Interfaces with Other Disciplines

Measuring super-efficiency in DEA in the presence of infeasibility

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Abstract

Super-efficiency data envelopment analysis (DEA) model is obtained when a decision making unit (DMU) under evaluation is excluded from the reference set. Because of the possible infeasibility of super-efficiency DEA model, the use of super-efficiency DEA model has been restricted to the situations where constant returns to scale (CRS) are assumed. It is shown that one of the input-oriented and output-oriented super-efficiency DEA models must be feasible for any efficient DMU under evaluation if the variable returns to scale (VRS) frontier consists of increasing, constant, and decreasing returns to scale DMUs. We use both input- and output-oriented super-efficiency models to fully characterize the super-efficiency. When super-efficiency is used as an efficiency stability measure, infeasibility means the highest super-efficiency (stability). If super-efficiency is interpreted as input saving or output surplus achieved by a specific efficient DMU, infeasibility does not necessarily mean the highest super-efficiency.

Keywords: Data envelopment analysis; Infeasibility; Super-efficiency

1. Introduction

Data envelopment analysis (DEA) was originally developed to measure the relative efficiency of peer decision making units (DMUs) in multiple input-multiple output settings (Charnes et al., 1978). DEA identifies an efficient frontier where all DMUs have a unity score. In order to discriminate the performance among efficient DMUs, based upon the CCR model, a super-efficiency DEA model in which a DMU under evaluation is excluded from the reference set was first developed by Banker and Gifford (1988) and Banker et al. (1989) (see also Andersen and Petersen, 1993). This super-efficiency DEA model was developed under (i) the DEA frontier exhibits constant returns to scale (CRS) and (ii) all inputs (or outputs) are simultaneously changed in the same proportion. When either of the conditions is not satisfied, infeasibility of the related linear program is very likely to occur (see, e.g., Seiford and Zhu, 1998a,b). As a result, we do not have a value associated with infeasibility to represent the super-efficiency, and the use of super-efficiency DEA is restricted.

Banker and Chang (2000) have demonstrated that the use of the super-efficiency model for
ranking efficient DMUs is inappropriate. However, in addition to ranking, super-efficiency concept has been used in other situations. For example, DEA sensitivity analysis (Charnes et al., 1992; Zhu, 1996, 2002), two-person ratio efficiency games (Rousseau and Semple, 1995), detecting influential observations (Banker et al., 1989; Wilson, 1995), and acceptance decision rules (Seiford and Zhu, 1998c), among others. Therefore, a study on infeasibility of super-efficiency DEA model is a worthwhile objective.

While the necessary and sufficient conditions for infeasibility in various super-efficiency DEA models are developed (Seiford and Zhu, 1999), no attempt has been made to solve the infeasibility problem. This is partly due to the fact that the meaning of super-efficiency has different interpretations.

In fact, an input-oriented super-efficiency DEA model measures the input super-efficiency when outputs are fixed at their current levels. An output-oriented super-efficiency DEA model measures the output super-efficiency when inputs are fixed at their current levels. From the different uses of the super-efficiency concept, we see that super-efficiency can be interpreted as the degree of efficiency stability or input saving/output surplus achieved by an efficient DMU. If super-efficiency is used as an efficiency stability measure, then based upon Seiford and Zhu (1998b), infeasibility means that an efficient DMU’s efficiency classification is stable to any input changes if an input-oriented super-efficiency DEA model is used (or any output changes if an output-oriented super-efficiency DEA model is used). Therefore, we can use $+\infty$ to represent the super-efficiency score. i.e., infeasibility means the highest super-efficiency.

Using the variable returns to scale (VRS) super-efficiency DEA model as an example, the current study discusses the situation when super-efficiency is interpreted as input saving or output surplus. The current study shows that if the VRS frontier has increasing, constant, and decreasing returns to scale (IRS, CRS, and DRS) DMUs, one of the input-oriented and output-oriented super-efficiency DEA models must be feasible. This implies that (i) if an efficient DMU does not possess any input super-efficiency (input saving), it must possess output super-efficiency (output surplus), and (ii) if an efficient DMU does not possess any output super-efficiency, it must possess input super-efficiency. We can use both input-oriented and output-oriented super-efficiency DEA models to fully characterize the super-efficiency.

