UFIFT-HEP-08-14

The Flavor Group $\Delta(6n^2)$

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Abstract

Many non-Abelian finite subgroups of $SU(3)$ have been used to explain the flavor structure of the Standard Model. In order to systematize and classify successful models, a detailed knowledge of their mathematical structure is necessary. In this paper we shall therefore look closely at the series of finite non-Abelian groups known as $\Delta(6n^2)$, its smallest members being $\mathcal{S}_3$ $(n = 1)$ and $\mathcal{S}_4$ $(n = 2)$. For arbitrary n, we determine the conjugacy classes, the irreducible representations, the Kronecker products as well as the Clebsch-Gordan coefficients.

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1 Introduction

The experimental observation that neutrinos have non-zero mass has opened up a window to physics beyond the Standard Model. Advances in the setup of several neutrino experiments furthermore revealed that the mixing in the lepton sector features two large and one small angle. Its numerical values lead to a Maki-Nakagawa-Sakata-Pontecorvo (MNSP) matrix which is intriguingly close to the so-called tri-bimaximal mixing pattern [1]

$$\mathcal{U}_{\text{tri-bi}} \;=\; \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \tag{1.1}$$

The most promising explanation for this remarkable fact seems to be the existence of an underlying discrete symmetry [2–12]. Having chosen a preferred non-Abelian finite symmetry group [13], the three generations of fermions as well as the (extended) Higgs sector particles are assigned to irreducible representations of the group. The theory can then be formulated by writing down all couplings which are allowed by the symmetry group. A good understanding of the group structure is therefore an essential first step in our endeavor to explain the physics of flavor.

The mathematical details of the groups of type $\Delta(3n^2)$ were worked out in Ref. [14]. It is the purpose of this article to give an analogous account for the groups of type $\Delta(6n^2)$. These two series of groups are particularly interesting since their smallest members $\Delta(12) = \mathcal{A}_4$ [3] and $\Delta(27)$ [5], as well as $\Delta(6) = \mathcal{S}_3$ [2] and $\Delta(24) = \mathcal{S}_4$ [4] have been applied successfully in numerous models of flavor. Discussing these finite subgroups of $SU(3)$ for arbitrary values of n might be helpful in constructing more constrained models with larger symmetry groups. We also hope that our general presentation will serve as a tool for a systematic investigation of these flavor groups.

In Section 2, we first determine the conjugacy classes of the groups $\Delta(6n^2)$. All irreducible representations are obtained from the six-dimensional induced representations in Section 3. A possible choice for labeling the six-dimensional irreps is shown in Appendix A. The Kronecker products, presented in Section 4, can then be calculated from the character table. Appendix B details two ways of obtaining them, including the explicit construction of the Clebsch-Gordan coefficients. We conclude in Section 5.

2 The Structure of $\Delta(6n^2)$

The group $\Delta(6n^2)$ is a non-Abelian finite subgroup of $SU(3)$ of order $6n^2$. It is isomorphic to the semidirect product of the $\mathcal{S}_3$, the smallest non-Abelian finite group, with $(\mathcal{Z}_n \times \mathcal{Z}_n)$ [15],

$$\Delta(6n^2) \;\sim\; (\mathcal{Z}_n \times \mathcal{Z}_n) \rtimes \mathcal{S}_3 \;.$$

With the above in mind we now give the presentation of $\Delta(6n^2)$:

$$a^3 \;=\; b^2 \;=\; (ab)^2 \;=\; c^n \;=\; d^n \;=\; 1, \tag{2.1}$$

$$cd \;=\; dc, \tag{2.2}$$

$$\begin{aligned} aca^{-1} &= c^{-1}d^{-1}, & ada^{-1} &= c, \\ bcb^{-1} &= d^{-1}, & bdb^{-1} &= c^{-1}. \end{aligned} \tag{2.3}$$

The elements a and b are the generators of the S_3 while c and d generate $(\mathcal{Z}_n \times \mathcal{Z}_n)$. The last two lines are consequences of the semidirect product.

Note that when given the generators of the two groups involved in a semidirect product it is usually possible to obtain completely different sets of consistent group conjugations. Each separate set correspond to a possible distinct group.

Finally we should mention that in general any element of the group, which we will make use of occasionally, can be written in the form

$$g \;=\; a^\alpha b^\beta c^\gamma d^\delta \;, \tag{2.4}$$

where α, β, γ, δ are integers.

2.1 Conjugacy Classes

Our next step is to determine the conjugacy classes of the group. Keeping in mind that the first class is always the identity class, $1C_1(e) = \{e\}$. To obtain the remaining classes we first need some classification principle, the following set of conjugations will give us such a principle.

$$\begin{array}{rclrcl} cac^{-1} &=& ac^{-1}d, & dad^{-1} &=& ac^{-1}d^{-2}\ , \\ cbc^{-1} &=& bc^{-1}d^{-1}, & dbd^{-1} &=& bc^{-1}d^{-1}\ , \\ bab^{-1} &=& a^2, & aba^{-1} &=& a^2b\ . \end{array} \tag{2.5}$$

The above shows that conjugation by the Abelian elements c and d on a and b will not alter their powers. The last line, along with Eq. (2.1)-(2.3), is significant in that it places elements of solely one and two powers of a in the same type of classes, and those of single powers in b with elements of a^2b and ab in another.

We now begin to find the elements in each class, starting with the classes formed by elements with no powers of either a nor b.

- **Elements with neither a nor b**

  We begin with the classes whose elements are of the form $c^\rho d^\sigma$, where $\rho, \sigma = 0, 1, ..., n-1$. We obtain the elements of this class by action of

  $$g\,(c^\rho d^\sigma)\,g^{-1}, \tag{2.6}$$

  where $g = a^\alpha b^\beta$, $\alpha = 0, 1, 2$, and $\beta = 0, 1$. The result is the following class of six elements

  $$\left\{c^\rho d^\sigma, c^{\sigma-\rho} d^{-\rho}, c^{-\sigma} d^{\rho-\sigma}, c^{-\sigma} d^{-\rho}, c^{\sigma-\rho} d^\sigma, c^\rho d^{\rho-\sigma}\right\}\ , \tag{2.7}$$

  the second and third obtained from conjugation with a on the first term, and the last three obtained by conjugating the first term with b and then repeated application of a.

  There are possible values for ρ and σ in which the above class may collapse to a smaller set of elements, providing, in effect, new classes. This collapse happens when any of the following conditions are met

  $$\rho + \sigma \ = \ 0 \bmod(n)\ , \quad 2\rho - \sigma \ = \ 0 \bmod(n)\ , \quad \rho - 2\sigma \ = \ 0 \bmod(n)\ . \tag{2.8}$$

  The above can be obtained by equating the first term to the fourth, the fifth, and then the sixth term. Each yields the same class of three elements of the form

  $$\left\{c^\rho d^{-\rho}, c^{-2\rho} d^{-\rho}, c^\rho d^{2\rho}\right\}\ , \tag{2.9}$$

  where $\rho = 1, 2, ..., n-1$. Closer inspection shows that the above can further collapse when

  $$3\rho \ = \ 0 \bmod(n)\ , \tag{2.10}$$

  or in other words if $n = 3\mathbb{Z}$, leading to two possibilities.

  (*i*) $n \neq 3\,\mathbb{Z}$. Here we have only two types of classes, those with six elements and those with three.

  $$n-1 \ :\ 3C_1^{(\rho)} \ = \ \left\{c^\rho d^{-\rho}, c^{-2\rho} d^{-\rho}, c^\rho d^{2\rho}\right\}, \qquad \rho = 1, 2, ..., n-1, \tag{2.11}$$

  $$\frac{n^2-3n+2}{6} \ :\ 6C_1^{(\rho,\sigma)} \ = \ \left\{c^\rho d^\sigma, c^{\sigma-\rho} d^{-\rho}, c^{-\sigma} d^{\rho-\sigma}, c^{-\sigma} d^{-\rho}, c^{\sigma-\rho} d^\sigma, c^\rho d^{\rho-\sigma}\right\}, \tag{2.12}$$

  where $\rho, \sigma = 0, 1, ..., n-1$, but excluding possibilities given by Eq. (2.8). The convention used here is that the quantity left of the colon is the number of classes of the kind on the right of the colon.

(*ii*) $n = 3\,\mathbb{Z}$. Eq. (2.10) now has two solutions, thus lowering the number of possible classes with three elements.

$$\begin{aligned}
2 \;&:\; 1C_1^{(\nu)} &=& \left\{c^\nu d^{2\nu}\right\}, \quad \nu = \tfrac{n}{3}, \tfrac{2n}{3}, &(2.13)\\
n-3 \;&:\; 3C_1^{(\rho)} &=& \left\{c^\rho d^{-\rho}, c^{-2\rho} d^{-\rho}, c^\rho d^{2\rho}\right\}, \quad \rho \neq \tfrac{n}{3}, \tfrac{2n}{3}, &(2.14)\\
\frac{n^2-3n+6}{6} \;&:\; 6C_1^{(\rho,\sigma)} &=& \left\{c^\rho d^\sigma, c^{\sigma-\rho} d^{-\rho}, c^{-\sigma} d^{\rho-\sigma}, c^{-\sigma} d^{-\rho}, c^{\sigma-\rho} d^{\sigma}, c^\rho d^{\rho-\sigma}\right\}, &(2.15)
\end{aligned}$$

where $\rho, \sigma = 0, 1, ..., n-1$, again excluding possibilities given by Eq. (2.8).

