
Max Dehn and the Origins of Topology and Infinite Group Theory

David Peifer

Abstract. This article provides a brief history of the life, work, and legacy of Max Dehn. The emphasis is put on Dehn's papers from 1910 and 1911. Some of the main ideas from these papers are investigated, including Dehn surgery, the word problem, the conjugacy problem, the Dehn algorithm, and Dehn diagrams. A few examples are included to illustrate the impact that Dehn's work has had on subsequent research in logic, topology, and geometric group theory.

1. INTRODUCTION. It is now one hundred years since Max Dehn published his papers *On the Topology of Three-Dimensional Spaces* (1910) [9] and *On Infinite Discontinuous Groups* (1911) [10]. At that time, topology was in its infancy. Group theory was also new and the study of general infinite groups had only just begun. Dehn's papers mark the beginning of two closely related research areas: the study of 3-manifolds in topology, and the study of infinite groups given by presentations in algebra. In these papers, Dehn brought together important ideas from the late 1800's concerning hyperbolic geometry, group theory, and topology. These papers are beautifully written and exhibit Dehn's gifted ability to use simple combinatorial diagrams to illustrate foundational connections between algebra and geometry. In the years since Dehn's passing, mathematicians have gone back to these two papers again and again to draw motivation and inspiration. This article is written as a celebration of these two papers and of Max Dehn's influence on one hundred years of mathematics.

Over the last century, Dehn's ideas have touched many areas of mathematics. There are connections between Dehn's work and early work in logic. Dehn's research motivated concepts in combinatorial group theory and low dimensional topology. Even decades after his death, Dehn's ideas have influenced recent research in geometric group theory by Mikhail Gromov, William Thurston, and others. Such a survey necessitates a need to mention precise mathematical results and use some advanced mathematical terms. Readers who have taken a course in group theory and one in topology will be familiar with most of the terminology. However, a student who has completed only an introductory course in algebra could understand many of the main ideas and will learn some recent mathematics history.

In a paper of this length, there is not room to give more than a sketch of most of the mathematics. An attempt has been made to illustrate the main ideas with simple examples and figures. References are given throughout the article for readers interested in more detailed mathematics background. Many of these are written at the advanced undergraduate level. General references are also provided to more advanced research and mathematical history texts. It is hoped that this article will only be a start and that readers will use the references to further their own investigation of Dehn and his legacy.

2. THE BEGINNING. Max Dehn was born on November 13, 1878, in Hamburg, Germany. He was one of eight siblings in a prosperous family. His father, Maximilian

<http://dx.doi.org/10.4169/amer.math.monthly.122.03.217>

MSC: Primary 01A60, Secondary 20-03

Moses Dehn, was a physician. Dehn's eldest brother became a distinguished lawyer, his other brothers went into business or traveled, and one of Dehn's sisters played violin in the Hamburg Opera orchestra. Dehn's early family life instilled in him a lifelong respect for the sciences, education, and the arts. Dehn attended gymnasium in Hamburg, then began his university work in Freiburg before moving to Göttingen. By the age of 21, with the dawning of a new century, Dehn was finishing his education and was beginning his career in mathematics.

Dehn received his doctorate in 1900 at the University of Göttingen working with David Hilbert. This was the year that Hilbert delivered his famous speech at the International Congress of Mathematicians, in Paris, and presented his list of problems for the next century. Geometry had been in turmoil a century earlier, when hyperbolic geometry offered an alternative to the Euclidean axioms. This put into question what had seemed basic geometric assumptions and motivated a closer look at the foundations of geometry. But by the time of Hilbert's speech, mathematicians were putting geometry, and all of mathematics, on the more rigorous foundations of set theory and modern logic. There had been great successes. Our understanding of geometry during the 1800's had grown from a simple Euclidean perspective to the more abstract and modern perspectives of mathematicians like Bernhard Riemann, David Hilbert, and others. Hilbert's speech marked a time of great pride and hope in the mathematics community.

A year earlier, in 1899, Hilbert had published his text book *Foundations of Geometry* [31], in which he provides a modern axiomatization of Euclidean geometry. Hilbert's system includes 20 axioms (21 in the first edition). One of Hilbert's axioms in particular would have caused concern to Euclid and other ancient Greek philosophers. The *Archimedean axiom* relies on a modern perspective of infinity that the Greeks (other than Archimedes) did not accept. A simple version of the axiom is that, given two lengths, a and b , in the geometry, there is a natural number $n \in \mathbb{N}$ such that $na > b$. This implies a type of actual infinity that the Greeks did not accept. Dehn's first two results are important historically because they point out places where this concept of infinity is needed.

For his thesis in 1900, Dehn showed that, in Euclidean geometry, to prove that the sum of the angles of a triangle does not exceed 180 degrees requires the Archimedean axiom. Note that Dehn had not only shown that there is a proof using the Archimedean axiom, but more importantly, that any proof of this theorem must at some point use the Archimedean axiom; the other 19 axioms will not suffice. Using modern logic and Hilbert's axioms for Euclidean geometry, Dehn had answered a question from antiquity. Dehn had shown that even some of the most seemingly obvious facts about Euclidean geometry require the modern concept of infinity, which the ancient Greeks did not embrace.

The following year, for his habilitation at the University of Münster, Dehn became the first to solve one of Hilbert's famous problems. Dehn solved Hilbert's third problem by showing that the Archimedean axiom is logically needed to prove that tetrahedra of equal base and height have equal volume. Note that the corresponding problem in two dimensions is easy and was known to the ancient Greeks. The formula for the area of a triangle is easy to prove geometrically by dissecting the triangle in a finite number of straight cuts to form a rectangle with the same base and one half the height of the original triangle. The proof that the area of a triangle is $A = \frac{1}{2}bh$ does not require the use of any infinite process or, therefore, the Archimedean axiom. The volume of a tetrahedron has a similar simple formula, $V = \frac{1}{3}bh$, where b is the area of the base and h the height. Essentially, Hilbert's third problem asked if the proof of this volume formula requires some kind of infinite process. Dehn proved that the answer is yes.

3. GROUPS AND GEOMETRY IN THE LATE 1800'S. As Dehn began his career, two new areas of mathematics were developing, *infinite group theory* and *topology*. Group theory was only a few decades old. Arthur Cayley first gave the abstract definition of a group in 1854. Specific groups had been studied before that time. Euler in 1761, and later Gauss in 1801, had studied modular arithmetic. Lagrange in 1770, and Cauchy in 1844, had studied permutations. Research by Abel and Galois preceded Cayley's definition by a few decades. Cayley unified these works under the field of *group theory* by providing an axiomatic abstract definition for a group. In 1878, Cayley wrote four papers highlighting his abstract point of view. In one of these papers, Cayley introduced a combinatorial graph associated to a group (with a given set of generators). The *Cayley graph* is a directed graph with exactly one vertex for each group element. Edges are labeled by group generators. If g is a group element and x a group generator, then there is an edge labeled by x from the vertex g to the vertex xg . This graph has some very nice properties. (For more details, see [30], [34], or [36].)

Although Cayley's work generalizes to the infinite case, Cayley was concerned primarily with finite groups. With finite groups the group operation can be completely described by a finite table or chart. When studying infinite groups, it can be difficult to describe the group operation precisely. In two papers from 1882 [22] and 1883 [21], Walther von Dyck introduced a way to present a group in terms of generators and relators. Given a set of group generators, *relators* are strings of generators and inverse generators that, under the group operation, result in the identity. For example, in $\mathbb{Z} \times \mathbb{Z}$, the free Abelian group generated by a and b , the string $aba^{-1}b^{-1}$ is equivalent to the identity, thus it is a relator in the group. A group *presentation* is a set of generators and a set of relators on the generators. The group defined by a presentation is the most general group with the given number of generators that satisfies the relators. (For more details, see [24] or [44].) Dyck's definition of a group presentation provided the needed tool to begin a general study of infinite groups. Figure 1 contains a few examples of the Cayley graphs and presentations for some familiar groups.

