non-recombining binomial method, the 10,000-step interpolated binomial method, and the approximation proposed by Bos and Vandermark (2002). This last method provides higher pricing errors for American put options, as in some cases, it overprices the true option value, while the
The interpolated binomial approach turns out to be accurate.

Table 2
American Call and Put Options With Dividend $D = 5$ ($S_0 = 100$, $T = 1$, $r = 0.05$, and $\sigma = 0.2$)

<table>
<thead>
<tr>
<th>$t_D$</th>
<th>$X$</th>
<th>HHL</th>
<th>American call</th>
<th>Interp. bin. ($n = 10,000$)</th>
<th>Non-rec. bin. ($n = 2,000$)</th>
<th>American put</th>
<th>Bos-Vand.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>non-rec. hyb. bin. ($n = 5,000$)</td>
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<td>32.6407</td>
<td>32.6411</td>
<td>0.3070</td>
<td>0.3071</td>
<td>0.2901</td>
</tr>
<tr>
<td></td>
<td>130</td>
<td>1.1799</td>
<td>1.1764</td>
<td>1.1767</td>
<td>30.8512</td>
<td>30.8515</td>
<td>30.0012</td>
</tr>
</tbody>
</table>

The authors also applied the model based on the interpolation procedure to the case of multiple dividends. Table 3 shows the results for the European call with multiple dividends. The authors have compared the non-reconnecting binomial method with $n = 2,000$ steps (only for the case with one and two dividends) and the interpolated binomial method with $n = 10,000$ steps (the results are consistent with those obtained by HHL). Different maturities (in years) have been considered, with dividends paid every year at $t_D = 0.5, 1.5$, and so on. In Table 4, the authors have reported the results for the at-the-money American options written on an asset with multiple dividends. Also in the case of more dividends, the interpolated binomial method proved to be very efficient.

Table 3
European Call Option With Multiple Dividends $D = 5$ Paid at Time $t_D \in \{0.5, 1.5, 2.5, 3.5, 4.5, 5.5\}$ ($S_0 = 100, X = 100$, $r = 0.05$, and $\sigma = 0.2$)

<table>
<thead>
<tr>
<th>$T$</th>
<th>Non-rec. bin. ($n = 2,000$)</th>
<th>Interp. bin. ($n = 10,000$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.7742</td>
<td>7.7741</td>
</tr>
<tr>
<td>2</td>
<td>10.7119</td>
<td>10.7122</td>
</tr>
<tr>
<td>3</td>
<td>12.7885</td>
<td>12.7885</td>
</tr>
<tr>
<td>4</td>
<td>14.4005</td>
<td>14.4005</td>
</tr>
<tr>
<td>5</td>
<td>15.7076</td>
<td>15.7076</td>
</tr>
<tr>
<td>6</td>
<td>16.7943</td>
<td>16.7943</td>
</tr>
</tbody>
</table>

Table 4
American Options With Multiple Dividends in the Interpolated 10,000-Step Binomial Method (With Parameters $S_0 = 100, X = 100$, $r = 0.05$, and $\sigma = 0.2$)

<table>
<thead>
<tr>
<th>$T$</th>
<th>American call</th>
<th>American put</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.1439</td>
<td>8.4412</td>
</tr>
<tr>
<td>2</td>
<td>11.2792</td>
<td>11.5904</td>
</tr>
<tr>
<td>3</td>
<td>13.3994</td>
<td>13.7399</td>
</tr>
<tr>
<td>4</td>
<td>15.0169</td>
<td>15.3834</td>
</tr>
<tr>
<td>5</td>
<td>16.3136</td>
<td>16.7035</td>
</tr>
<tr>
<td>6</td>
<td>17.3824</td>
<td>17.7938</td>
</tr>
</tbody>
</table>

Note. A cash dividend $D = 5$ is paid at the date $t_D \in \{0.5, 1.5, 2.5, 3.5, 4.5, 5.5\}$ for different maturities.
Implied Volatilities When the Underlying Asset Pays Discrete Dividends

In this section, the authors derive implied volatilities of listed equity options of the IDEM. The authors consider options of American type. For this aim, with reference to American call options, HHL Formula (9) can be numerically inverted, in order to compute the implied volatilities from the prices of American call options written on stocks which pay cash dividends. It is worth noting that the computation and numerical inversion of Formula (9) entail some drawbacks concerning the approximation of the integral, in order to obtain accurate results. In particular, difficulties arise when considering dividends paid very near in the future or very close to the maturity of option. Truncation of the interval of integration has also to be chosen carefully. In the numerical experiments discussed below, the authors used both HHL method and interpolated binomial approach, in order to obtain prices and volatilities of American call options written on single dividend-paying stock. Whereas in the case of American put options and multiple dividends (This is the case for example of Stmicroelectronics (STM), which pays dividends quarterly), the authors used only the interpolated binomial method.