- **Elements with a and a^2**

Here we consider an element of the form $ac^\rho d^\sigma$ with $\rho, \sigma = 0, 1, ..., n-1$, when it is conjugated first by the element $g_\alpha = a^\alpha c^\gamma d^\delta$ (where $\alpha = 0, 1, 2$ and $\gamma, \delta = 0, 1, ..., n-1$) and then by the element b. As we had noted in the beginning of this section conjugation by b results in a and a^2 belonging to the same type of classes. When $\alpha = 0$ we get

$$\begin{aligned}
g_0 a c^\rho d^\sigma g_0^{-1} &= a c^{\rho-\gamma-\delta} d^{\sigma+\gamma-2\delta} &(2.16)\\
&= a c^{\rho+\sigma-y_0-3x_0} d^{y_0}, &(2.17)
\end{aligned}$$

where we have used the redefinitions $x_0 \equiv \delta$ and $y_0 \equiv \sigma + \gamma - 2\delta$, with $x_0, y_0 = 0, 1, ..., n-1$. Applying g_α on $ac^\rho d^\sigma$ but for $\alpha = 1$ and then once again for $\alpha = 2$, we find we can summarize the three cases of α as

$$g_\alpha a c^\rho d^\sigma g_\alpha^{-1} = a c^{\rho+\sigma-y_\alpha-3x_\alpha} d^{y_\alpha}, \tag{2.18}$$

where $y_\alpha = -\rho - \sigma + y_{\alpha-1} + 3x_{\alpha-1}$ and $x_\alpha = x_{\alpha-1} - y_\alpha$. Since both x_α and y_α take all values $x_\alpha, y_\alpha = 0, 1, ..., n-1$, we can write the elements with a single power of a belonging to the same class as

$$a c^{\tau-y-3x} d^y, \qquad x, y = 0, 1, ..., n-1\ , \tag{2.19}$$

for a given choice of τ where $\tau \equiv \rho + \sigma$. Finally, the conjugation of the above with the element b will produce the elements with a^2,

$$b a c^{\tau-y-3x} d^y b^{-1} = a^2 c^{-y} d^{y+3x-\tau}. \tag{2.20}$$

So far we have new classes parametrized by τ:

$$C_2^{(\tau)} = \left\{a c^{\tau-y-3x} d^y,\ a^2 c^{-y} d^{y+3x-\tau} |\ \ x, y = 0, 1, ..., n-1\right\}, \quad \tau = 0, 1, ..., n-1\ . \tag{2.21}$$

We can now check for conditions which would cause over counting of elements in the same class. If we hold y constant for any two elements with the same power in a, we find we can over count when $3\,(x - x') = 0 \bmod(n)$. We then have to consider once again the two separate conditions for n.

(*i*) $n \neq 3\,\mathbb{Z}$. We do not have to worry about redundant terms caused by x. Quick inspection of Eq.(2.21) reveals that $\tau - y - 3x$ could be replaced by redefinition $z \equiv \tau - y - 3x$. Notice that when y has been chosen, we are still free to choose values for x, thus generating *all* possible values for z mod (n), irrespective of initial choice of τ. Our class is therefore no longer parametrized by τ:

$$1\ :\ 2n^2 C_2 = \left\{a c^z d^y,\ a^2 c^{-y} d^{-z} |\ \ z, y = 0, 1, ..., n-1\right\}\ . \tag{2.22}$$

(ii) $n = 3\,\mathbb{Z}$. Here the variable x, as mentioned, can produce over counting in the same class. So it must be limited to $x = 0, 1, ..., (n-3)/3$ to avoid it. Thus we allow y to take on all values, and in doing so we need to set the values for τ, which now parametrizes the different classes:

$$3 \; : \; \frac{2n^2}{3} C_2^{(\tau)} = \{ac^{\tau-y-3x}d^y,\; a^2c^{-y}d^{y+3x-\tau} |\; y = 0, 1, ..., n-1,\;\; x = 0, 1, ..., \tfrac{n-3}{3}\}, \quad \tau = 0, 1, 2\,. \tag{2.23}$$

- **Elements with b**

  The exercise here is similar to the one above, with the sole exception that we begin with an element of the form $bc^\rho d^\sigma$, but nevertheless conjugating the term with g_α as before. Conjugations with $\alpha = 0$ results in

$$g_0 bc^\rho d^\sigma g_0^{-1} = bc^{\rho-\gamma-\delta} d^{\sigma-\delta-\gamma} = bc^{\rho'+x} d^x \ , \tag{2.24}$$

  where $\rho' = \rho - \sigma$, $x = \sigma - \delta - \gamma$, and $x = 0, 1, ..., n-1$. Now, conjugating $bc^\rho d^\sigma$ with g_α for the cases $\alpha = 1, 2$, we get one term involving ab and another a^2b. The final result are classes parametrized by ρ':

$$n \; : \; 3nC_3^{(\rho')} = \left\{bc^{\rho'+x}d^x,\; a^2bc^{-\rho'}d^{-x-\rho'},\; abc^{-x}d^{\rho'} |\;\; x = 0, 1, ..., n-1\right\}, \rho' = 0, 1, ..., n-1\ . \tag{2.25}$$

  The parameterization of the class comes about when one chooses a value for ρ and σ, the ρ' value becomes fixed but not x. Finally unlike previous types of classes the above is independent on whether or not $n = 3\mathbb{Z}$.

This completes the derivation of the class structure of the group $\Delta(6n^2)$.

The results can be summarized as:

(i) $n \neq 3\,\mathbb{Z}$. Five types of classes

$$1C_1, \quad 3C_1^{(\rho)}, \quad 6C_1^{(\rho,\sigma)}, \quad 2n^2C_2, \quad 3nC_3^{(\rho)} \tag{2.26}$$

adding up to $1 + (n-1) + \frac{n^2-3n+2}{6} + 1 + n$ distinct classes.

(ii) $n = 3\,\mathbb{Z}$. Six types of classes

$$1C_1, \quad 1C_1^{(\nu)}, \quad 3C_1^{(\rho)}, \quad 6C_1^{(\rho,\sigma)}, \quad \tfrac{2n^2}{3}C_2^{(\tau)}, \quad 3nC_3^{(\rho)}, \tag{2.27}$$

resulting in $1 + 2 + (n-3) + \frac{n^2-3n+6}{6} + 3 + n$ different classes.

3 Irreducible Representations

We shall systematically construct all the irreducible representations of $\Delta(6n^2)$. The explicit construction will allow us to determine the character table by simple application of taking the trace. Keep in mind that we shall not list the representations in numerical order by their dimension, but instead we shall list by order in which they were obtained.

- **One-dimensional Representations**

  The semidirect product produces very strong constraints on c and d, reducing the set of possible one dimensional representations to simply two:

$$\mathbf{1_1} \quad : \quad a = b = c = d = 1, \tag{3.1}$$
$$\mathbf{1_2} \quad : \quad a = c = d = 1, \; b = -1, \tag{3.2}$$

true regardless of n.

- **Six-dimensional Representations**

  The method of induced representation[1] shows conclusively that there exist six-dimensional representations. The Abelian elements (c and d) are 6×6 diagonal matrices whose entries are powers (l and k) of the nth root of unity, given by $\eta \equiv e^{2\pi i/n}$ with $l, k = 0, 1, ..., n-1$.

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \tag{3.3}$$

where

$$a_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.4}$$

$$c_1 = d_2^{-1} = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-l-k} \end{pmatrix}, \qquad c_2 = d_1^{-1} = \begin{pmatrix} \eta^{l+k} & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}. \tag{3.5}$$

Care must be taken, in that the above terms lend themselves to over counting. To observe this we note that there exist similarity transformations that leave both a and b alone while only exchanging the diagonal entries of both c and d. Two matrices that perform such transformations are

$$V = \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix}, \quad V^3 = 1, \tag{3.6}$$

$$T = \begin{pmatrix} 0 & t_1 \\ t_1 & 0 \end{pmatrix}, \quad T^2 = 1, \tag{3.7}$$

where

$$t_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

When we apply the above transformation on c we get the following set of six possible labeling exchanges

$$\begin{pmatrix} k \\ l \end{pmatrix}, \begin{pmatrix} -k-l \\ k \end{pmatrix}, \begin{pmatrix} l \\ -k-l \end{pmatrix} \begin{pmatrix} -l \\ -k \end{pmatrix}, \begin{pmatrix} k+l \\ -l \end{pmatrix}, \begin{pmatrix} -k \\ k+l \end{pmatrix}. \tag{3.9}$$