2. Super-efficiency and infeasibility

A DEA model evaluates the performance of a set of DMUs, \( \{\text{DMU}_j : j = 1, 2, \ldots, n\} \), which produce multiple outputs by utilizing multiple inputs. Each DMU has a set of \( s \) output measures, \( y_{rj} \) \( (r = 1, 2, \ldots, s) \) and a set of \( m \) input measures, \( x_{ij} \) \( (i = 1, 2, \ldots, m) \). A DEA model which exhibits VRS can be written as (Banker et al., 1984)

\[
\begin{align*}
\min & \quad \theta_o \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_o x_{io}, \quad i = 1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, 2, \ldots, s, \\
& \quad \sum_{j=1}^{n} \lambda_j = 1, \\
& \quad \lambda_j \geq 0, \quad j = 1, \ldots, n,
\end{align*}
\]

where, \( x_{io} \) and \( y_{ro} \) are respectively the \( i \)th input and \( r \)th output for a DMU\( _o \) under evaluation.

The VRS super-efficiency DEA model related to (1) can be expressed as (Seiford and Zhu, 1999)

\[
\begin{align*}
\min & \quad \theta_{o}^{\text{VRS-super}} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_{o}^{\text{VRS-super}} x_{io}, \quad i = 1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, 2, \ldots, s, \\
& \quad \sum_{j=1}^{n} \lambda_j = 1, \\
& \quad \theta_{o}^{\text{VRS-super}} \geq 0, \\
& \quad \lambda_j \geq 0, \quad j \neq o,
\end{align*}
\]
where the DMU\(_o\) under evaluation is excluded from the reference set.

If we drop
\[
\sum_{j=1, j \neq o}^{n} \lambda_j = 1,
\]
we obtain a super-efficiency DEA model under CRS. While the CRS super-efficiency DEA model is usually feasible, \(^1\) model (2) is not.

Note that model (2) is an input-oriented super-efficiency DEA model. We can also have an output-oriented version.

\[
\max \phi^{VRS-super}_o
\]

s.t. \[
\sum_{j=1, j \neq o}^{n} \lambda_j x_{ij} \leq x_{io}, \quad i = 1, 2, \ldots, m,
\]
\[
\sum_{j=1, j \neq o}^{n} \lambda_j y_{rj} \geq y_{ro}, \quad r = 1, 2, \ldots, s,
\]
\[
\sum_{j=1, j \neq o}^{n} \lambda_j = 1,
\]
\[
\phi^{VRS-super}_o \geq 0,
\]
\[
\lambda_j \geq 0, \quad j \neq o.
\] (3)

The following theorem indicates a relationship between models (2) and (3) with respect to the infeasibility.

**Theorem 1.** If model (2) is infeasible and DMU\(_o\) is CRS-inefficient, then model (3) must be feasible.

**Proof.** Recall that Seiford and Zhu (1999) show that when model (2) is infeasible, then DMU\(_o\) must exhibit CRS or DRS.

Now, if DMU\(_o\) is inefficient under the CRS assumption, it must exhibit DRS.

Therefore,
\[
\left\{ \begin{array}{l}
\sum_{j=1, j \neq o}^{n} \lambda_j^* x_{ij} \leq \phi^{CRS}_o x_{io}, \\
\sum_{j=1, j \neq o}^{n} \lambda_j^* y_{rj} \geq y_{ro},
\end{array} \right.
\] (4)

where \(\phi^{CRS}_o < 1\) is the CRS efficiency score and
\[
\sum_{j=1, j \neq o}^{n} \lambda_j^* > 1
\]
representing optimal solutions from the CRS DEA model (Charnes et al., 1978). \(^2\)

Let
\[
\lambda_j' = \frac{\lambda_j^*}{\sum_{j=1, j \neq o}^{n} \lambda_j^*},
\]
then (4) becomes
\[
\left\{ \begin{array}{l}
\sum_{j=1, j \neq o}^{n} \lambda_j' x_{ij} \leq \phi^{CRS}_o x_{io}, \\
\sum_{j=1, j \neq o}^{n} \lambda_j' y_{rj} \geq y_{ro},
\end{array} \right.
\] (5)

Model (5) indicates that the output-oriented super-efficiency DEA model (3) is feasible. \(\square\)

Similar to Theorem 1, we have

**Theorem 2.** If model (3) is infeasible and DMU\(_o\) is CRS-inefficient, then model (2) must be feasible.

Now, if DMU\(_o\) is also efficient under CRS, i.e., DMU\(_o\) exhibits CRS, we have

**Theorem 3.** Both models (2) and (3) are infeasible if and only if DMU\(_o\) is the only VRS efficient DMU.

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\(^1\) Zhu (1996) indicates that the (input-oriented) CRS super-efficiency DEA model is always feasible unless certain patterns of zero data entries are present in the inputs. Therefore, if one assumes all data are positive, then the (input-oriented) CRS super-efficiency DEA model is always feasible.