Usually, the derivative pricing theory assumes that stocks pay known dividends, both in size and timing. Such an assumption might be too strong in some cases. In the applications considered in this section, the authors know the amount of announced dividends and the ex-dividend date.

Let the authors also observe that dividend policies are not uniform for all corporations. With reference to common stocks in the Financial Times and Stock Exchange (FTSE) MIB index for the year 2010 (hence when considering the dividend which will be paid in 2011), there are some companies that pay no dividends at all, some other companies pay a dividend, or the remainder of the dividends already paid in the end of 2010: For instance, Eni paid a dividend of 0.50 euro in September 2010 as an anticipation of the dividends for the year 2010 and would pay a dividend of 0.50 euro in June 2011. A few stocks pay quarterly dividends, but most stocks pay only once or twice a year. Dividends can be paid in cash: Normally in euro, but sometimes dividends are also in dollars (such as for STM and Tenaris). Hence, one has to evaluate currency risk. Alternatively, dividends are paid issuing new shares of stock (in a number, which is proportional to the shares already held), or could be a mixture of stocks and cash. For example, on April 18, Parmalat pays a dividend of 0.036 and will distribute a new share of stock every 20 shares already owned. Taking into account in the model all such different dividend policies is a tough task.

As a first example, let the authors consider the American call and put options written on ENEL stock, which pays parts of the annual dividend in June. At the trading date of April 1, 2011, the underlying price is $S_t = 4.48$. A dividend $D = 0.18$ is announced to be paid on June 20 ($t_D - t = 0.2192$). Note that a dividend of 0.1 has already been paid in November 2010. The expiration of options is on the September 16. The risk-free interest rate is assumed to be $r = 0.0125$ (approximately the 3-month EURIBOR). Implied volatilities are obtained in the interpolated binomial method with 2,000 steps. Figures 2 and 3 show the results and the typical smile effect of implied volatilities respectively from call and put ENEL options.

The authors have considered bid-ask prices and the bid-ask average prices.

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5 The FTSE MIB (Milano Italia Borsa) (the S&P/MIB prior to June 2009) is the benchmark stock market index for the Borsa Italiana, the Italian national stock exchange, which superseded the MIB-30 in September 2004.
6 Ente Nazionale per l’energia Elettrica (ENEL) is an Italian electric utility company, the second-largest in Europe by market capitalization.
7 Euro Interbank Offered Rate.
Figure 2. Implied volatilities of American call options prices on ENEL stocks with maturity September 16, 2011 \((S_t = 4.48, D = 0.18, t = April 1, 2011, \text{ and } t_D = June 20, 2011)\).

Figure 3. Implied volatilities of American put options prices on ENEL stocks with maturity September 16, 2011 \((S_t = 4.48, D = 0.18, t = April 1, 2011, \text{ and } t_D = June 20, 2011)\).

Considering now American call and put options written on STM stock, which pays a dividend of $0.4 during the year, \(D = $0.1\) is thus paid quarterly (on the dates: \(t_1 = May 23; t_2 = August 22; t_3 = November 28, 2011; \text{ and } t_4 = February 20, 2012\)). Let the euro/dollar exchange rate be approximately 1.45 (even if this assumption is not realistic, in this example, the authors hold the exchange rate constant), then each dividend in euro is \(D = 0.069\). At the trading date April 1, 2011, the underlying price is \(S_t = 8.695\).

Figures 4 and 5 show the implied volatilities for the options with maturity June 17, 2011. The risk-free interest rate is assumed to be \(r = 0.01\) (approximately the 1-month EURIBOR).
Figure 4. Implied volatilities of American call options prices on STM stocks with maturity June 17, 2011 ($S_t = 8.695$, $D = $0.1, $t = 1$ April 2011, and $t_D = May 23, 2011$).

Figure 5. Implied volatilities of American put options prices on STM stocks with maturity June 17, 2011 ($S_t = 8.695$, $D = $0.1, $t = 1$ April 2011, and $t_D = May 23, 2011$).

Implied volatilities are obtained by numerically inverting the interpolated binomial method with 2,000 steps (Such a number of steps is in general sufficient to obtain accurate results for the volatility). In the computations, the authors have considered option prices calculated as an average between the bids and ask prices. Let the authors observe that, in some cases, it is not possible to determine the implied volatilities due to mispricing of options with respect to the theoretical model. For example, the authors may have bid prices for call options lower than the immediate exercise value $S_t - X$. The authors also found bid prices for put options which violate the condition$^8$:

$^8$ Inequality (16) holds in a frictionless market.
\[ P_t \geq \max(D e^{-r(T-t)} + X e^{-r(T-t)} - S_t, 0) \]  \hspace{1cm} (16)

This may happen in practice, because such prices may correspond to options that have not been traded for some time or the trading volume is low.