We can arrive at the above six cases by first using the definition

$$M_s^p \equiv \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^p \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^s, \quad \text{with} \quad p = 0, 1, 2, \quad s = 0, 1, \tag{3.10}$$

so that we may write the six pairs above in the form

$$M_s^p \begin{pmatrix} k \\ l \end{pmatrix}, \tag{3.11}$$

[1] Details of the method can be found described in [14]. The coset here contains six points: (e, z), (a^2, z), (a, z), (b, z), (ab, z), (a^2b, z). The action of c and d remain the same: $cz = \eta^l z$, $dz = \eta^{-k-l} z$.

for the choices of p and s. Notice that we can reproduce the exponents of the elements of the classes parametrized by (ρ, σ) (Eq. (2.6)) by application of $(\rho,\sigma)M_s^p$. The six-dimensional representation will in general be labeled by a pair of numbers (k,l), but we want to avoid the possible labeling ambiguity that arises as mentioned above. To do so we assume the existence of a mapping $\widetilde{\ }$ that gives the standard representative:

$$\widetilde{\begin{pmatrix} k \\ l \end{pmatrix}} \quad \longmapsto \quad \text{either} \quad \begin{pmatrix} k \\ l \end{pmatrix}, \quad \begin{pmatrix} -k-l \\ k \end{pmatrix}, \quad \begin{pmatrix} l \\ -k-l \end{pmatrix}, \quad \begin{pmatrix} -l \\ -k \end{pmatrix}, \quad \begin{pmatrix} k+l \\ -l \end{pmatrix}, \text{ or } \begin{pmatrix} -k \\ k+l \end{pmatrix} \ . \tag{3.12}$$

We now must ask whether the six dimensional representation is in fact irreducible or not. To begin we note that the non-Abelian generators a and b both form representation of the S_3. Taking the trace of both a and b we find that $Tr(a) = 0$ and $Tr(b) = 0$. Then looking at the irreducible representations of S_3 (underlined), we notice that the only possible decompositions of a and b is

$$a, b \ \rightarrow \ \underline{\mathbf{2}} + \underline{\mathbf{2}} + \underline{\mathbf{1}} + \underline{\mathbf{1}'} \ . \tag{3.13}$$

The above fact however does not immediately make it clear in what way the decomposition will take place, but does show that it is perhaps possible for some choices of l and k. A similarity transformation may in fact explicitly show the above breakdown of a and b but in doing so c and d will in general be non-diagonal. In the case of non-diagonal generators c, d we will find conditions on the parameters k, l that will cause the generators c, d to also break down according to Eq. (3.13). It is these restrictions that limit us from taking all possible values of l and k in the following list of six-dimensional irreducible representations. The cases in which $l = 0$, $k = 0$, $l + k = 0 \bmod(n)$, produce both three and two-dimensional representations, while the case $l = k = n/3, 2n/3$ produces another set of two-dimensional representations. What we will find is that there are two cases for the six-dimensional representations:

(i) $n \neq 3\,\mathbb{Z}$. The notation that follows will list the irreducible representation with its parameterization, and subsequently the conditions which would make it reducible.

$$\mathbf{6}_{\widetilde{(k,l)}}, \qquad \text{reducible if} \begin{cases} n & : \ \ k+l = \ 0 \bmod(n) \\ n-1 & : \ \ k = 0 \ \ , \ \ l \neq 0 \\ n-1 & : \ \ l = 0 \ \ , \ \ k \neq 0 \end{cases} \ . \tag{3.14}$$

Excluding the above cases for l and k it is easy to see that there should be $n^2 - 2(n-1) - n = (n-1)(n-2)$ different representations, however remembering Eq. (3.12) it is easy to see that we need to divide this factor by six i.e. $\frac{(n-1)(n-2)}{6}$.

(ii) $n = 3\,\mathbb{Z}$.

$$\mathbf{6}_{\widetilde{(k,l)}}, \qquad \text{reducible if} \begin{cases} 2 & : \ \ (k,l) = (\frac{n}{3}, \frac{n}{3}), (\frac{2n}{3}, \frac{2n}{3}) \\ n & : \ \ k+l = \ 0 \bmod(n) \\ n-1 & : \ \ k = 0 \ \ , \ \ l \neq 0 \\ n-1 & : \ \ l = 0 \ \ , \ \ k \neq 0 \end{cases} \ . \tag{3.15}$$

Similar to the $n \neq 3\mathbb{Z}$ case with the exception of the exclusion of the first line in the above conditions. This means we have $n^2 - 2 - n - 2(n-1) = n(n-3)$ possibilities and again because of the over counting issue mentioned above we really have $\frac{n(n-3)}{6}$ possible irreducible representations.

With Eq. (3.12) in mind we shall from now on omit the $\widetilde{\ }$ symbol and simply write $\mathbf{6}_{(k,l)}$. In the case where the restrictions are violated, it can be shown that the now reducible representations break up into three-dimensional and two-dimensional representations.

- **Three-dimensional Representations**

  Recall that a representation is reducible if it can be found to be block diagonal. Looking at the six-dimensional representation, one cannot help but notice that the 3×3 block structure of all generators (a,b,c,d) is suggestive. With this realization, we shall attempt to diagonalize a and b simultaneously (via similarity transformation). The general matrix found that diagonalizes a and b is of the form:

$$S = \begin{pmatrix} \mathbb{1} & \mathbf{e} \\ \mathbb{1} & -\mathbf{e} \end{pmatrix}, \quad \mathbf{e} = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}, \tag{3.16}$$

  where $\mathbf{e}$ matrices are the order two elements belonging to S_3 in its three-dimensional representation. In block diagonalizing a and b we spoil the diagonal property of c and d. However, it may be possible to find conditions for k and l that would once again make the representation block diagonal. In fact there exist three possible conditions, each for the choice of $\mathbf{e}$ made. The conditions are either

$$k + l = 0 \text{ mod}(n), \qquad l = 0, \qquad k = 0, \tag{3.17}$$

  respectively. Excluding $k = l = 0$ which will be shown to give even smaller irreducible representations, the diagonalization produces two three-dimensional irreps listed below for the choice $k + l = 0 \text{ mod}(n)$:

$$a_{(3_1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; b_{(3_1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; c_{(3_1)} = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; d_{(3_1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^{-l} \end{pmatrix}, \tag{3.18}$$

$$a_{(3_2)} = a_{(3_1)}, \quad b_{(3_2)} = -b_{(3_1)}, \quad c_{(3_2)} = c_{(3_1)}, \quad d_{(3_2)} = d_{(3_1)} . \tag{3.19}$$

  Once again η is the nth root of unity, and $l = 1, 2, ..., n-1$. The other two choices in Eq. (3.17) are related to Eq. (3.18)-(3.19) by similarity transformations. So we see that the number of three dimensional representations in this case is simply $2(n-1)$.

- **Two-dimensional Representations**

  The two-dimensional representations are obtained from two distinct origins. Where one is obtained from the conditions that make three-dimensional representation reducible, the other two-dimensional representations arise directly from the six-dimensional representation. It is important to be aware that the first case occurs for all values of n.

  (*i*) $\forall n$. In the previous section we excluded the case when $l = k = 0$, but if this occurs both c and d become nothing more than the identity matrix. We are then left with two three-dimensional representations of S_3. However S_3, according to its character table, has no irreducible three-dimensional representations. Looking both at Eq. (3.13) and the character table of S_3, the only possible decompositions are $\mathbf{3_1} \to \underline{\mathbf{2}} + \mathbf{1_1}$ and $\mathbf{3_2} \to \underline{\mathbf{2}} + \mathbf{1_2}$. The difference between the $\mathbf{3_1}$ and the $\mathbf{3_2}$ is that the trace of b are opposite signs which simply translates in having different one-dimensional representations in the decomposition. In either case, they both contain the same two-dimensional representation, $\mathbf{2_1} = \underline{\mathbf{2}}$:

$$a_{(2_1)} = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad b_{(2_1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c_{(2_1)} = d_{(2_1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.20}$$

  where $\omega = e^{2\pi i/3}$.

  (*ii*) $n = 3\,\mathbb{Z}$. The three-dimensional representation was primarily motivated by the block diagonal structure found in the six-dimensional representation. However, it is possible that other reductions may in fact exist. To find these reductions we perform a similarity transformation on both a and b with the following results

$$a' = UaU^{-1} = \begin{pmatrix} a_{(2_1)} & 0 & 0 \\ 0 & a_{(2_1)} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix}, \quad b' = UbU^{-1} = \begin{pmatrix} b_{(2_1)} & 0 & 0 \\ 0 & b_{(2_1)} & 0 \\ 0 & 0 & b_{(2_1)} \end{pmatrix}, \tag{3.21}$$

where all the entries are 2×2 matrices, the ones on the diagonal defined in Eq. (3.20). The last entry in a' is an identity matrix to ensure that the trace of a' is identical to the trace of a. With the choices specified above, one finds constraints on U. When U, under the newly obtained constraints, is applied to c and d, we find that in general these generators are no longer diagonal. To remove off diagonal terms, excluding the case where $k = l = 0$, it's necessary that

$$k = l = n/3, 2n/3, \tag{3.22}$$

so that we may replace η^k for ω. The end result being three two-dimensional representations, with $k = l = n/3$ (we could have chosen $k = l = 2n/3$ but it produces equivalent results):

$$a_{(2_2)} = a_{(2_1)}, \quad b_{(2_2)} = b_{(2_1)}, \quad c_{(2_2)} = d_{(2_2)} = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \tag{3.23}$$

$$a_{(2_3)} = a_{(2_1)}, \quad b_{(2_3)} = b_{(2_1)}, \quad c_{(2_3)} = d_{(2_3)} = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \tag{3.24}$$

$$a_{(2_4)} = \mathbb{1}, \quad b_{(2_4)} = b_{(2_1)}, \quad c_{(2_4)} = d_{(2_4)} = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}. \tag{3.25}$$

Notice that in the last representation $a_{(2_4)}$ and $b_{(2_4)}$ are not an irreducible representation of S_3. Now, including the two-dimensional representation from the $\forall n$ case, we see that in total we have $1 + 3 = 4$ two-dimensional representations.