Geometers in the late 1800's developed analytic tools and models for understanding and studying hyperbolic geometry. One such model is the Poincaré disk model of the hyperbolic plane. (This model was discovered by Eugenio Beltrami.) In this model, the hyperbolic plane is mapped to the interior of a Euclidean disk. The mapping distorts hyperbolic distance by making lengths appear smaller as they approach the boundary of the disk. With this metric, the boundary of the disk can be thought of as hyperbolic points at infinity. It can be shown that shortest paths (called geodesics) appear in this model as portions of Euclidean circles that intersect the boundary circle at right angles. Figure 2 shows the Poincaré disk model with a tessellation by octagons of equal area. Notice that the octagon edges are geodesic lines.

Out of the efforts made in the 1800's to understand geometry the new discipline of topology was being born. Topology can be understood as the study of the shape of spaces, without regard to the exact distances between points. Topology is geometry without the metric. By the end of the 1800's, mathematicians had already made important connections between the algebraic properties of groups and the geometric and topological properties of spaces. In 1872, Felix Klein outlined his *Erlangen program* in which symmetry groups were used to classify and organize geometries. In 1884, Sophus Lie began studying a new type of group which included an additional topological structure. This work led to the theory of *Lie groups*. Henri Poincaré, in 1895, defined the fundamental group of a space and showed that it was a topological invariant of the space. This started the study of *algebraic topology*. From the beginnings of our modern abstract view of spaces, ideas from group theory and geometry have been deeply intertwined.

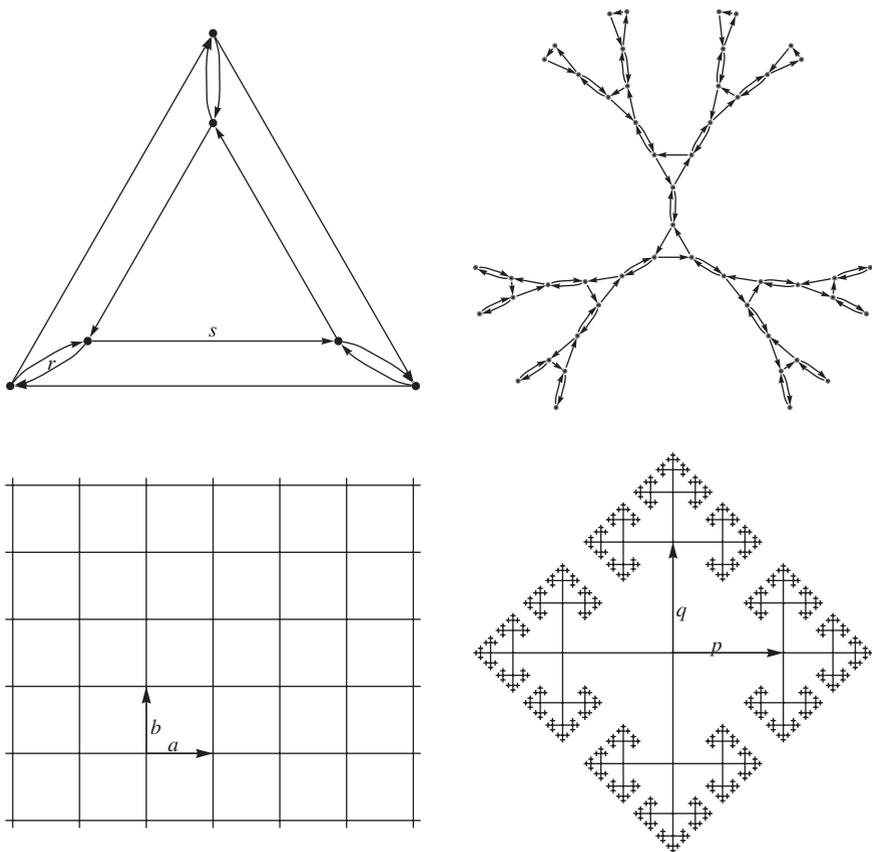


Figure 1. The Cayley graphs of $D_3 = \langle s, r \mid s^3, r^2, srsr \rangle$ (the dihedral group of order 6), $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle x, y \mid x^2, y^3 \rangle$ (the free product of \mathbb{Z}_2 and \mathbb{Z}_3), $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle$ (the free Abelian group of rank 2), and $F_2 = \langle p, q \rangle$ (the free group on two generators). The Cayley graph for D_3 is finite. The other three figures represent portions of infinite Cayley graphs. A few edges are labeled to indicate the general pattern.

To define Poincaré’s fundamental group of a space, first pick a base point in the space, x_0 . We will assume the space is path connected, so that the choice of x_0 is inconsequential. Now consider the set of paths in the space that begin and end at x_0 . We define two paths as equivalent if one path can be continuously deformed into the other while keeping the endpoints fixed at x_0 . (This deformation is called a *homotopy* and the two equivalent paths are said to be *homotopic*.) The group elements are the equivalence classes of paths. The product of two paths, a and b , is the path that follows a first and then follows b . For example, on a sphere, all paths can be deformed to the base point x_0 . Thus there is one equivalence class of paths and the fundamental group is trivial. However on the torus, where paths can not be continuously deformed across the hole in the middle or the interior of the torus, the fundamental group is $\mathbb{Z} \times \mathbb{Z}$. For more information on fundamental groups see [37] or [48].

Homeomorphisms play a key role in topology. A *homeomorphism* is a continuous function that is invertible and has a continuous inverse. In topology, two spaces are considered equivalent if there is a homeomorphism mapping one space to the other. In this case, we say the spaces are homeomorphic. Manifolds are one of the most important types of spaces studied in topology. An *n-manifold without boundary* is a space that satisfies two properties. First of all, it must be a *Hausdorff space*. This means that for any two distinct points, x and y in the space, there exists a pair of disjoint

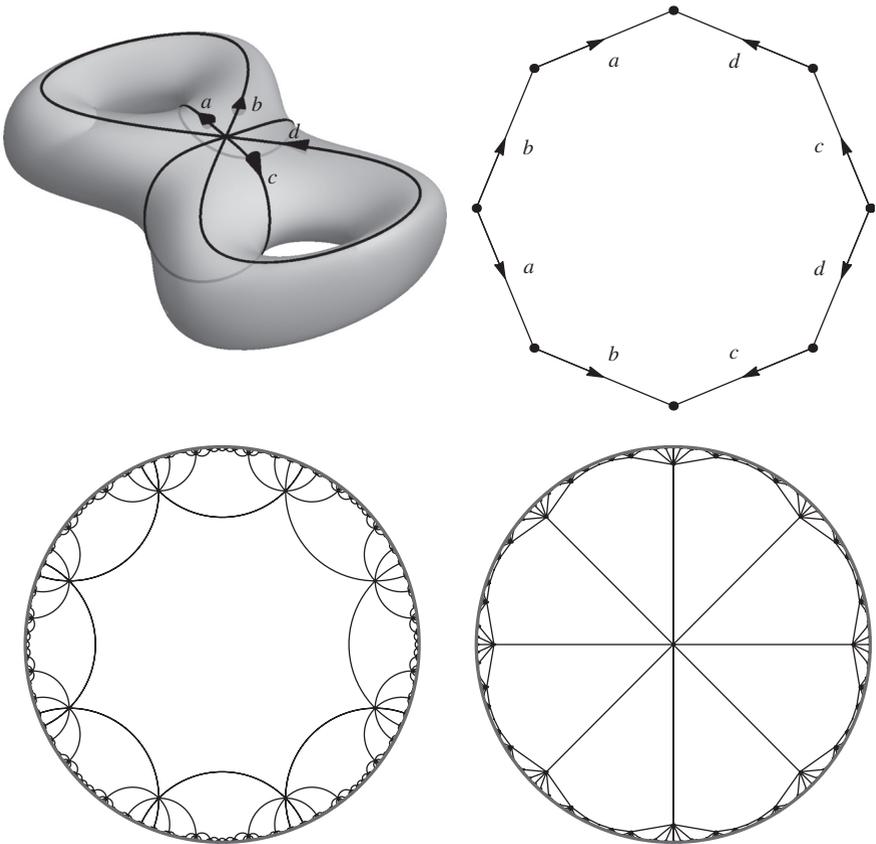


Figure 2. The surface of genus two S is topologically equivalent to the octagon with edges identified as shown. The top row shows the unfolding of this octagon. The bottom left figure is the hyperbolic plane \mathbb{H}^2 tiled by a regular octagon. (In topological terms, \mathbb{H}^2 is the universal covering space of S .) The figure on the bottom right is the Cayley graph of the fundamental group of S . Notice it is the dual graph of the tiling by octagons of \mathbb{H}^2 .