\(^2\) Since DMU\(_o\) is CRS inefficient, (7) is equivalent to the set of original constraints in the CRS DEA model (Charnes et al., 1978).
Proof. Suppose both models (2) and (3) are infeasible. We have that under the condition of VRS, DMU_o has the largest outputs and smallest inputs. Therefore, DMU_o dominates all other DMUs and their convex combinations. This indicates that DMU_o is the only efficient DMU under both VRS and CRS.

Now, suppose DMU_o is the only VRS efficient DMU, then it must also be the only CRS efficient DMU. Thus, for each of other DMUs under CRS evaluation, we have $x_{io} < x_{ij} (j \neq o)$ and $y_{ro} < y_{rj} (j \neq o)$. Further, when

$$\sum_{j=1}^{n} \lambda_j = 1,$$

we have

$$\sum_{j=1}^{n} \lambda_j x_{ij} > x_{io} \quad \text{and} \quad \sum_{j=1}^{n} \lambda_j y_{rj} < y_{rj}.$$

This implies both models (2) and (3) are infeasible. \(\square\)

Theorems 1–3 show that one of the input-oriented and output-oriented VRS super-efficiency DEA models must be feasible if the VRS frontier contains IRS, CRS and DRS DMUs. Note that Theorem 3 describes a very rare situation which often does not exist in real world data sets.

Now, suppose model (2) is feasible when an efficient DMU_o is under evaluation. The optimal value of $\hat{\theta}_o^{\text{VRS-super}}$ indicates that the inputs of DMU_o can be increased to reach a level that is used by other DMUs or by the convex combination of other DMUs.

Consider the five DMUs (A, B, C, D, and H) with one output and one input in Fig. 1. When model (2) is applied to DMU B, we have $\hat{\theta}_B^{\text{VRS-super}} = \frac{31}{31}$, indicating that DMU B can increase its input to $\frac{31}{31} ( = \frac{3}{3} \times 3)$, the input level used by B—a convex combination of A and C. This possible input increase can actually be viewed as an input saving achieved by DMU B compared to the remaining DMUs.

When model (2) is applied to DMU D, model (2) is infeasible. Although DMU D remains efficient under the condition of VRS when its input increases, this input increase does not represent any input savings after passing H' which uses the input level of H. Because H's input level is the largest. Thus, the super-efficiency of D should be compared to H'—a radial DEA projection. To achieve this, we solve model (2) in the following format

$$\min_{\lambda} \hat{\theta}_o^{\text{VRS-super}}$$

s.t. \(\sum_{j=1}^{n} \lambda_j x_{ij} \leq \hat{\theta}_o^{\text{VRS-super}} x_{io}, \quad i = 1, 2, \ldots, m,\)

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \hat{y}_ro = y_{ro}, \quad r = 1, 2, \ldots, s, \quad (6)$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0, \quad j \neq o,$$

where $\hat{y}_{rj} = \phi^*_o y_{rj}$ and $\phi^*_o$ is the optimal value to the following output-oriented VRS DEA model

$$\phi^*_o = \max \phi_o$$

s.t. \(\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}, \quad i = 1, 2, \ldots, m,\)

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \phi_o y_{ro}, \quad r = 1, 2, \ldots, s, \quad (7)$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$\lambda_j \geq 0.$$
Applying model (6) is equivalent to applying model (2) after all inefficient DMUs are projected onto the VRS frontier via proportional output augmentation through model (7).  

Applying model (6) to DMU D yields \( \hat{\phi}_o^{\text{VRS-super}} = 1.1 \). Note that model (6) may still be infeasible. For example, if we do not have DMU H in Fig. 1, model (6) is infeasible when DMU D is under evaluation. In such cases, we say that DMU \( o \) does not indicate super-efficiency in terms of input saving, since DMU \( o \) cannot be moved onto the frontier constructed from the remaining DMUs via input increases, indicating that DMU \( o \) has the largest input levels given the current output levels. We therefore denote \( \hat{\phi}_o^{\text{VRS-super}} = 1 \) when model (6) is infeasible, indicating zero input super-efficiency which means zero input saving for DMU \( o \).

Let \( \gamma_o \) represent the score for characterizing the super-efficiency in terms of input saving. We have

\[
\gamma_o = \begin{cases} 
\hat{\phi}_o^{\text{VRS-super}} & \text{if model (2) is feasible}, \\
\hat{\phi}_o^{\text{VRS-super}} & \text{if model (2) is infeasible and model (6) is feasible}, \\
1 & \text{if model (6) is infeasible}.
\end{cases}
\]  

(8)

Note that \( \gamma_o \geq 1 \). If \( \gamma_o > 1 \), a specific efficient DMU \( o \) has input super-efficiency. If \( \gamma_o = 1 \), DMU \( o \) does not have input super-efficiency.