There are no more irreducible representations as can be seen by performing the following sum

$$6n^2 = \sum_{irrep\ i} d_i^2, \tag{3.26}$$

where d_i is the dimension of the representation. Performing the sum for $n \neq 3\mathbb{Z}$ we find

$$\frac{(n-1)(n-2)}{6} \times 6^2 + 2(n-1) \times 3^2 + 1 \times 2^2 + 2 \times 1^2 = 6n^2, \tag{3.27}$$

and for $n = 3\mathbb{Z}$ we have

$$\frac{n(n-3)}{6} \times 6^2 + 2(n-1) \times 3^2 + 4 \times 2^2 + 2 \times 1^2 = 6n^2. \tag{3.28}$$

We can derive the $\Delta(6n^2)$ character table by taking traces over the relevant matrices, for both $n \neq 3\,\mathbb{Z}$ and $n = 3\,\mathbb{Z}$. The results are displayed in Table 1, using the classes and representations just derived, including the restrictions on the parameters (ρ, σ) and (k, l).

Looking at the character table above, notice that the entries for the class C_1, $C_1^{(\rho)}$, and $C_1^{(\nu)}$ can all be obtained by suitable substitutions in the parameters of $C_1^{(\rho,\sigma)}$. A fact that will be a source of great simplification in the next section.

4 Kronecker Products

In order to build models, it is important to know which products of representations will lead to invariant quantities, associated with the singlet representations. So it is necessary to determine all the Kronecker products of the irreducible representations. In general the Kronecker products are obtained by

Table 1: The character tables of $\Delta(6n^2)$ for the cases (a) $n \neq 3\mathbb{Z}$ and (b) $n = 3\mathbb{Z}$. Note ρ and σ take on different values depending on the class, $\tau = 0, 1, 2$, $\nu = \frac{n}{3}, \frac{2n}{3}$, $p = 0, 1, 2$, $s = 0, 1$, $r = 1, 2$, $\omega = e^{2\pi i/3}$, and $l, k = 1, 2, ..., n-1$.

(a) $n \neq 3\mathbb{Z}$	$1C_1$	$3C_1^{(\rho)}$	$6C_1^{(\rho,\sigma)}$	$2n^2C_2$	$3nC_3^{(\rho)}$
$\mathbf{1_1}$	1	1	1	1	1
$\mathbf{1_2}$	1	1	1	1	-1
$\mathbf{2_1}$	2	2	2	-1	0
$\mathbf{3_1}_{(l)}$	3	$\sum_p \eta^{(\rho,-\rho)M_0^p\binom{-l}{l}}$	$\sum_p \eta^{(\rho,\sigma)M_0^p\binom{-l}{l}}$	0	$\eta^{-\rho l}$
$\mathbf{3_2}_{(l)}$	3	$\sum_p \eta^{(\rho,-\rho)M_0^p\binom{-l}{l}}$	$\sum_p \eta^{(\rho,\sigma)M_0^p\binom{-l}{l}}$	0	$-\eta^{-\rho l}$
$\mathbf{6}_{(k,l)}$	6	$\sum_{p,s} \eta^{(\rho,-\rho)M_s^p\binom{k}{l}}$	$\sum_{p,s} \eta^{(\rho,\sigma)M_s^p\binom{k}{l}}$	0	0

(b) $n = 3\mathbb{Z}$	$1C_1$	$1C_1^{(\nu)}$	$3C_1^{(\rho)}$	$6C_1^{(\rho,\sigma)}$	$\frac{2n^2}{3}C_2^{(\tau)}$	$3nC_3^{(\rho)}$
$\mathbf{1_1}$	1	1	1	1	1	1
$\mathbf{1_2}$	1	1	1	1	1	-1
$\mathbf{2_1}$	2	2	2	2	-1	0
$\mathbf{2_2}$	2	2	2	$\sum_r \omega^{(\rho+\sigma)r}$	$\sum_r \omega^{(2+\tau)r}$	0
$\mathbf{2_3}$	2	2	2	$\sum_r \omega^{(\rho+\sigma)r}$	$\sum_r \omega^{(1+\tau)r}$	0
$\mathbf{2_4}$	2	2	2	$\sum_r \omega^{(\rho+\sigma)r}$	$\sum_r \omega^{(\tau)r}$	0
$\mathbf{3_1}_{(l)}$	3	$\sum_p \eta^{(\nu,-\nu)M_0^p\binom{-l}{l}}$	$\sum_p \eta^{(\rho,-\rho)M_0^p\binom{-l}{l}}$	$\sum_p \eta^{(\rho,\sigma)M_0^p\binom{-l}{l}}$	0	$\eta^{-\rho l}$
$\mathbf{3_2}_{(l)}$	3	$\sum_p \eta^{(\nu,-\nu)M_0^p\binom{-l}{l}}$	$\sum_p \eta^{(\rho,-\rho)M_0^p\binom{-l}{l}}$	$\sum_p \eta^{(\rho,\sigma)M_0^p\binom{-l}{l}}$	0	$-\eta^{-\rho l}$
$\mathbf{6}_{(k,l)}$	6	$\sum_{p,s} \eta^{(\nu,-\nu)M_s^p\binom{k}{l}}$	$\sum_{p,s} \eta^{(\rho,-\rho)M_s^p\binom{k}{l}}$	$\sum_{p,s} \eta^{(\rho,\sigma)M_s^p\binom{k}{l}}$	0	0

$$\mathbf{r} \otimes \mathbf{s} \quad = \quad \sum_{\mathbf{t}} d(\mathbf{r}, \mathbf{s}, \mathbf{t}) \ \mathbf{t} \ , \tag{4.1}$$

where $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{t}$ are irreducible representations, and the sum is taken over all irreducible representations. One approach of solving for the the integer numerical factors $d(\mathbf{r}, \mathbf{s}, \mathbf{t})$ would be to calculate it from the character table by using

$$d(\mathbf{r}, \mathbf{s}, \mathbf{t}) \quad = \quad \frac{1}{N} \sum_i n_i \cdot \chi_i^{[\mathbf{r}]} \, \chi_i^{[\mathbf{s}]} \, \overline{\chi}_i^{[\mathbf{t}]} \ , \tag{4.2}$$

where N is the order of the group; i labels a class of n_i elements and character χ_i. $\overline{\chi}_i$ denotes the complex conjugate character. Another approach, discussed in Appendix B, obtains the same results, by explicitly building the products. Below we shall only summarize the results for all Kronecker products, relegating the details to Appendix B.

(i) $n \neq 3\,\mathbb{Z}$.

Looking at Table 1a, we note that there are: 2 one-dimensional, 1 two-dimensional, $2(n-1)$ three-dimensional, and $\frac{(n-1)(n-2)}{6}$ six-dimensional representations.