neighborhoods such that x is in one and y is in the other. The second property is that each point has a neighborhood homeomorphic to an open Euclidean n -ball. That is, in at least some small neighborhood of each point, the space looks like Euclidean n -space. The Möbius band is an example of a 2-manifold with a single boundary component. A *surface* is a 2-manifold without boundary. A surface is said to be *orientable* if it does not contain a homeomorphic copy of the Möbius band. An important problem in topology is to classify the n -manifolds up to homeomorphism. By the late 1800's mathematicians understood the classification for 2-manifolds. For example, they knew that the closed orientable surfaces are homeomorphic to a sphere, a torus, or a shape that looks like a multiholed torus. The number of holes, g , is called the genus of the surface.

Of particular importance for Dehn were the connections Klein and Poincaré made between hyperbolic geometry and the fundamental groups of closed orientable surfaces, or the *surface groups*. In his papers of 1895 and 1904 [41], Poincaré had already established a deep understanding of this connection. Poincaré understood that a surface of genus $g > 0$ could be represented by a $4g$ -gon, with a specific gluing pattern on the edges. Copies of this $4g$ -gon can tessellate a Euclidean or a hyperbolic plane. In the case of the torus ($g = 1$), the tessellation is the standard square tiling of the Euclidean plane. For all other values of $g > 1$, a hyperbolic plane is required.

The elements of the surface group act on the hyperbolic plane by translations that map the net of $4g$ -gons onto itself. In modern terms we would say that the surface group acts freely on this hyperbolic tessellation by translations. In 1904 [41], Poincaré understood that the hyperbolic plane is the universal cover for this group action. The details are more than we have space for here. The point is that Poincaré had already noticed this deep connection and that Dehn had knowledge and respect for Poincaré's work. Figure 2 illustrates Poincaré's ideas for a surface of genus two. The figure includes the fundamental group generators, the associated octagon, the hyperbolic tessellation, and the group Cayley graph.

4. DEHN'S CAREER BEFORE WWI. Dehn's work helped to direct and establish topology as a discipline in its earliest years. With co-author Poul Heegaard in 1907, Dehn published the first comprehensive article on topology (known at that time as *analysis situs*) [20]. The article appeared in the German Encyclopedia of Mathematical Sciences. Their article contained the first rigorous proof of the classification of surfaces. Both Möbius and Dyck had published incomplete proofs years earlier. It should be noted that in their paper, Dehn and Heegaard had assumed that all surfaces can be triangulated. This fact was not actually proven until years later by Tibor Radó [43]. The triangulation assumption led to the kind of combinatorial cutting and pasting argument that exemplified Dehn's style.

Dehn's mathematics was at a peak during the years 1910–1914. During that time he published a series of papers that pulled together diverse ideas in group theory, geometry, and topology. These papers are steeped in the ideas of Dehn's contemporaries. Dehn's contribution was his ability to bring these ideas together to form a comprehensive program and to provide tools and techniques for future study. Dehn's 1910–1914 papers have been translated into English by John Stillwell [19]. These translations, from 1987, include Stillwell's notes on the historical context and the modern significance of Dehn's work.

As Dehn's earliest work had reached back to geometry of the past, Dehn's work during this period would reach forward, asking questions that would occupy the thoughts of mathematicians for the next hundred years. In particular, two of Dehn's papers from this prewar period stand out as some of the most important works of the last century. Dehn's 1910 paper, *On the Topology of Three-Dimensional Spaces* [9], marks the beginning of the search to classify 3-manifolds. Dehn's 1911 paper, *On Infinite Discontinuous Groups* [10], began the study of infinite groups given by finite presentations.

Dehn's paper of 1910 provided the first steps in a comprehensive program to study 3-manifolds. Only a few years before, Dehn and Heegaard had presented their proof of the classification of 2-manifolds. Dehn's 1910 paper marked the beginning of a program to find a similar classification of 3-manifolds. Several of the ideas and techniques from Dehn's paper have continued to play an important role in the study of 3-manifolds even to this day.

Early on, Dehn recognized the importance of *knot theory* in the study of 3-manifolds. A *knot* is a simply connected closed curve in \mathbb{R}^3 . That is, a knot is a curve in \mathbb{R}^3 that begins and ends at the same point, but otherwise does not intersect itself. We are concerned here with what are called *tame knots*. These knots can be realized as simply connected closed polygonal curves made up of a finite number of line segments. This restriction eliminates wild behavior that would result in a knot with infinite tangling.

In his 1910 paper, Dehn introduces a technique now known as *Dehn surgery*. This is a process that can be used to generate new 3-manifolds from old ones. The process involves drilling a knot out from inside a 3-manifold M , more accurately, removing a

tubular neighborhood of a knot from M . This increases the boundary of M by adding an additional boundary component T . Note that T is equivalent to a torus surface. Now glue a solid torus back into the empty space left after removing the knot from M . Essentially a solid torus is being removed from M and another is being glued in its place. However, the second solid torus is glued in so that the meridian of its surface is mapped to a nontrivial simple closed curve (other than the original meridian) on T . Think of this as twisting the solid torus before it is glued in place.

An important result from Dehn's 1910 paper has to do with homology theory. In his early work in topology, Poincaré had developed homology theory as a tool to help distinguish manifolds. If a manifold can be given a certain type of "cellular decomposition," then Poincaré could associate a list of numbers to the cellular structure. He was able to show that these numbers were independent of the specific cellular decomposition and were the same for homeomorphic spaces. (This work generalized earlier works of Leonhard Euler and Enrico Betti.) In the modern point of view, Poincaré's numbers are associated with a list of Abelian groups called the homology groups for the space. Although Poincaré knew that his homology theory would not distinguish all manifolds, early on he thought that it could be used to identify the 3-sphere. (The 3-sphere is the set of points in \mathbb{R}^4 that are exactly one unit from the origin.) In a paper from 1904 [41], Poincaré found a 3-manifold that (in modern terms) had the same homology groups as the 3-sphere, yet had a nontrivial fundamental group, and thus was not homeomorphic to the 3-sphere. Today, a 3-manifold that has the same homology groups as the 3-sphere, yet has a nontrivial fundamental group, is called a *homology sphere*. In his 1910 paper, Dehn used his surgery to create an infinite set of examples of homology spheres. Before Dehn's work, only Poincaré's original example was known.

It should also be pointed out that in the 1904 paper mentioned above, Poincaré makes his famous conjecture. Poincaré had shown that homology theory could not distinguish the 3-sphere. He then wondered if homotopy theory and the fundamental group could be used instead. The *Poincaré conjecture* asserts his conviction that the fundamental group is sufficient to distinguish the 3-sphere. That is, Poincaré conjectured that if a 3-manifold has a trivial fundamental group, then it is homeomorphic to the 3-sphere.