### Multiple Dividends

![Graph](image)

**Figure 6.** Implied volatilities of American call options prices on STM stocks with maturity September 16, 2011 \((S_t = 8.695, D = \$0.1, t = \text{April 2011}, t_1 = \text{May 23, 2011}, \text{and } t_2 = \text{August 22, 2011})\).

![Graph](image)

**Figure 7.** Implied volatilities of American put options prices on STM stocks with maturity September 16, 2011 \((S_t = 8.695, D = \$0.1, t = \text{April 1, 2011}, t_1 = \text{May 23, 2011}, \text{and } t_2 = \text{August 22, 2011})\).

With reference to the options written on STM, if the authors consider longer maturities, then the authors have to take into account multiple dividend payments. For example, if considering the options that expire on
September 16, 2011, the authors have two dividends at the dates $t_1$ and $t_2$, and for options with maturity December 16, 2011, the authors have three dividends at the dates $t_1$, $t_2$, and $t_3$. While for the options that expire in March 2012, the authors have four dividends. Figures 6 and 7 show the results for the maturity of September for call and put options respectively. Similar results are reported in Figures 8 and 9 for the maturity of December. The risk-free rates used in the calculations are the 3-month and 6-month EURIBOR. Implied volatilities explain the typical smile effect and become flatter for longer maturities.

**Figure 8.** Implied volatilities of American call options prices on STM stocks with maturity December 16, 2011 ($S_t = 8.695$, $D = 0.1$, $t = April 1, 2011$, $t_1 = May 23, 2011$, $t_2 = August 22, 2011$, and $t_3 = November 28, 2011$).

**Figure 9.** Implied volatilities of American put options prices on STM stocks with maturity December 16, 2011 ($S_t = 8.695$, $D = 0.1$, $t = April 1, 2011$, $t_1 = May 23, 2011$, $t_2 = August 22, 2011$, and $t_3 = November 28, 2011$).

**Dividends Forecasts**
Making forecasts on dividends paid on stocks is relevant for investors, as dividends can have an important impact on the total return of their investments. In this section, the authors analyze some methodologies to extract information on dividend uncertainty from observable option prices. A fundamental aspect when valuing index and stock options correctly is the knowledge of the amount and the timing of the cash dividends that will be paid before the option expiration.

In absence of arbitrage, put-call parity relationship must occur between the price of European call and put options. Such a relation is independent of a pricing model, and therefore, it can be used to test the market efficiency. First, it should be noted that for each pair of European call and put options with the same strike and maturity, implied dividends can be computed using a modified version of the well-known parity relationship. This technique is straightforward and does not depend on the assumptions about the dynamics of the underlying price. Nevertheless, its use is limited to European options.

As an alternative, the numerical inversion of pricing methods, such as an interpolated binomial approach analyzed in Section 2, can be used to derive implied dividends from market data. By equating the observed market prices and the corresponding theoretical option values, one has to solve a problem in two unknowns: the implied volatility and the implied dividend. The authors propose to fix the volatility by using a model-free implied volatility. In particular, in order to compute implied volatilities, one can apply a procedure similar to VIX\(^9\), based on a set of at-the-money and out-of-the-money call and put options in the two nearest-term expiration months. The authors apply such a procedure to obtain implied dividends from options on stocks listed in the IDEM.

First, the authors consider European options on indexes and information on cash dividends using put-call parity. Then, the authors focus on American options written on a single stock.

Predictions of Cash Dividends Using Put-Call Parity

Using no arbitrage arguments, it is easy to prove the put-call parity relation:

\[ c_0 - p_0 = S_0 - De^{-rT} - Xe^{-rT} \]  

(17)

where \( p_0 \) and \( c_0 \) are the current premia of European put and call options respectively. In the case of multiple dividends, \( De^{-rT} \) is replaced by the sum of the present values of the future dividends.

Theoretically, one can obtain the implied dividend \( D \) in Equation (17) from any pair of option premia with the same strike and maturity. Nevertheless, implementing the put-call relationship faces several theoretical and practical problems that can be conveniently mitigated. The practical use of the put-call relationship requires an estimation of the risk-free rate. Option literature employs either LIBOR\(^10\) or treasure \( T \)-note rate. \( T \)-notes are the safest traded investments, but only governments can borrow at this rate. On the other hand, LIBOR rate can be subjected to credit risk. Therefore, it is not totally clear which interest rate one can use in order to implement the model. Another problem concerns the bid-ask quote convention for trading stocks and options. To mitigate the noise introduced by the bid-ask spread, one can use the quote midpoints. A further drawback is the necessity to transform the index points into dividends paid on the single stocks. Overcoming all these issues may be a difficult task.

Although most index options are of European type, options on single stock are normally of American type.

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\(^9\) VIX is a trademarked ticker symbol for the Chicago Board Options Exchange (CBOE) market volatility index, a popular measure of the implied volatility of S&P 500 index options.