$$
\begin{aligned}
\mathbf{1_2} \otimes \mathbf{1_2} &= \mathbf{1_1} \\
\mathbf{1_2} \otimes \mathbf{2_1} &= \mathbf{2_1} \\
\mathbf{1_2} \otimes \mathbf{3_1}_{(l)} &= \mathbf{3_2}_{(l)} \\
\mathbf{1_2} \otimes \mathbf{3_2}_{(l)} &= \mathbf{3_1}_{(l)} \\
\mathbf{1_2} \otimes \mathbf{6}_{(k,l)} &= \mathbf{6}_{(k,l)} \\
\mathbf{2_1} \otimes \mathbf{2_1} &= \mathbf{1_1} + \mathbf{1_2} + \mathbf{2_1} \\
\mathbf{2_1} \otimes \mathbf{3_1}_{(l)} &= \mathbf{3_1}_{(l)} + \mathbf{3_2}_{(l)} \\
\mathbf{2_1} \otimes \mathbf{3_2}_{(l)} &= \mathbf{3_1}_{(l)} + \mathbf{3_2}_{(l)} \\
\mathbf{2_1} \otimes \mathbf{6}_{(k,l)} &= \mathbf{6}_{(k,l)} + \mathbf{6}_{(k,l)} \\
\mathbf{3_1}_{(l)} \otimes \mathbf{3_1}_{(l')} &= \mathbf{3_1}_{(l+l')} + \mathbf{6}_{\widetilde{(l,-l')}} \\
\mathbf{3_1}_{(l)} \otimes \mathbf{3_2}_{(l')} &= \mathbf{3_2}_{(l+l')} + \mathbf{6}_{\widetilde{(l,-l')}} \\
\mathbf{3_1}_{(l)} \otimes \mathbf{6}_{(k',l')} &= \mathbf{6}_{\widetilde{\begin{pmatrix} k' \\ l'-l \end{pmatrix}}} + \mathbf{6}_{\widetilde{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}}} + \mathbf{6}_{\widetilde{\begin{pmatrix} l+k' \\ l' \end{pmatrix}}} \\
\mathbf{3_2}_{(l)} \otimes \mathbf{3_2}_{(l')} &= \mathbf{3_1}_{(l+l')} + \mathbf{6}_{\widetilde{(l,-l')}} \\
\mathbf{3_2}_{(l)} \otimes \mathbf{6}_{(k',l')} &= \mathbf{6}_{\widetilde{\begin{pmatrix} k' \\ l'-l \end{pmatrix}}} + \mathbf{6}_{\widetilde{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}}} + \mathbf{6}_{\widetilde{\begin{pmatrix} l+k' \\ l' \end{pmatrix}}} \\
\mathbf{6}_{(k,l)} \otimes \mathbf{6}_{(k',l')} &= \sum_{p,s} \mathbf{6}_{\left(\widetilde{\begin{pmatrix} k \\ l \end{pmatrix} + M_s^p \begin{pmatrix} k' \\ l' \end{pmatrix}}\right)}
\end{aligned}
$$

(*ii*) $n = 3\,\mathbb{Z}$.

Now in Table 1b, we note that there are: 2 one-dimensional, 4 two-dimensional, $2(n-1)$ three-dimensional, and $\frac{n(n-1)}{6}$ six-dimensional representations.

$$
\begin{aligned}
\mathbf{1_2} \otimes \mathbf{1_2} &= \mathbf{1_1} \\
\mathbf{1_2} \otimes \mathbf{2_1} &= \mathbf{2_1} \\
\mathbf{1_2} \otimes \mathbf{2_2} &= \mathbf{2_2} \\
\mathbf{1_2} \otimes \mathbf{2_3} &= \mathbf{2_3} \\
\mathbf{1_2} \otimes \mathbf{2_4} &= \mathbf{2_4} \\
\mathbf{1_2} \otimes \mathbf{3_1}_{(l)} &= \mathbf{3_2}_{(l)} \\
\mathbf{1_2} \otimes \mathbf{3_2}_{(l)} &= \mathbf{3_1}_{(l)} \\
\mathbf{1_2} \otimes \mathbf{6}_{(k,l)} &= \mathbf{6}_{(k,l)} \\
\mathbf{2_1} \otimes \mathbf{2_1} &= \mathbf{1_1} + \mathbf{1_2} + \mathbf{2_1} \\
\mathbf{2_1} \otimes \mathbf{2_2} &= \mathbf{2_3} + \mathbf{2_4} \\
\mathbf{2_1} \otimes \mathbf{2_3} &= \mathbf{2_2} + \mathbf{2_4} \\
\mathbf{2_1} \otimes \mathbf{2_4} &= \mathbf{2_2} + \mathbf{2_3} \\
\mathbf{2_1} \otimes \mathbf{3_1}_{(l)} &= \mathbf{3_1}_{(l)} + \mathbf{3_2}_{(l)} \\
\mathbf{2_1} \otimes \mathbf{3_2}_{(l)} &= \mathbf{3_1}_{(l)} + \mathbf{3_2}_{(l)} \\
\mathbf{2_1} \otimes \mathbf{6}_{(k,l)} &= \mathbf{6}_{(k,l)} + \mathbf{6}_{(k,l)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{2_2} \otimes \mathbf{2_2} &= \mathbf{1_1} + \mathbf{1_2} + \mathbf{2_2} \\
\mathbf{2_2} \otimes \mathbf{2_3} &= \mathbf{2_1} + \mathbf{2_4} \\
\mathbf{2_2} \otimes \mathbf{2_4} &= \mathbf{2_1} + \mathbf{2_3} \\
\mathbf{2_2} \otimes \mathbf{3_{1}}_{(l)} &= \mathbf{6}_{\widetilde{\binom{2n/3-l}{2n/3+l}}} \\
\mathbf{2_2} \otimes \mathbf{3_{2}}_{(l)} &= \mathbf{6}_{\widetilde{\binom{2n/3-l}{2n/3+l}}} \\
\mathbf{2_2} \otimes \mathbf{6}_{(k,l)} &= \mathbf{6}_{\widetilde{\binom{k+n/3}{l+n/3}}} + \mathbf{6}_{\widetilde{\binom{k+2n/3}{l+2n/3}}} \\
\mathbf{2_3} \otimes \mathbf{2_3} &= \mathbf{1_1} + \mathbf{1_2} + \mathbf{2_3} \\
\mathbf{2_3} \otimes \mathbf{2_4} &= \mathbf{2_1} + \mathbf{2_2} \\
\mathbf{2_3} \otimes \mathbf{3_{1}}_{(l)} &= \mathbf{6}_{\widetilde{\binom{n/3-l}{n/3+l}}} \\
\mathbf{2_3} \otimes \mathbf{3_{2}}_{(l)} &= \mathbf{6}_{\widetilde{\binom{n/3-l}{n/3+l}}} \\
\mathbf{2_3} \otimes \mathbf{6}_{(k,l)} &= \mathbf{6}_{\widetilde{\binom{k+n3}{l+n/3}}} + \mathbf{6}_{\widetilde{\binom{k+2n/3}{l+2n/3}}} \\
\mathbf{2_4} \otimes \mathbf{2_4} &= \mathbf{1_1} + \mathbf{1_2} + \mathbf{2_4} \\
\mathbf{2_4} \otimes \mathbf{3_{1}}_{(l)} &= \mathbf{6}_{\widetilde{\binom{n/3-l}{n/3+l}}} \\
\mathbf{2_4} \otimes \mathbf{3_{2}}_{(l)} &= \mathbf{6}_{\widetilde{\binom{n/3-l}{n/3+l}}} \\
\mathbf{2_4} \otimes \mathbf{6}_{(k,l)} &= \mathbf{6}_{\widetilde{\binom{k+n/3}{l+n/3}}} + \mathbf{6}_{\widetilde{\binom{k+2n/3}{l+2n/3}}} \\
\mathbf{3_{1}}_{(l)} \otimes \mathbf{3_{1}}_{(l')} &= \mathbf{3_{1}}_{(l+l')} + \mathbf{6}_{\widetilde{(l,-l')}} \\
\mathbf{3_{1}}_{(l)} \otimes \mathbf{3_{2}}_{(l')} &= \mathbf{3_{2}}_{(l+l')} + \mathbf{6}_{\widetilde{(l,-l')}} \\
\mathbf{3_{1}}_{(l)} \otimes \mathbf{6}_{(k',l')} &= \mathbf{6}_{\widetilde{\binom{k'}{l'-l}}} + \mathbf{6}_{\widetilde{\binom{k'-l}{l'+l}}} + \mathbf{6}_{\widetilde{\binom{l+k'}{l'}}} \\
\mathbf{3_{2}}_{(l)} \otimes \mathbf{3_{2}}_{(l')} &= \mathbf{3_{1}}_{(l+l')} + \mathbf{6}_{\widetilde{(l,-l')}} \\
\mathbf{3_{2}}_{(l)} \otimes \mathbf{6}_{(k',l')} &= \mathbf{6}_{\widetilde{\binom{k'}{l'-l}}} + \mathbf{6}_{\widetilde{\binom{k'-l}{l'+l}}} + \mathbf{6}_{\widetilde{\binom{l+k'}{l'}}} \\
\mathbf{6}_{(k,l)} \otimes \mathbf{6}_{(k',l')} &= \sum_{p,s} \mathbf{6}_{\widetilde{\left(\binom{k}{l} + M_s^p \binom{k'}{l'}\right)}}
\end{aligned}
$$

As a final note, occasionally the right hand side of a Kronecker product will contain terms that are reducible. As a general rule, one can go through Section 3, in particular Eqs. (3.17) and (3.22), and determine how certain representations become reducible and what irreducible representations they break up into. Below we list all possible cases explicitly:

$$\mathbf{6}_{(-l,l)},\ \mathbf{6}_{(0,-l)},\ \mathbf{6}_{(l,0)} \ \rightarrow\ \mathbf{3_{1}}_{(l)} + \mathbf{3_{2}}_{(l)}, \tag{4.3}$$

$$\mathbf{3_{r}}_{(l)} \ \rightarrow\ \mathbf{2_1} + \mathbf{1}_r, \qquad r = 1, 2, \tag{4.4}$$

and additionally for the case $n = 3\mathbb{Z}$

$$\mathbf{6}_{(n/3,n/3)},\ \mathbf{6}_{(2n/3,2n/3)} \ \rightarrow\ \mathbf{2_2} + \mathbf{2_3} + \mathbf{2_4}. \tag{4.5}$$

5 Conclusion

In order to carry out a systematic investigation of how a non-Abelian finite group can be used to successfully explain the remarkable tri-bimaximal structure in the neutrino mixing, a thorough knowledge of the finite group is essential. In this article we have studied the class structure of the group $\Delta(6n^2)$

and its irreducible representations. Introducing a compact notation which is valid for arbitrary n, the Kronecker products as well as the Clebsch-Gordan coefficients have been derived. Our results can be used to easily calculate the group invariants (e.g. trilinear Yukawa couplings) of one particular group defined by the value for n, in particular going beyond the smallest groups with $n = 1, 2$. On the other hand, comparing cases with different n might unveil important common or distinguishing features which are crucial in building viable flavor models. Our study complements similar work [14] on the group $\Delta(3n^2)$.