In his 1910 paper, Dehn made important connections between group theory and topology. In this paper, Dehn uses Poincaré's fundamental group to study knots. Notice that every knot is homeomorphic to the unit circle. Therefore to a topologist, who sees homeomorphic things as equivalent, the knot itself is not very interesting. What distinguishes knots to a topologist is the space around the knot, called the *knot complement*. To construct the complement of a knot, start with the knot embedded in \mathbb{R}^3 . Remove from \mathbb{R}^3 a tubular neighborhood of the knot. The 3-manifold that results is called the knot complement. For any given knot, the associated *knot group* is the fundamental group of the knot complement. In the 1910 paper, Dehn proved that any nontrivial knot has a noncommutative knot group. To prove this result, he used a lemma now known as the *Dehn lemma*. Dehn's proof of the lemma contained a gap that went undetected until 1929 when it was noticed by Hellmuth Kneser. Kneser informed Dehn of the problem and the two corresponded but could not fix the proof. A correct proof of Dehn's lemma was finally given by Christos Dimitriou Papakyriakopoulos in 1957 [40].

In his 1911 paper, *On Infinite Discontinuous Groups*, Dehn developed a comprehensive perspective for studying infinite groups given by finite presentations. A presentation is finite if both the set of generators and the set of relators are finite. Dehn had seen that infinite groups frequently come up in topology. For example, the surface groups are infinite and the fundamental groups of the homology spheres are infinite

(except for Poincaré's original example, which has order 120). Dehn realized that a study of infinite groups was needed.

In the 1911 paper, Dehn presented the *word problem*, *conjugacy problem*, and *isomorphism problem* for groups given by finite presentations. He actually considered these problems in his 1910 paper and would address them again in 1912. However, the 1911 paper is where Dehn lays out his general program. The term *word* is meant to indicate a finite string of generators and inverse generators. Notice that, if we were to perform the group operation, each word corresponds to a specific group element. However, each group element corresponds to infinitely many words. The empty string is included as a word and corresponds to the identity in the group. Simply stated, the word problem for a group (given by a presentation) asks for a process that can determine, in a finite number of steps, if a word represents the identity in the group. A solution to the word problem is an algorithm that can take as input any word and determine whether or not it is equivalent to the empty word. The algorithmic nature of all three of Dehn's problems foreshadowed and motivated ideas in logic and complexity theory that would emerge a few decades later.

In more modern terms, Dehn's word problem asks whether the set of words that are equivalent to the identity is a recursive set. This is about the simplest question one could ask about a group. Can we recognize the identity? The conjugacy and isomorphism problems are similar in nature. Both ask for algorithms to answer fundamental questions about group presentations. The conjugacy problem asks for an algorithm that can determine if two words represent conjugate elements in the group. The isomorphism problem asks for an algorithm to determine if two presentations represent the same group.

If one is considering Poincaré's fundamental group of a topological space, then the word and conjugacy problems correspond to questions about the space. The word problem corresponds to determining whether or not a loop in the space can be continuously deformed to a point; that is, is the loop homotopic to a point? The conjugacy problem corresponds to determining whether or not a pair of curves in the space can be continuously deformed to each other; that is, are the curves homotopic? Over the last century, the word, conjugacy, and isomorphism problems have become a centerpiece in the study of groups given by presentations.

One of Dehn's great insights was to use geometric properties of the Cayley graph of a group to answer algebraic questions about the group. By considering each edge in the Cayley graph as an unit interval, the Cayley graph can be given a metric. This is known as the *word metric*. The distance between two vertices (or group elements) is the shortest edge path between the vertices. Note that this path will be labeled by a word, that is, a string of generators and inverse generators. (Traversing an edge backwards corresponds to an inverse generator.) Thus Dehn had found a distance function for the group. The distance between two group elements is the length of the word that labels the shortest path between them in the Cayley graph.

Dehn was also able to find a kind of area function for a group. As Dyck's work made clear, any word equal to the identity in the group must be the product of conjugates of relators for the group. Dehn found a combinatorial diagram that represented this fact (known as a Dehn diagram, van Kampen diagram, or small cancellation diagram). This diagram is planar, simply connected, and made up of smaller regions that are each labeled by the group relators. The boundary for the whole diagram is labeled by the word that is equivalent to the identity. Figure 3 gives an example of a Dehn diagram for a word equivalent to the identity in $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle$. Using this diagram, Dehn had essentially defined area in the group. A word equivalent to the identity forms a closed path (or loop) in the Cayley graph. The area enclosed by

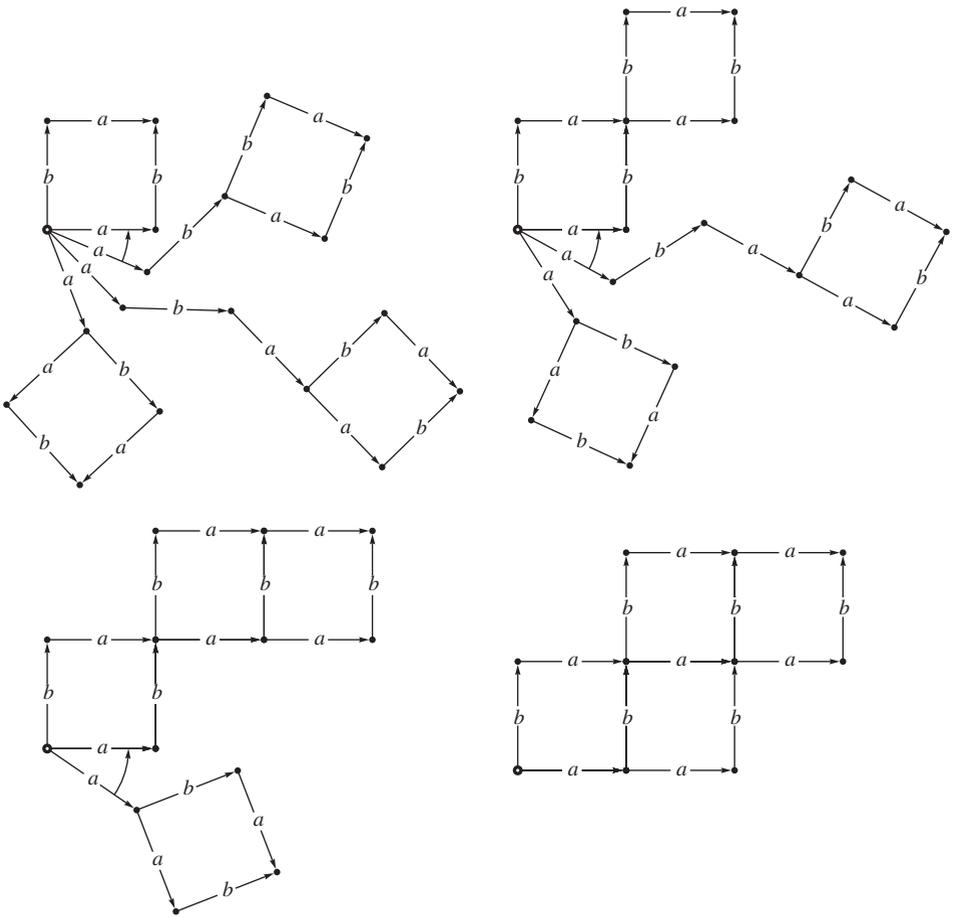


Figure 3. Consider the word $w = a^2 b a b a^{-2} b^{-1} a^{-1} b^{-1}$ in the group $\mathbb{Z} \times \mathbb{Z}$ given by the presentation $\langle a, b \mid aba^{-1}b^{-1} \rangle$. The word w is equivalent to the identity in $\mathbb{Z} \times \mathbb{Z}$. This follows because, in the free group on $\{a, b\}$, w is equivalent to a product of conjugates of the relator, $w' = [a(aba^{-1}b^{-1})a^{-1}][aba(aba^{-1}b^{-1})a^{-1}b^{-1}a^{-1}][ab(aba^{-1}b^{-1})b^{-1}a^{-1}][(aba^{-1}b^{-1})]$. (The words w' and w are equivalent in the free group because repeated cancellation of inverse pairs, $aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b$, reduces w' to w .) The word w' labels the boundary, going around counter-clockwise, of the top left diagram. Notice that the regions are labeled by the relator and the edge paths connecting these to a base point are labeled by the conjugating elements. By folding that diagram along identical labeled edges, one creates a diagram with boundary labeled by $w = a^2 b a b a^{-2} b^{-1} a^{-1} b^{-1}$ and each interior region labeled by the relator. The bottom right is a Dehn diagram for w .