\(^10\) The London interbank offered rate is the average interest rate estimated by leading banks in London that they would be charged if borrowing from other banks.
As is well-known, Parity (17) does not hold for American options, due to the possibility of early exercise, which cannot be completely ruled out when the strategies are established. If the options are of American style, the following double inequality holds:

\[
S_0 - D e^{-rT} - X \leq C_0 - P_0 \leq S_0 - X e^{-rT} \tag{18}
\]

where \(C_0\) and \(P_0\) are the current prices of an American call and put options respectively. Note that the first inequality in (18) can be used to obtain a lower bound for the expected dividend:

\[
D e^{-rT} \geq S_0 - X + P_0 - C_0 \tag{19}
\]

with \(D \geq 0\).

**Implied Dividends**

In this section, the aim is to derive implied dividends from market information about option prices. The authors assume that the time at which dividends are paid is announced (or can be estimated), but their amount is unknown.

If dividends are declared, as seen in previous section, one can obtain implied volatilities from option prices. In the case of unknown dividends, Formula (9) or the numerical procedure described in Section 3 can be used to derive implied dividends from market data. By equating the observed market prices and the corresponding theoretical option values, the authors have to solve an equation in two unknowns: the implied volatility and the implied dividend. The authors then suggest to fix the volatility by using a model-free implied volatility (see for instance, Demeterfi, Derman, Kamal, & Zou, 1999; Jiang & Tian, 2007) \(\hat{\sigma}\) obtained with a procedure similar to the one applied by CBOE to compute VIX. Such a volatility index is based on a set of at-the-money and out-of-the-money call and put options in the two nearest-term expiration months and provides a measure of the expected stock market volatility over the next 30 calendar days. Calibration can be an alternative solution to the problem.

**Empirical Experiments**

The authors introduce a short empirical study based on options written on stocks listed in the IDEM. A set of option prices is observed, and the authors assume that such prices contain all relevant information concerning the underlying assets. At the same time in each section, the negotiable expirations are the four quarterly expirations (March, June, September, and December), the two nearest monthly expirations and the four 6-month maturities (June and December) of the two years following the current year, for a total of 10 expirations. New issued options are quoted on the first trading day following the expiration. The expiration day is the third Friday of the month, in which the option expires. Ex-dividend dates are usually the third Monday of the month, in which a dividend payment is declared. Dividend settlement is normally three days after the ex-dividend date.

The empirical analysis relates to quotations at the trading date April 1, 2011. Similar results have been obtained considering other trading dates. First, the authors have considered American options written on Eni stock with maturity June 17, 2011. Model-free implied volatility \(\hat{\sigma}\) has been computed using a set of at-the-money and out-of-the-money options expiring in April and May.

As a second experiment, the authors have computed the implied dividend from American options written on Generali stock, traded on April 21 with maturity June 17, 2011. Dividends will be paid on May 23. Model-free implied volatility \(\hat{\sigma}\) has been computed using a set of at-the-money and out-of-the-money options expiring in May and June.
Figures 10 and 11 show the dividends obtained using the interpolated binomial method (with 5,000 steps), based on a set of put option prices. It is interesting to observe the behaviors of implied dividends which show a smile effect or a little skewed shape. Implied dividends can be compared with the declared dividends, which in the case of ENI are 0.5 (A dividend of 0.5 has been paid in September 2010) with 0.45 for Generali. In the examples reported here, implied dividends are able to capture the announced ones.

**Figure 10.** Implied dividends of American put options on ENI stocks ($S_t = 17.42$, $t = April 1, 2011$, $r = 0.0125$, $\sigma = 0.2126$, $t_0 = May 23$, and $T = June 17$).

**Figure 11.** Implied dividends of American put options on Generali stocks ($S_t = 15.43$, $t = April 21, 2011$, $r = 0.012$, $\sigma = 0.2585$, $t_0 = May 23$, and $T = June 17$).

**Concluding Remarks**

In this contribution, the authors studied American options on stocks which pay discrete dividends. Making forecasts on dividend paid on stocks is an important skill for investors. The authors obtained implied volatilities and implied dividends considering the prices of options traded on the IDEM, using different numerical methodologies. In particular, when both the volatility and the dividend are unknown, the authors suggest to fix
the volatility using a model-free estimate and to obtain the implied dividends by the numerical inversion of pricing methods. For further researches, pricing models in the presence of discrete dividends can also be extended in order to consider stochastic volatility, jumps, and stochastic interest rates, non-standard payoffs: Exotic options trade in over-the-counter (OTC) equity markets and are also embedded in warrants and other derivatives. The approaches above described for options with cash dividends are interesting, in which they could also be used to evaluate real options (e.g., real investment opportunities), when the underlying offers known discrete payouts.

References