Acknowledgments

The work of CL is supported by the University of Florida through the Institute for Fundamental Theory and that of JAE is supported by the Department of Energy Grant No. DE-FG02-97ER41029. We are grateful for helpful discussions with Salah Nasri and Pierre Ramond.

Appendix

A Labeling the Six-Dimensional Representations

It was shown in Ref. [14] that the labeling of the three-dimensional irreducible representations of $\Delta(3n^2)$ by (k, l) is ambiguous. A possible resolution by choosing a particular standard representative was presented in three tables. A similar ambiguity arises in the case of $\Delta(6n^2)$ where the six-dimensional irreducible representations are labeled by (k, l). Due to Eq. (3.12) we obtain the following sets of possible standard representatives, indicated by dots on the $n \times n$ grid. We distinguish between three cases: (i_1) $n = 3z + 1$ with $z \in \mathbb{Z}$, (i_2) $n = 3z + 2$, and (ii) $n = 3z$.

(i_1) $n = 3z + 1$.

(k,l)	0	1	·	·	z	·	·	·	$2z$	·	·	·	$3z$
0													
1		·	·	·	·	·	·	·					
·		·	·	·	·	·	·						
·		·	·	·	·	·							
z		·	·	·	·								
·													
·													
·													
$2z$													
·													
·													
·													
$3z$													

(i_2) $n = 3z + 2$.

(k,l)	0	1	·	·	z	·	·	·	$2z$	·	·	·	$3z$	$3z{+}1$
0														
1		·	·	·	·	·	·	·	·					
·		·	·	·	·	·	·	·						
·		·	·	·	·	·	·							
z		·	·	·	·	·								
·														
·														
·														
$2z$														
·														
·														
·														
$3z$														
$3z+1$														

(*ii*) $n = 3z$.

(k,l)	0	1	·	·	z	·	·	·	$2z$	·	·	$3z$–1
0												
1		·	·	·	·	·	·					
·		·	·	·	·	·						
·		·	·	·	·							
z		·	·	·								
·												
·												
·												
$2z$												
·												
·												
$3z-1$												

B Product of Irreducible Representations: Details

There are two ways of obtaining the quantities $d(\mathbf{r},\mathbf{s},\mathbf{t})$. The first method involves the use of Eq. (4.2) in conjunction with the Character Table 1. The calculations will not be explicitly performed as in [14]. The second method relies on deriving the actual representations of the products explicitly, providing us immediately the Kronecker products and also the Clebsch-Gordan coefficients.

(*B.i*) Products via the Character Table.

First note that the classes $C_1^{(\rho)}$, and C_1, (also $C_1^{(\nu)}$ for the $n = 3\mathbb{Z}$ case) can all be obtained from $C_1^{(\rho,\sigma)}$, by the appropriate choices of the parameters ρ and σ. For the case that $n \neq 3\mathbb{Z}$ we can formally express the sum over classes $C_1^{(\rho,\sigma)}$ as

$$\sum_{C_1^{(\rho,\sigma)}} = \frac{1}{6}\left(\sum_{\rho,\sigma=0}^{n-1} \; - \sum_{\substack{\rho+\sigma \;=\; 0 \bmod(n) \\ 2\rho-\sigma \;=\; 0 \bmod(n) \\ \rho-2\sigma \;=\; 0 \bmod(n)}} \; - \sum_{\rho=\sigma=0} \right). \tag{B.1}$$

The second term on the right hand side contains the three conditions of Eq. (2.8) on ρ and σ, all of which lead to the same class $C_1^{(\rho)}$ of three elements. The factor of 1/6 comes from the six distinct elements within one class, see Eq. (2.7). Looking at Eq. (4.2) we find that we may now write the terms involving the classes C_1, $C_1^{(\rho)}$, and $C_1^{(\rho,\sigma)}$ in a compact way

$$\chi^{[\mathbf{r}]}_{C_1}\,\chi^{[\mathbf{s}]}_{C_1}\,\overline{\chi}^{[\mathbf{t}]}_{C_1} + \sum_{C_1^{(\rho)}} 3\cdot\chi^{[\mathbf{r}]}_{C_1^{(\rho)}}\,\chi^{[\mathbf{s}]}_{C_1^{(\rho)}}\,\overline{\chi}^{[\mathbf{t}]}_{C_1^{(\rho)}} + \sum_{C_1^{(\rho,\sigma)}} 6\cdot\chi^{[\mathbf{r}]}_{C_1^{(\rho,\sigma)}}\,\chi^{[\mathbf{s}]}_{C_1^{(\rho,\sigma)}}\,\overline{\chi}^{[\mathbf{t}]}_{C_1^{(\rho,\sigma)}} = \sum_{\rho,\sigma=0}^{n-1} \chi^{[\mathbf{r}]}_{C_1^{(\rho,\sigma)}}\,\chi^{[\mathbf{s}]}_{C_1^{(\rho,\sigma)}}\,\overline{\chi}^{[\mathbf{t}]}_{C_1^{(\rho,\sigma)}}. \tag{B.2}$$

The above line means that the classes C_1, $C_1^{(\rho)}$ and $C_1^{(\rho,\sigma)}$ can be neatly written as a sum over all n^2 pairs of (ρ,σ). For the case that $n = 3\mathbb{Z}$ a similar expression occurs

$$\chi^{[\mathbf{r}]}_{C_1}\,\chi^{[\mathbf{s}]}_{C_1}\,\overline{\chi}^{[\mathbf{t}]}_{C_1} + \sum_{C_1^{(\nu)}} \chi^{[\mathbf{r}]}_{C_1^{(\nu)}}\,\chi^{[\mathbf{s}]}_{C_1^{(\nu)}}\,\overline{\chi}^{[\mathbf{t}]}_{C_1^{(\nu)}} + \sum_{C_1^{(\rho)}} 3\cdot\chi^{[\mathbf{r}]}_{C_1^{(\rho)}}\,\chi^{[\mathbf{s}]}_{C_1^{(\rho)}}\,\overline{\chi}^{[\mathbf{t}]}_{C_1^{(\rho)}} + \sum_{C_1^{(\rho,\sigma)}} 6\cdot\chi^{[\mathbf{r}]}_{C_1^{(\rho,\sigma)}}\,\chi^{[\mathbf{s}]}_{C_1^{(\rho,\sigma)}}\,\overline{\chi}^{[\mathbf{t}]}_{C_1^{(\rho,\sigma)}}$$

$$= \sum_{\rho,\sigma=0}^{n-1} \chi^{[\mathbf{r}]}_{C_1^{(\rho,\sigma)}}\,\chi^{[\mathbf{s}]}_{C_1^{(\rho,\sigma)}}\,\overline{\chi}^{[\mathbf{t}]}_{C_1^{(\rho,\sigma)}}. \tag{B.3}$$

Applying these relations, it is straightforward, though still tedious, to determine the integers $d(\mathbf{r},\mathbf{s},\mathbf{t})$ of Eq. (4.2).

$(B.ii)$ Explicitly Building Products of Irreducible Representations.

Here we form explicitly the representations produced by taking products of irreducible representations. The investigation will be performed in some detail on the two separate cases for n.

(i) $n \neq 3\mathbb{Z}$. We first need to form several vector spaces, one for each irreducible representation (with the exception of the trivial one-dimensional representations). The spaces become useful when we explicitly show how each transforms according to the generators a, b, and c:[2]

$$\begin{aligned}
\mathbf{6}_{(k,l)} \quad &: \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ x_1 \\ x_6 \\ x_4 \\ x_5 \end{pmatrix}_a, \quad \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}_b, \quad \begin{pmatrix} \eta^l x_1 \\ \eta^k x_2 \\ \eta^{-k-l} x_3 \\ \eta^{k+l} x_4 \\ \eta^{-l} x_5 \\ \eta^{-k} x_6 \end{pmatrix}_c, \\
\mathbf{3}_{1(l)} \quad &: \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}_a, \quad \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}_b, \quad \begin{pmatrix} \eta^l x_1 \\ \eta^{-l} x_2 \\ x_3 \end{pmatrix}_c, \\
\mathbf{3}_{2(l)} \quad &: \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}_a, \quad \begin{pmatrix} -x_3 \\ -x_2 \\ -x_1 \end{pmatrix}_b, \quad \begin{pmatrix} \eta^l x_1 \\ \eta^{-l} x_2 \\ x_3 \end{pmatrix}_c, \\
\mathbf{2}_1 \quad &: \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_a, \quad \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_b, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_c.
\end{aligned} \tag{B.4}$$

The first product below will demonstrate the procedure in defining the product spaces, but its worth noting that we always begin with one term. The importance of this is that each term will require us to refer to the above list to determine how that first term transforms under the action of any generator. These transformations will in general lead to new terms that should also be included in a product space. In this way we build a set of terms that together define a space of an irreducible representation.