this loop is the minimum number of regions needed to construct a Dehn diagram for the word. Or put more algebraically, this says that a word equivalent to the identity encloses an area equal to the minimum number of relators needed to write the word as a product of conjugates of those relators. Dehn had found a way to relate the algebraic structure of words and group elements to the geometric structure of the group's Cayley graph. (For more details, see [3], [34], or [36].)

An important result in Dehn's 1911 paper is the solution to the word and conjugacy problems for the surface groups. These results actually appear in three of Dehn's papers at this time. The earliest proof relies more on hyperbolic geometry and his final proof is almost entirely combinatorial. Dehn's proof uses the tessellations of the hyperbolic plane studied by Klein and Poincaré. With the concepts of distance and area defined above, Dehn was able to relate the algebraic word problem to the geometry of

these regular tessellations. Using hyperbolic geometry, Dehn proved that, for the surface groups (of genus 2 or more and given by the standard presentation), any nonempty word equal to the identity can be converted to a shorter word also equivalent to the identity. In fact, Dehn showed that if a nonempty word is equivalent to the identity, then the word must contain a subword that is more than half of a relator. By replacing this subword with the inverse of the shorter half of the relator, the word can be shortened and will still represent the same group element. Determining if such a “shortening” exists can be done in a finite number of steps. Thus Dehn had found a solution to the word problem. Take a word; determine whether or not it can be shortened. If it can be, then shorten it. Now check if this shortened word can again be shortened. Since the original word had only finite length and each step shortens the word, this process must end. If the word represents the identity, then this process will end with the empty word. If the process stops at a nonempty word, then the original word does not represent the identity. Today any solution to the word problem that guarantees words are shortened at each step is called a *Dehn algorithm*.

In another paper, from 1914 [12], Dehn returned to the study of knots. Again he was able to translate topological problems to algebraic questions by studying the knot group (i.e., the fundamental group of the complement of the knot). By analyzing the geometry of the Cayley graph for the trefoil knot group, Dehn was able to prove that the right and left trefoil knots are not equivalent. Two knots are equivalent if there is a continuous deformation of \mathbb{R}^3 that takes one knot to the other. Furthermore, at each stage the deformation must be an embedding of the knot in \mathbb{R}^3 . Intuitively, this means that, in \mathbb{R}^3 , one knot can be deformed to the other without ever breaking the knot or passing it through itself. Knot theorists call this type of deformation an *ambient isotopy*.

As his research career flourished, Dehn’s personal and professional life also prospered. In 1911 he left the University of Münster to become Extraordinarius at the University of Kiel. Shortly after, Dehn met and married his wife, Toni Landau. In 1913, Dehn became Ordinarius (Chair) at the technical university, Technische Hochschule, in Breslau and stayed at this position until 1921. While living in Breslau, the Dehns had three children: Helmut Max born in 1914, Marie born in 1915, and Eva born in 1919. During the First World War, from 1915 to 1918, Dehn took a mandatory break from mathematics and served in the German Army. He worked as a surveyor and then a coder/decoder. Finally in 1921, Dehn and his family settled down in Frankfurt where Dehn became Ordinarius (Chair) at the University of Frankfurt.

5. DEHN’S CAREER AFTER WWI. Over his career, and particularly in his later years, Dehn’s professional goals seem not to have included personal recognition, as much as to further the progress of mathematics. Every account of Dehn by former colleagues and students portray him as a kind and generous individual. The papers discussed above not only brought together many diverse ideas of the time, but also helped to direct further studies. There are several examples of mathematicians (such as Dehn’s students Magnus and Nielsen), who after providing a proof for a given result, give credit to Dehn for having a previously unpublished proof.

Carl Siegel, an analytic number theorist, has written a wonderful article about the History Seminar that Dehn led from 1922 until 1935 [47]. Siegel was a colleague of Dehn’s during this time. For the seminar, students and faculty met to read and discuss historical works of mathematics. The participants read (in their original, or oldest known, texts) the works of Archimedes, Ptolemy, Descartes, Kepler, Riemann, and more. Siegel describes this time in Frankfurt as a kind of mathematical Camelot. The city of Frankfurt was rich with culture and layered with beautiful gardens. Dehn is reported to have been an amateur naturalist. He preferred the Socratic method of

teaching. On occasion he gave his lectures while taking his students through the local botanic gardens. Each semester, Dehn led hikes for faculty and students in the nearby mountains. Siegel and others have described Dehn as a true scholar of everything. Dehn knew several languages (including Greek and Latin) and he loved and studied music and the arts. Dehn knew the classics well and had a deep knowledge of ancient philosophy. According to Siegel, Dehn's time in Frankfurt had its wonderful moments; however, things would change.

As a German of Jewish descent in the first half of the 1900's, Dehn's life was greatly affected by the turmoil of the two World Wars. Dehn served in the German Army in WWI. He lived through the rise of the Nazi Party. In 1935, Dehn was forced by the Nazis to resign his position as Chair of Mathematics at Frankfurt. Nearing the end of his career and sure to lose his pension and property, Dehn was reluctant to leave Germany. By this time he had sent his children to study in England and the United States. Dehn was arrested the day of Kristallnacht (November 10, 1938). However, he was released that night by the Chief of Police because "the jails were too full." The next day the Dehns began their escape to Norway. In Norway, Dehn was able to take a vacated position at the Technical University in Trondheim. However, in 1940, when the Germans invaded Norway, the Dehns again had to go into hiding. With the help of colleagues, including Siegel and Hellinger, Dehn and his wife eventually crossed the German patrolled Norwegian-Swedish border and began their journey to the United States. U-boats in the Atlantic Ocean prevented any travel in that direction. They were forced to take the long road through Finland, the Soviet Union, and Japan, then across the Pacific to San Francisco. In his article [47] Siegel gives a personal account of these difficult times. There is also a nice article comparing and contrasting the Trans-Siberian escapes of Kurt Gödel and Max Dehn [8].

Dehn's work in the United States started with a short term position at the University of Idaho, in Pocatello. The university had offered him a temporary position, allowing Dehn and his wife to immigrate to the US. However, because of the difficult financial times, they could not hire Dehn permanently. Because of the lack of funds for research facilities during the Depression and because of Dehn's rather late immigration, it was difficult for him to find a suitable position. Siegel suggests that, under financial strain, "prestigious universities thought it unbecoming to offer [Dehn] an ill-paying position, and found it best simply to ignore his presence." After a year and a half in Idaho, Dehn moved to Chicago for a position at the Illinois Institute of Technology. Siegel reports that Dehn "never could adjust to the turbulence of the big city." Dehn then turned his attention to liberal education. He taught for one year at St. John's College, in Annapolis, Maryland. Siegel suggests that Dehn caused friction with the administration when he criticized the curriculum and that Dehn was again looking for a new position.