When model (6) is infeasible, the super-efficiency is actually reflected in DMU’s outputs via output surplus. Suppose again that H is not included in Fig. 1. Fig. 1 indicates that the super-efficiency of DMU D is represented by its output rather than its input. Since given the current input level, DMU D achieves an output surplus of 0.5 if DMU C uses the input level of DMU D. Thus, we should also use model (3) to characterize the super-efficiency via output super-efficiency signaled by output surplus.

Consider DMU D in Fig. 1 (assume that H is not included), we have \( \theta_{\text{D}}^{\text{VRS-super}} = 0.9 \). i.e., DMU D’s output super-efficiency score is 0.9.

Similar to model (6), we may also adjust the input values in model (6) by the input-oriented VRS DEA model (1) when model (3) is feasible.

\[
\begin{align*}
\max & \quad \hat{\phi}_o^{\text{VRS-super}} \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \tilde{x}_{io}, \quad i = 1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} \lambda_j y_{ij} \geq \hat{\phi}_o^{\text{VRS-super}} y_{io}, \quad r = 1, 2, \ldots, s, \\
& \quad \sum_{j=1}^{n} \lambda_j = 1, \\
& \quad \lambda_j \geq 0, \quad j \neq o,
\end{align*}
\]  

(9)

where \( \tilde{x}_{ij} = 0^* x_{ij} \) and \( 0^* \) is the optimal value to model (1) when DMU \( j \) is under evaluation.

Both models (3) and (9) are infeasible when DMU A is under evaluation, indicating that DMU A does not have super-efficiency in its output. In fact, no other DMUs produce less output than DMU A does. Thus, we denote \( \phi_o^{\text{VRS-super}} = \hat{\phi}_o^{\text{VRS-super}} = 1 \) when model (9) is infeasible, indicating zero output super-efficiency which means zero output surplus for DMU \( o \).

Let \( \tau_o \) represent the score for characterizing the output super-efficiency, we have

\[
\tau_o = \begin{cases} 
\phi_o^{\text{VRS-super}} & \text{if model (3) is feasible}, \\
\phi_o^{\text{VRS-super}} & \text{if model (3) is infeasible and model (9) is feasible}, \\
1 & \text{if model (9) is infeasible}.
\end{cases}
\]  

(10)

Note that \( \tau_o \leq 1 \). If \( \tau_o < 1 \), a specific efficient DMU \( o \) has output super-efficiency. If \( \tau_o = 1 \), DMU \( o \) does not have output super-efficiency.

From the above discussion we see that super-efficiency is represented by only the input saving or by only the output surplus when infeasibility occurs. Theorems 1–3 provide a basis for employing both input-oriented and output-oriented VRS super-efficiency models to fully characterize the super-efficiency that are inherent in DMU \( o \)’s inputs or outputs.
We may integrate $\gamma_o$ and $\tau_o$ into one super-efficiency score. For example, we may select $w_c$ and $w_s$ such that $w_c + w_s = 1$ and define
\[
S_o = w_c\gamma_o + w_s\frac{1}{\tau_o}
\]
(11)
or
\[
\hat{S}_o = w_c\frac{1}{\gamma_o} + w_s\tau_o.
\]
(12)
Obviously, $S_o \geq 1$ and $\hat{S}_o \leq 1$. Larger $S_o$ or smaller $\hat{S}_o$ indicates higher super-efficiency performance.\footnote{Note that under the condition of CRS, we have $\gamma_o = 1/\tau_o$. Thus, $S_o = \hat{S}_o = \gamma_o = 1/\tau_o$, i.e., either the input-oriented CRS super-efficiency DEA model or the output-oriented CRS super-efficiency DEA model is sufficient in ranking the efficient DMUs.}

3. Conclusions

The current paper discusses the relationship between super-efficiency and the infeasibility of super-efficiency DEA model. It is shown that in order to fully characterize the super-efficiency, both input-oriented and output-oriented super-efficiency DEA models are needed. Although the discussion is based upon VRS super-efficiency DEA models, all the developments can be applied to other super-efficiency DEA models (Seiford and Zhu, 1999). In particular, the current study can also benefit the calculation of the Malmquist productivity index (Färe et al., 1994). As noted in Chen (2000), most of the applications on DEA-based Malmquist productivity analysis are based upon the CRS assumption. There are a few applications of Malmquist productivity indexes using VRS DEA models in literature, but the authors neither provide the detail computation results nor mention the occurrence of infeasibility. Using the results from the current paper, we are able to extend the DEA-based Malmquist productivity analysis into the VRS technology.

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References