- $\mathbf{2}_1 \otimes \mathbf{2}_1 = \mathbf{1}_1 + \mathbf{1}_2 + \mathbf{2}_1$. We demonstrate here how to construct the representations of the products, keeping in mind that the remainder products will follow along similar lines. First we start with a vector x, which transforms according to the first $\mathbf{2}_1$, and then we include y, which transforms as according to the other $\mathbf{2}_1$. Picking any term that is a product of these two spaces, we prefer to start with $x_1 y_1$, we note by Eq. (B.4) that

$$x_1 y_1 \mapsto_a w^2 x_1 y_1, \quad \mapsto_b x_2 y_2, \quad \mapsto_c x_1 y_1. \tag{B.5}$$

The transformations have led us to conclude that whatever vector space $x_1 y_1$ may occupy, $x_2 y_2$ must also. Thus similarly $x_2 y_2$ maps according to

$$x_2 y_2 \mapsto_a w x_2 y_2, \quad \mapsto_b x_1 y_1, \quad \mapsto_c x_2 y_2. \tag{B.6}$$

We now see that no new term has been produced, we can safely conclude that we have all the terms of this new vector space, and so we once again look at Eq. (B.4) and find that

$$\mathbf{2}_1 \quad : \quad \begin{pmatrix} x_2 y_2 \\ x_1 y_1 \end{pmatrix}, \tag{B.7}$$

as can be verified by applying the actions of the generators. Similarly we may look at the remainder possible combinations $x_1 y_2$ and $x_2 y_1$. If we perform the same analysis, we may guess that we have another two-dimensional representation. However on closer

[2] We omit the discussion of the generator d as it is not independent of a, b, and c, see Eq. (2.3)

inspection we would find that this two-dimensional representation is actually reducible. The two terms can be shown to occur in two linear combinations that form vector spaces of one-dimensional irreducible representations:

$$\mathbf{1}_1 \quad : \quad x_1y_2 + x_2y_1, \tag{B.8}$$

and

$$\mathbf{1}_2 \quad : \quad x_1y_2 - x_2y_1, \tag{B.9}$$

up to some normalization constant. From here on, we follow the same method to obtain the vector spaces that make up the Kronecker products, however we shall neglect the work and list only the results. Lastly, for those representations that do depend on a parameter (e.g. $\mathbf{3}_{1(l)}$) we determine the value of the parameter by noting how the representations transform under the action of c and then finding the corresponding values of the parameters as based on Eq. (B.4).

- $\mathbf{2}_1 \otimes \mathbf{3}_{1(l)} = \mathbf{3}_{1(l)} + \mathbf{3}_{2(l)}$. Vector x will transform as a $\mathbf{2}_1$ and y as a $\mathbf{3}_{1(l)}$.

$$\mathbf{3}_{1(l)} \quad : \quad \begin{pmatrix} x_1y_1 \\ \omega x_1y_2 \\ \omega^2 x_1y_3 \end{pmatrix} + \begin{pmatrix} \omega^2 x_2y_1 \\ \omega x_2y_2 \\ x_2y_3 \end{pmatrix}, \qquad \mathbf{3}_{2(l)} \quad : \quad \begin{pmatrix} x_1y_1 \\ \omega x_1y_2 \\ \omega^2 x_1y_3 \end{pmatrix} - \begin{pmatrix} \omega^2 x_2y_1 \\ \omega x_2y_2 \\ x_2y_3 \end{pmatrix}. \tag{B.10}$$

- $\mathbf{2}_1 \otimes \mathbf{3}_{2(l)} = \mathbf{3}_{1(l)} + \mathbf{3}_{2(l)}$. Vector x will transform as a $\mathbf{2}_1$ and y as a $\mathbf{3}_{2(l)}$.

$$\mathbf{3}_{1(l)} \quad : \quad \begin{pmatrix} x_1y_1 \\ \omega x_1y_2 \\ \omega^2 x_1y_3 \end{pmatrix} - \begin{pmatrix} \omega^2 x_2y_1 \\ \omega x_2y_2 \\ x_2y_3 \end{pmatrix}, \qquad \mathbf{3}_{2(l)} \quad : \quad \begin{pmatrix} x_1y_1 \\ \omega x_1y_2 \\ \omega^2 x_1y_3 \end{pmatrix} + \begin{pmatrix} \omega^2 x_2y_1 \\ \omega x_2y_2 \\ x_2y_3 \end{pmatrix}. \tag{B.11}$$

- $\mathbf{2}_1 \otimes \mathbf{6}_{(k,l)} = \mathbf{6}_{(k,l)} + \mathbf{6}_{(k,l)}$. Vector x will transform as a $\mathbf{2}_1$ and y as a $\mathbf{6}_{(k,l)}$.

$$\mathbf{6}_{(k,l)} \quad : \quad \begin{pmatrix} x_1y_1 \\ \omega x_1y_2 \\ \omega^2 x_1y_3 \\ x_2y_4 \\ \omega x_2y_5 \\ \omega^2 x_2y_6 \end{pmatrix}, \qquad \mathbf{6}_{(k,l)} \quad : \quad \begin{pmatrix} x_2y_1 \\ \omega^2 x_2y_2 \\ \omega x_2y_3 \\ x_1y_4 \\ \omega^2 x_1y_5 \\ \omega x_1y_6 \end{pmatrix}. \tag{B.12}$$

- $\mathbf{3}_{1(l)} \otimes \mathbf{3}_{1(l')} = \mathbf{3}_{1(l+l')} + \mathbf{6}_{\widetilde{(l,-l')}}$. Vector x will transform as a $\mathbf{3}_{1(l)}$ and y as a $\mathbf{3}_{1(l')}$.

$$\mathbf{3}_{1(l+l')} \quad : \quad \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}, \qquad \mathbf{6}_{(-l,l-l')} \quad : \quad \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_3y_2 \\ x_2y_1 \\ x_1y_3 \end{pmatrix}, \tag{B.13}$$

where we note that $(l, -l')$ is related to $(-l, l-l')$ via the matrix transformation M_s^p. Recall we want to use the standard values explaining the tilde used in the product.

- $\mathbf{3}_{1(l)} \otimes \mathbf{3}_{2(l')} = \mathbf{3}_{2(l+l')} + \mathbf{6}_{\widetilde{(l,-l')}}$. Vector x will transform as a $\mathbf{3}_{1(l)}$ and y as a $\mathbf{3}_{2(l')}$.

$$\mathbf{3}_{2(l+l')} \quad : \quad \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}, \qquad \mathbf{6}_{(-l,l-l')} \quad : \quad \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ -x_3y_2 \\ -x_2y_1 \\ -x_1y_3 \end{pmatrix}, \tag{B.14}$$

where $(l, -l')$ is related to $(-l, l-l')$ via the matrix transformation M_s^p.

- $\mathbf{3}_{1(l)} \otimes \mathbf{6}_{(k',l')} = \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} k' \\ l'-l \end{smallmatrix}\right)}} + \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} k'-l \\ l'+l \end{smallmatrix}\right)}} + \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} l+k' \\ l' \end{smallmatrix}\right)}}$. Vector x will transform as a $\mathbf{3}_{1(l)}$ and y as a $\mathbf{6}_{(k',l')}$.

$$\mathbf{6}_{\left(\begin{smallmatrix} l'-l \\ l-k'-l' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \\ x_3y_6 \\ x_2y_4 \\ x_1y_5 \end{pmatrix}, \qquad \mathbf{6}_{\left(\begin{smallmatrix} k'-l \\ l'+l \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \\ x_3y_4 \\ x_2y_5 \\ x_1y_6 \end{pmatrix}, \qquad \mathbf{6}_{\left(\begin{smallmatrix} -l-k'-l' \\ l+k' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_3y_5 \\ x_2y_6 \\ x_1y_4 \end{pmatrix}, \tag{B.15}$$

where $(l'-l, l-k'-l')$ is related to $(k', l'-l)$ by the matrix transformation M_s^p, and similarly $(-l-k'-l', l+k')$ is related to $(l+k', l')$.

- $\mathbf{3}_{2(l)} \otimes \mathbf{3}_{2(l')} = \mathbf{3}_{1(l+l')} + \mathbf{6}_{\widetilde{(l,-l')}}$. Vector x will transform as a $\mathbf{3}_{2(l)}$ and y as a $\mathbf{3}_{2(l')}$.

$$\mathbf{3}_{1(l+l')} : \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}, \qquad \mathbf{6}_{(-l,l-l')} : \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_3y_2 \\ x_2y_1 \\ x_1y_3 \end{pmatrix}, \tag{B.16}$$

where we note that $(l, -l')$ is related to $(-l, l-l')$ via the matrix transformation M_s^p.