In March of 1944, Dehn was invited to give two talks at Black Mountain College (BMC), in North Carolina. The next year he was offered a position. Dehn spent the last seven years of his life teaching at BMC. This small college, tucked away in the southern Appalachian mountains, was an experiment in John Dewey's philosophy of liberal education. Despite its remote location and small size, BMC had a profound influence on modern culture. Founded in 1933, BMC attracted faculty from the Bauhaus Art School in Germany, which had been closed by Hitler because of its Jewish influence. Other German exiled artists, composers, and writers followed. Soon American artists began visiting and teaching at BMC. BMC is now known for its significant influence on modern art. During the 24 year life of the college, several important artists, writers, musicians, and dancers taught or studied at Black Mountain College. The list includes: Ruth Asawa, Joseph and Anni Albers, John Cage, Merce Cunningham, Walter Gropius, Franz Kline, Willem de Kooning, Robert Motherwell, Charles Olson,

Pat Passlof, Robert Rauschenberg, M. C. Richards, Dorothea Rockburne, Kenneth Snelson, and Jack Tworokov.

The college also had connections to important scientists. Albert Einstein and Carl Jung served on the advisory board. Two of Einstein's postdocs, Nathan Rosen and Peter Bergmann, spent the year after their work in Princeton teaching at BMC. Buckminster Fuller taught at BMC during two summer sessions. He built his first geodesic dome on the campus. Natasha Goldowski, one of the highest ranking women scientists on the Manhattan Project, taught at BMC for six years.

Although BMC seems an unlikely place to find a great mathematician, all accounts by his family and friends indicate that this was the perfect place for Dehn. At BMC, Dehn taught classes in mathematics, Greek, Latin, and ancient philosophy. He continued his socratic style of teaching. There are reports of his lectures being given while hiking to local waterfalls. Siegel states that the "thickly-wooded hills where one could find the rarest of wildflowers" helped to lift Dehn's spirits from the distress of the previous years.

Dehn continued to write throughout his career, producing several papers each decade. Dehn's later works include philosophic and historic papers. Wilhelm Magnus, one of Dehn's most distinguished students, provides a wonderful overview of Dehn's career and writings [35]. Magnus writes of what mathematics meant to Dehn, "It was something he felt responsible for, something that was important not only per se but also as a part of a greater whole which included the arts, philosophy, and history." While the 1910–1914 years stand out as his most prolific period, Dehn continued to prove significant theorems in topology and infinite group theory throughout his entire life. In his paper of 1938 [13], Dehn found generating sets for the mapping class groups. Here he introduced the idea now known as the *Dehn twist*, which continues to play an important role in the study of mapping class groups today. This paper was written fairly late in his career, after his forced retirement by the Nazis. During the 1940's, among a few other papers, Dehn wrote five short historical papers that appeared in the *American Mathematical Monthly* [14], [15], [16], [17], and [18].

In all, Dehn had nine Ph.D. students. Of Dehn's students, the most famous mathematicians are Eduard Ott-Heinrich Keller, Jacob Nielsen, Wilhelm Magnus, and Ruth Moufang. Keller did work in topology and algebraic geometry, Nielsen and Magnus continued Dehn's work on infinite groups, and Moufang did work in geometry. In his later years in the United States, Dehn continued to encourage a love for mathematics in his students. During his BMC years, Dehn visited the University of Wisconsin, Madison, where he supervised a Ph.D. student. Two of Dehn's undergraduate students at BMC went on to get Ph.D.'s in mathematics. Peter Nemenyi received his Ph.D. from Princeton University and went on to teach statistics. Another BMC student, Truman MacHenry, is now an emeritus mathematics faculty at York University in Canada. MacHenry did most of his graduate work with Peter Hilton at the University of Manchester in England. After funding issues forced him to return to the states, MacHenry studied with Magnus (Dehn's former student) at the Courant Institute of Mathematical Sciences and eventually received his Ph.D. from Adelphi University, working with Donald Solitar. MacHenry is currently doing research in the area of algebraic combinatorics. In a recent letter to the author, MacHenry states that, "The roots of this work lie in lessons that I had with Dehn on symmetric polynomials at BMC." Another BMC student, Dorothea Rockburne, who is now a New York based artist, credits Dehn as her biggest artistic influence.

Max Dehn died in 1952 from an embolism at the age of 73. The stories say that he was working to mark trees to be protected from loggers on the BMC property when he fell ill. His ashes now lie under a thicket of rhododendron on the old BMC campus.

6. INFLUENCES. During his lifetime and after, Dehn's work has continued to have a profound influence on mathematics. For example, Dehn's word problem has played an important role in modern logic. Recall that logic was thriving when Dehn wrote his 1911 paper. In fact, Bertrand Russell and Alfred Whitehead had recently published the first two volumes of *Principia Mathematica*. The goal of that work was to set forth an axiomatic system from which all of mathematics could be deduced. There was a sense in the mathematics community, at that time, that every mathematics problem could be solved, one must only work hard enough to find the solution.

However, in 1911, when Dehn published his paper containing the word problem, modern logicians had only an intuitive definition of an algorithm. A solution to Dehn's word problem required an algorithm. In the years that followed, logicians developed theories of computability and solvability. In particular, during the 1920's and 1930's Alan Turing, Alonzo Church, and Emil Post developed theoretical machines and functions that captured the intuitive idea of an algorithm. This work suggested that there could be a problem in mathematics that no algorithm can solve. In this context, a problem is considered to be an infinite set of statements. For example, for each natural number n , one could ask, is n prime? To solve the problem one must find an algorithm that takes as input n and answers yes or no depending on whether or not n is prime. If an unsolvable problem did exist, Dehn's word problem was a perfect candidate to consider. Shortly after Dehn's death, in 1955 and 1957, Peter Sergeevich Novikov and William Werner Boone independently proved that the word problem for finitely presented groups is an unsolvable problem, [39] and [2]. They had shown that there are finitely presented groups for which no algorithm can solve the word problem. Dehn's word problem for groups was the first "real" mathematics problem to be proven unsolvable.

Throughout the last century, mathematicians have continued to study infinite groups given by presentations. Work in this area was typically called *combinatorial group theory* (up until the 1980's). There are several good texts on this work, see [34], [36], and [48]. Although many other interesting questions emerged, over the years, Dehn's three problems have continued to play a central role. Although there are groups for which the word problem is unsolvable, the word problem and conjugacy problem can be solved for many specific classes of groups. In the 1960's, mathematicians including Martin Greendlinger, Roger Lyndon, and Paul Schupp generalized Dehn's original work with Dehn diagrams to provide a combinatorial approach for solving the word and conjugacy problems for a wider class of groups. This area of research is known as *small cancellation theory*, see [34] and [45].

Knot theorists still use techniques Dehn developed in his original work with the trefoil knot. One important problem in knot theory has been to show that knot groups can be used to distinguish knots. A general result is in fact not possible. There are nonequivalent knots that have the same knot group. However, if one considers only prime knots, then the knot groups can be used to distinguish between knots. Similar to the way that the integers can be factored into prime numbers, knots can be "factored" into prime knots. A *prime knot* is a nontrivial knot that is not the sum of two nontrivial knots. (For details on prime knots and knot sums, see [1].) In the 1960's and 1970's, research by Friedhelm Waldhausen, Klaus Johannson, and others began work to show that if two prime knots have complements with the same fundamental group, then their complements are homeomorphic, see [33] and [52]. This work was completed in 1985 with a paper entitled *Dehn Surgery on Knots*, by Marc Culler, Cameron Gordon, John Luecke, and Peter Shalen [7]. Four years later, in 1989, Gordon and Luecke showed that if two prime knots have homeomorphic complements, then the knots are equivalent [26]. Together these results showed that prime knots are determined by the fundamental group of their complements.