- $\mathbf{3}_{2(l)} \otimes \mathbf{6}_{(k',l')} = \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} k' \\ l'-l \end{smallmatrix}\right)}} + \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} k'-l \\ l'+l \end{smallmatrix}\right)}} + \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} l+k' \\ l' \end{smallmatrix}\right)}}$. Vector x will transform as a $\mathbf{3}_{2(l)}$ and y as a $\mathbf{6}_{(k',l')}$.

$$\mathbf{6}_{\left(\begin{smallmatrix} l'-l \\ l-k'-l' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \\ -x_3y_6 \\ -x_2y_4 \\ -x_1y_5 \end{pmatrix}, \qquad \mathbf{6}_{\left(\begin{smallmatrix} k'-l \\ l'+l \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \\ -x_3y_4 \\ -x_2y_5 \\ -x_1y_6 \end{pmatrix}, \qquad \mathbf{6}_{\left(\begin{smallmatrix} -l-k'-l' \\ l+k' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ -x_3y_5 \\ -x_2y_6 \\ -x_1y_4 \end{pmatrix}, \tag{B.17}$$

where $(l'-l, l-k'-l')$ is related to $(k', l'-l)$ by the matrix transformation M_s^p, and similarly $(-l-k'-l', l+k')$ is related to $(l+k', l')$.

- $\mathbf{6}_{(k,l)} \otimes \mathbf{6}_{(k',l')} = \sum_{p,s} \mathbf{6}_{\widetilde{\left(\left(\begin{smallmatrix} k \\ l \end{smallmatrix}\right) + M_s^p \left(\begin{smallmatrix} k' \\ l' \end{smallmatrix}\right)\right)}}$ Vector x will transform as a $\mathbf{6}_{(k,l)}$ and y as a $\mathbf{6}_{(k',l')}$.

$$\mathbf{6}_{\left(\begin{smallmatrix} k+k' \\ l+l' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \\ x_4y_4 \\ x_5y_5 \\ x_6y_6 \end{pmatrix}, \qquad \mathbf{6}_{\left(\begin{smallmatrix} k-k'-l' \\ l+k' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_4y_5 \\ x_5y_6 \\ x_6y_4 \end{pmatrix}, \qquad \mathbf{6}_{\left(\begin{smallmatrix} k+l' \\ l-l'-k' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \\ x_4y_6 \\ x_5y_4 \\ x_6y_5 \end{pmatrix},$$

$$\mathbf{6}_{\left(\begin{smallmatrix} k-k' \\ l+k'+l' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_4 \\ x_2y_6 \\ x_3y_5 \\ x_4y_1 \\ x_5y_3 \\ x_6y_2 \end{pmatrix}, \qquad \mathbf{6}_{\left(\begin{smallmatrix} k+k'+l' \\ l-l' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_5 \\ x_2y_4 \\ x_3y_6 \\ x_4y_2 \\ x_5y_1 \\ x_6y_3 \end{pmatrix}, \qquad \mathbf{6}_{\left(\begin{smallmatrix} k-l' \\ l-k' \end{smallmatrix}\right)} : \begin{pmatrix} x_1y_6 \\ x_2y_5 \\ x_3y_4 \\ x_4y_3 \\ x_5y_2 \\ x_6y_1 \end{pmatrix}. \tag{B.18}$$

(*ii*) $n = 3\mathbb{Z}$. The only major difference between the case here and case (*i*) is the addition of three more two-dimensional representations. Here, we limit ourselves to all products which involve the irreducible representation $\mathbf{2_2}$. The products involving $\mathbf{2_3}$ and $\mathbf{2_4}$ are similar in structure and can be determined by the reader following similar lines. To begin, the additional two-dimensional representations transform under a, b, and c as:

$$\begin{aligned}
\mathbf{2}_2 \quad &: \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_a, \quad \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_b, \quad \begin{pmatrix} \omega^2 x_1 \\ \omega x_2 \end{pmatrix}_c, \\
\mathbf{2}_3 \quad &: \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_a, \quad \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_b, \quad \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_c, \\
\mathbf{2}_4 \quad &: \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_a, \quad \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}_b, \quad \begin{pmatrix} \omega x_1 \\ \omega^2 x_2 \end{pmatrix}_c.
\end{aligned} \tag{B.19}$$

- $\mathbf{2}_2 \otimes \mathbf{2}_1 = \mathbf{2}_3 + \mathbf{2}_4$. The quantity x transforms as a $\mathbf{2}_2$ and y as a $\mathbf{2}_1$.

$$\mathbf{2}_3 \quad : \quad \begin{pmatrix} x_2 y_2 \\ x_1 y_1 \end{pmatrix}, \qquad \mathbf{2}_4 \quad : \quad \begin{pmatrix} x_2 y_1 \\ x_1 y_2. \end{pmatrix}. \tag{B.20}$$

- $\mathbf{2}_r \otimes \mathbf{2}_r = \mathbf{1}_1 + \mathbf{1}_2 + \mathbf{2}_r$. We can generalize since the result is true regardless of the choice of $r = 1, 2, 3, 4$. The quantity x and y both transform as $\mathbf{2}_r$.

$$\mathbf{2}_r \quad : \quad \begin{pmatrix} x_2 y_2 \\ x_1 y_1 \end{pmatrix}. \qquad \mathbf{1}_1 \quad : \quad x_1 y_2 + x_2 y_1, \qquad \mathbf{1}_2 \quad : \quad x_1 y_2 - x_2 y_1, \tag{B.21}$$

- $\mathbf{2}_2 \otimes \mathbf{2}_3 = \mathbf{2}_1 + \mathbf{2}_4$. The quantity x transforms as a $\mathbf{2}_2$ and y as a $\mathbf{2}_3$.

$$\mathbf{2}_1 \quad : \quad \begin{pmatrix} x_2 y_2 \\ x_1 y_1 \end{pmatrix}, \qquad \mathbf{2}_4 \quad : \quad \begin{pmatrix} x_1 y_2 \\ x_2 y_1. \end{pmatrix}. \tag{B.22}$$

- $\mathbf{2}_2 \otimes \mathbf{2}_4 = \mathbf{2}_1 + \mathbf{2}_3$. The quantity x transforms as a $\mathbf{2}_2$ and y as a $\mathbf{2}_4$.

$$\mathbf{2}_1 \quad : \quad \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}, \qquad \mathbf{2}_3 \quad : \quad \begin{pmatrix} x_1 y_2 \\ x_2 y_1. \end{pmatrix}. \tag{B.23}$$

- $\mathbf{2}_2 \otimes \mathbf{3}_{1(l)} = \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} 2n/3 - l \\ 2n/3 + l \end{smallmatrix}\right)}}$. The quantity x transforms as a $\mathbf{2}_2$ and y as a $\mathbf{3}_{1(l)}$.

$$\mathbf{6}_{\left(\begin{smallmatrix} 2n/3 - l \\ 2n/3 + l \end{smallmatrix}\right)} \quad : \quad \begin{pmatrix} x_1 y_1 \\ \omega x_1 y_2 \\ \omega^2 x_1 y_3 \\ x_2 y_3 \\ \omega x_2 y_2 \\ \omega^2 x_2 y_1 \end{pmatrix}. \tag{B.24}$$

- $\mathbf{2}_2 \otimes \mathbf{3}_{2(l)} = \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} 2n/3 - l \\ 2n/3 + l \end{smallmatrix}\right)}}$. The quantity x transforms as a $\mathbf{2}_2$ and y as a $\mathbf{3}_{2(l)}$.

$$\mathbf{6}_{\left(\begin{smallmatrix} 2n/3 - l \\ 2n/3 + l \end{smallmatrix}\right)} \quad : \quad \begin{pmatrix} x_1 y_1 \\ \omega x_1 y_2 \\ \omega^2 x_1 y_3 \\ -x_2 y_3 \\ -\omega x_2 y_2 \\ -\omega^2 x_2 y_1 \end{pmatrix}. \tag{B.25}$$

- $\mathbf{2}_2 \otimes \mathbf{6}_{(k,l)} = \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} k + n/3 \\ l + n/3 \end{smallmatrix}\right)}} + \mathbf{6}_{\widetilde{\left(\begin{smallmatrix} k + 2n/3 \\ l + 2n/3 \end{smallmatrix}\right)}}$. The quantity x transforms as a $\mathbf{2}_2$ and y as a $\mathbf{6}_{(k,l)}$.

$$\mathbf{6}_{\binom{k+n/3}{l+n/3}} \quad : \quad \begin{pmatrix} x_2 y_1 \\ \omega^2 x_2 y_2 \\ \omega x_2 y_3 \\ x_1 y_4 \\ \omega^2 x_1 y_5 \\ \omega x_1 y_6 \end{pmatrix}, \qquad \mathbf{6}_{\binom{k+2n/3}{l+2n/3}} \quad : \quad \begin{pmatrix} x_1 y_1 \\ \omega x_1 y_2 \\ \omega^2 x_1 y_3 \\ x_2 y_4 \\ \omega x_2 y_5 \\ \omega^2 x_2 y_6 \end{pmatrix}. \tag{B.26}$$

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