In many ways, Dehn's 1910 paper initiated the systematic study of 3-manifolds. While work in this area continued for decades, it was not until the 1960's and 1970's that 3-manifolds (or more generally low dimensional topology) again had the spotlight. William Thurston was a leader in this surge of research. From his graduation at the University of California at Berkeley, in 1972, until his untimely death in 2012, Thurston's writings have helped to organize and direct research on 3-manifolds, see for example [50] and [51]. Thurston's geometrization conjecture is the classification that Dehn sought. This conjecture classifies 3-manifolds. In 1983, Thurston received the Fields Medal in Mathematics for his work in this area. Dehn's initial steps towards this classification were finally fully realized in 2003 when the proof of Thurston's geometrization conjecture was emailed to the mathematics community by Grigori Perelman. By proving Thurston's conjecture, Perelman had also shown that the Poincaré conjecture is true. As Poincaré had suspected one hundred years earlier, if a 3-manifold has a trivial fundamental group then it is homeomorphic to a 3-sphere.

Though he moves far beyond Dehn's humble beginnings, Thurston's research in 3-manifolds directly builds on and uses ideas from Dehn. Thurston's style even resembled that of Dehn in his ability to use simple combinatorial diagrams to make difficult mathematics seem intuitively obvious. Other topologists have also been motivated by Dehn's work. James Cannon's research in the early 1980's on 3-manifold theory was motivated by the early papers of Poincaré and Dehn [4]. Dehn's studies on mapping class groups and the Dehn twist are also still relevant in topology today.

In the late 1980's and 1990's, motivated by research on 3-manifolds, a number of topologists and geometers again brought their attention to infinite groups given by presentations. This new work on infinite groups is now called *geometric group theory*. A main goal in Dehn's 1911 paper had been to use geometry, in particular hyperbolic geometry, to study infinite groups. This new research is in many ways the realization of Dehn's dream. The geometer Mikhail Gromov has led the way. Gromov's theory of *hyperbolic groups* can be seen as a grand extrapolation on Dehn's work with surface groups. In fact, in his paper of 1987, *On Hyperbolic Groups* [28], Gromov proves that his hyperbolic groups are exactly the groups which have a presentation that has a Dehn algorithm for solving the word problem. Gromov's theory takes full advantage of Dehn's word metric on the Cayley graph. Gromov used his extensive knowledge of hyperbolic geometry to find global geometric properties of triangles in the Cayley graph that generalize Dehn's algorithm for surface groups. Gromov uses the idea of "area" that Dehn's work had suggested. Vaguely speaking, Gromov is able to show that area, in groups given by his definition, is linear with respect to boundary length. This is how area works in hyperbolic geometry. In Euclidean geometry, area is quadratic with respect to boundary length. Just as Dehn's papers provided a grand program and avenues for future research in his time, Gromov's work is doing the same now.

While Gromov was developing his ideas, another group of topologists, including James Cannon, David Epstein, and William Thurston, were working with a new class of infinite groups they called *automatic groups*, see [23]. Automatic groups are essentially the groups with a presentation for which there is a finite set of simple algorithms (specifically, finite state automata) that can build any finite size piece of the group's Cayley graph. These groups capture the algorithmic nature so prominent in Dehn's papers. From the definition, it can easily be shown that automatic groups have a solvable word problem. What is interesting and surprisingly beautiful is that this algorithmic definition also leads to nice global geometric structure in the Cayley graph for the group. For more information on geometric group theory, also see [3], [6], or [25].

It has now been over one hundred years since Max Dehn launched his programs for studying 3-manifolds and infinite groups. Dehn's ideas have laid the foundations for research in these areas ever since. In particular, Dehn's 1910 and 1911 papers are still relevant and important today. These papers have played a role in research decade after decade. Hopefully this article has encouraged you, the reader, to search out and read more about Dehn. For more information on Dehn's life, besides the articles mentioned above, see [49] and [53]. For more about Dehn's time at Black Mountain College, see [46]. For more details on the mathematics of early work in topology and infinite group theory the following are recommended [5] and [48].

ACKNOWLEDGMENT. The author would like to thank John Stillwell, Joseph Gallian, and the reviewers for their thoughtful comments on earlier versions of this paper. An additional thank you to Mark McClure for creating the figures.

REFERENCES

1. C. C. Adams, *The Knot Book. An Elementary Introduction to the Mathematical Theory of Knots*. Revised reprint of the 1994 original. American Mathematical Society, Providence, RI, 2004.
2. W. W. Boone, Certain simple, unsolvable problems of group theory. V, VI. *Nederl. Akad. Wetensch. Proc. Ser. A. 60 = Indag. Math.* **19** (1957) 22–27, 227–232.
3. M. R. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319, Springer-Verlag, Berlin, 1999, <http://dx.doi.org/10.1007/978-3-662-12494-9>.
4. J. W. Cannon, The combinatorial structure of cocompact discrete hyperbolic groups, *Geom. Dedicata* **16** no. 2 (1984) 123–148, <http://dx.doi.org/10.1007/bf00146825>.
5. B. Chandler, W. Magnus, *The History of Combinatorial Group Theory*. A case study in the history of ideas. Studies in the History of Mathematics and Physical Sciences, 9, Springer-Verlag, New York, 1982, <http://dx.doi.org/10.1007/978-1-4613-9487-7>.
6. M. Coornaert, T. Delzant, A. Papadopoulos, *Géométrie Et théorie des Groupes*. (French) [Geometry and Theory of Groups] *Les Groupes Hyperboliques de Gromov*. [Gromov Hyperbolic Groups] With an English summary. Lecture Notes in Mathematics, 1441, Springer-Verlag, Berlin, 1990.
7. M. Culler, C. McA. Gordon, J. Luecke, P. B. Shalen, Dehn surgery on knots. *Bull. Amer. Math. Soc. (N.S.)* **13** no. 1 (1985) 43–45.
8. J. W. Dawson Jr., Max Dehn, Kurt Gödel, and the trans-Siberian escape route, *Notices Amer. Math. Soc.* **49** no. 9 (2002) 1068–1075.
9. M. Dehn, Über die topologie des dreidimensionalen raumes. (German) *Math. Ann.* **69** no. 1 (1910) 137–168.
10. ———, Über unendliche diskontinuierliche gruppen. (German) *Math. Ann.* **71** no. 1 (1911) 116–144.
11. ———, Transformation der kurven auf zweiseitigen flächen. (German) *Math. Ann.* **72** no. 3 (1912) 413–421.
12. ———, Die beiden kleeblattschlingen. (German) *Math. Ann.* **75** no. 3 (1914) 402–413.
13. ———, Die Gruppe der abbildungsklassen. Das arithmetische feld auf flächen. (German) *Acta Math.* **69** no. 1 (1938) 135–206.
14. ———, Mathematics, 400 B.C.–300 B.C. *Amer. Math. Monthly* **50** (1943) 411–414.
15. ———, Mathematics, 300 B.C.–200 B.C. *Amer. Math. Monthly* **51** (1944) 25–31.
16. ———, Mathematics, 200 B.C.–600 A.D. *Amer. Math. Monthly* **51** (1944) 149–157.
17. ———, Recent publications: Die mathematische denkweise, *Amer. Math. Monthly* **54** no. 7 (1947) 424–426.
18. ———, Recent publications: A concise history of mathematics. *Amer. Math. Monthly* **56** no. 9 (1949) 643–644.
19. ———, *Papers on Group Theory and Topology*. Translated from the German and with introductions and an appendix by John Stillwell. With an appendix by Otto Schreier. Springer-Verlag, New York, 1987.
20. M. Dehn, P. Heegaard, Analysis situs, *Encykl. Math. Wiss.*, vol 3, AB3, Leipzig, 1907, 153–220.
21. W. Dyck, Gruppentheoretische studien. (German) *Math. Ann.* **20** no. 1 (1882) 1–44.
22. ———, Gruppentheoretische studien. II. Ueber die zusammensetzung einer gruppe discreter operationen, über ihre primitivität und transitivität. (German) *Math. Ann.* **22** no. 1 (1883) 70–108.
23. D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett Publishers, Boston, MA, 1992.

24. J. A. Gallian, *Contemporary Abstract Algebra*. Eighth edition. Brooks/Cole, Cengage Learning, 2012.
25. É. Ghys, P. de la Harpe, *Sur les Groupes Hyperboliques d'après Mikhael Gromov* (French) Panorama. Progr. Math., 83, Birkhäuser Boston, Boston, MA, 1990.
26. C. McA. Gordon, J. Luecke, Knots are determined by their complements, *J. Amer. Math. Soc.* **2** no. 2 (1989) 371–415.
27. M. Gromov, Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.* No. 53 (1981) 53–73.
28. ———, Hyperbolic groups, in *Essays in Group Theory*, Math. Sci. Res. Inst. Publ., 8, Springer, New York, 1987, 75–263.
29. ———, Asymptotic invariants of infinite groups, in *Geometric Group Theory*, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993.
30. I. Grossman, W. Magnus, *Groups and Their Graphs*. New Mathematical Library. Random House, New York; L. W. Singer, New York, 1964.
31. D. Hilbert, *Foundations of Geometry*. Second edition. Translated from the tenth German edition by Leo Unger. Open Court, LaSalle, IL, 1971.
32. J. E. Hopcroft, J. D. Ullman, *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley Series in Computer Science. Addison-Wesley Publishing, Reading, MA, 1979.
33. K. Johansson, Équivalences d'homologie des variétés de dimension 3. (French) *C.R. Acad. Sci. Paris Sér.*, A-B 281 no. 23 (1975) 1009–1010.
34. R. C. Lyndon, P. E. Schupp, *Combinatorial Group Theory*. Reprint of the 1977 edition. Classics in Mathematics, Springer-Verlag, Berlin, 2001.
35. W. Magnus, Max Dehn, *Math. Intelligencer* **1** no. 3 (1978/79) 132–143.
36. W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory*. Reprint of the 1976 second edition. Dover Publications, Mineola, NY, 2004.
37. J. R. Munkres, *Topology*. Prentice-Hall, Englewood Cliffs, NJ, 1975.
38. P. S. Novikov, Unsolvability of the conjugacy problem in the theory of groups, (Russian) *Izv. Akad. Nauk SSSR. Ser. Mat.* **18** (1954) 485–524.
39. ———, On the algorithmic unsolvability of the word problem in group theory, (Russian) *Trudy Mat. Inst. im. Steklov. no. 44. Izdat. Akad. Nauk SSSR*, Moscow, 1955.
40. C. D. Papakyriakopoulos, On Dehn's lemma and the asphericity of knots. *Proc. Nat. Acad. Sci. U.S.A.* **43** (1957) 169–172.
41. H. Poincaré, *Papers on Topology. Analysis Situs and its Five Supplements*. Translated and with an introduction by John Stillwell. History of Mathematics, 37. American Mathematical Society, Providence, RI; London Mathematical Society, London, 2010.
42. ———, *Papers on Fuchsian Functions*. Translated from the French and with an introduction by John Stillwell. Springer-Verlag, New York, 1985.
43. T. Radó, Über den begriff der Riemannsche fläche, *Acta Univ. Szeged* **2** (1924) 101–121.
44. D. J. S. Robinson, *A Course in the Theory of Groups*. Second edition. Graduate Texts in Mathematics, 80, Springer-Verlag, New York, 1996.
45. P. E. Schupp, On Dehn's algorithm and the conjugacy problem, *Math. Ann.* **178** (1968) 119–130.
46. R. B. Sher, Max Dehn and Black Mountain College, *Math. Intelligencer* **16** no. 1 (1994) 54–55.
47. C. L. Siegel, On the history of the Frankfurt mathematics seminar, *Math. Intelligencer* **1** no. 4 (1978/79) 223–230.
48. J. Stillwell, *Classical Topology and Combinatorial Group Theory*. Second edition. Graduate Texts in Mathematics, 72, Springer-Verlag, New York, 1993, <http://dx.doi.org/10.1007/978-1-4612-4372-4>.
49. ———, Max Dehn, in *History of Topology*, North-Holland, Amsterdam, 1999, 965–978.
50. W. P. Thurston, *Three-dimensional Geometry and Topology*. Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton Univ. Press, Princeton, NJ, 1997, <http://dx.doi.org/10.5860/choice.35-3367>.
51. ———, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc. (N.S.)* **6** no. 3 (1982) 357–381.
52. F. Waldhausen, Recent results on sufficiently large 3-manifolds. in *Algebraic and Geometric Topology – Stanford 1976*, Proc. Sympos. Pure Math., vol 32, Part 2, American Mathematical Society, Providence, RI, 1978, 21–38.
53. B. H. Yandell, *The Honors Class. Hilbert's Problems and Their Solvers*. A K Peters, Natick, MA, 2002, <http://dx.doi.org/10.5860/choice.39-5863>.

DAVID PEIFER received his Ph.D. in 1992 from the University of Illinois, Urbana-Champaign. His research work is in geometric group theory. For the past fourteen years, he has been the Chair of Mathematics at the

University of North Carolina at Asheville, a public liberal arts college. He is currently a Board Member of the Black Mountain College Museum and Art Center, a museum dedicated to preserving and continuing the legacy of Black Mountain College. When not doing mathematics or his administrative duties, he also enjoys mountain biking and playing the guitar.

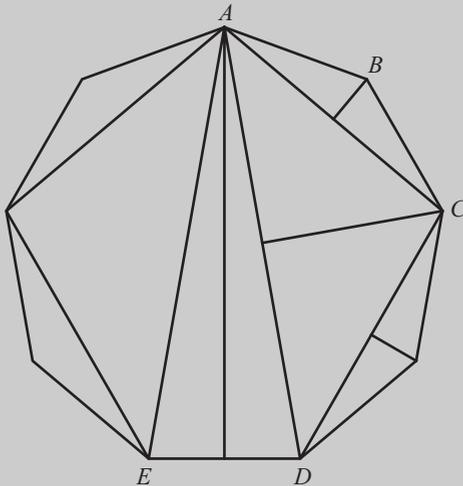
Department of Mathematics, University of North Carolina at Asheville, Asheville, NC 28804
 dpeifer@unca.edu

A Geometric Proof of Morrie’s Law

We aim to give a geometric proof of the so-called Morrie’s law [1], which reads

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right) = \frac{1}{8}.$$

For this purpose, we analyze the regular enneagon with unit edge length in which some of its diagonals and angle bisectors are considered, as shown in the figure below.



Each interior angle of the enneagon equals $7\pi/9$. Since half the angle B is equal to $7\pi/18$, then $\angle BAC = \pi/9$ and thus $|AC| = 2\cos(\pi/9)$. Similarly, $\angle CAD = 2\pi/9$ which yields $|AD| = 2|AC|\cos(2\pi/9)$. Finally, note that $\angle ADE = 4\pi/9$. It follows that $1 = |ED| = 2|AD|\cos(4\pi/9) = 2^3\cos(\pi/9)\cos(2\pi/9)\cos(4\pi/9)$, so we are done.

REFERENCE

1. W. A. Beyer, J. D. Louck, D. Zeilberger, A generalization of a curiosity that Feynman remembered all his life, *Math. Mag.* **69** (1996) 43–44.

—Submitted by Samuel G. Moreno and Esther M. García-Caballero

Departamento de Matemáticas, Universidad de Jaén, 23071 Jaén, Spain

<http://dx.doi.org/10.4169/amer.math.monthly.122.03.233>
 MSC: Primary 51M04