Some Concepts and Theorems of Uncertain Random Process

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Abstract

In order to deal with a system with both randomness and uncertainty, chance theory has been built, and uncertain random variable has been proposed as a generalization of random variable and uncertain variable. Correspondingly, as a generalization of both stochastic process and uncertain process, this paper will propose an uncertain random process. In addition, some special types of uncertain random processes such as stationary increment uncertain random process and uncertain random renewal process will also be discussed.

Keywords: uncertain random variable, uncertain process, stochastic process, uncertain random process

1 Introduction

Probability theory has been used to model random phenomena for a long time. A premise of applying probability is that the obtained probability distribution is close enough to the real frequency. However, due to economic reasons or technical difficulties, we are often lack of observed data to estimate the distribution via statistics. In this case, some domain experts are invited to evaluate their belief degree of the possible events. According to the prospect theory by the Nobelist Kahneman and Tversky [3], the belief degree usually has a much larger range than the real frequency, and as a result, probability theory is not applicable (Liu [9]). In order to deal with this problem, an uncertainty theory was founded by Liu [5] in 2007 and refined by Liu [8] in 2010 as a branch of axiomatic mathematics.

In order to indicate the belief degree that an event happens, a concept of uncertain measure was proposed by Liu [5] in 2007 as a set function satisfying normality, duality, subadditivity and product axioms. Uncertain variable, as a counterpart of random variable, was also proposed to model an uncertain quantity. During the past six years, many researchers contributed a lot in the area of uncertainty theory. For example, Peng and Iwamura [15] gave a sufficient and necessary condition for a function being the


Sometimes, randomness and uncertainty exist simultaneously in a complex system. In order to describe such a system, a concept of chance space was proposed by Liu [12] as a generalization of probability space and uncertainty space, and a concept of chance measure was also presented to indicate the event in the chance space. Meanwhile, Liu [12] provided uncertain random variable to model the quantity behaving randomly and uncertainly. In addition, the chance distribution, expected value and variance of an uncertain random variable were also provided. Following that, Liu [13] gave an operational law of uncertain random variables, and Yao and Gao [19] verified a law of large numbers for uncertain random variables.

As an application of chance theory, uncertain random programming was first proposed by Liu [13] as a branch of mathematical programming involving uncertain random variables. Then as some special cases, Zhou et al [21], Qin [16], Ke [4] proposed uncertain random multi-objective programming, uncertain random goal programming and uncertain random multi-level programming, respectively. In addition, uncertain random risk analysis was invented by Liu [14], and uncertain random reliability analysis was invented by Wen and Kang [17].

In this paper, we will propose uncertain random process. The rest of this paper is structured as follows. The next section is intended to introduce some concepts about uncertain variable and uncertain random variable. Then a concept of uncertain random process as well as its sample path, first hitting time is given in Section 3. As a special type of uncertain random process, stationary increment uncertain random process is proposed in Section 4. After that, uncertain random renewal process is given, and its average renewal rate is studied in Section 5. At last, some remarks are given in Section 6.
2 Preliminary

Probability theory is a branch of axiomatic mathematics to deal with the frequency, and uncertainty theory is a branch of axiomatic mathematics to deal with human’s belief degree. In order to model a system with both randomness and uncertainty, a chance theory was founded by Liu [12, 13] in 2013. In this section, we will introduce some useful definitions about uncertain variable and uncertain random variable.

2.1 Uncertain Variable

Definition 1. (Liu [5]) Let $\mathcal{L}$ be a $\sigma$-algebra on a nonempty set $\Gamma$. A set function $M : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality Axiom) $M\{\Gamma\} = 1$ for the universal set $\Gamma$.

Axiom 2: (Duality Axiom) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda$.

Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$  

Besides, the product uncertain measure on the product $\sigma$-algebra $\mathcal{L}$ is defined by Liu [7] as follows,

Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ Then the product uncertain measure $M$ is an uncertain measure satisfying

$$M\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\}$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 2. (Liu [5]) An uncertain variable $\xi$ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers $\mathbb{R}$, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by $\Phi(x) = M\{\xi \leq x\}$ for any real number $x$, and the expected value $E[\xi]$ is defined by

$$E[\xi] = \int_{0}^{+\infty} M\{\xi \geq x\}dx - \int_{-\infty}^{0} M\{\xi \leq x\}dx$$

provided that at least one of the two integrals is finite. For an uncertain variable $\xi$ with a regular uncertainty distribution $\Phi$, Liu [8] proved that if $E[\xi]$ exists, then

$$E[\xi] = \int_{0}^{+\infty} (1 - \Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx = \int_{0}^{1} \Phi^{-1}(\alpha)d\alpha.$$
Definition 3. (Liu [7]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if

$$M\left\{\bigcap_{i=1}^m (\xi_i \in B_i)\right\} = \bigwedge_{k=1}^m M\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Theorem 1. (Liu [8]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$$

is an uncertain variable with inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

2.2 Uncertain Random Variable

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and $(\Omega, \mathcal{A}, \Pr)$ be a probability space. Then

$$(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr) = (\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \Pr)$$

is called a chance space.

Definition 4. (Liu [12]) Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ be a chance space, and $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event. Then the chance measure $\text{Ch}$ of $\Theta$ is defined by

$$\text{Ch}\{\Theta\} = \int_0^1 \Pr(\omega \in \Omega | M\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq r) \, dr.$$ 

Liu [12] verified that the chance measure $\text{Ch}$ satisfies normality, duality, and monotonicity properties, that is, i) $\text{Ch}\{\Gamma \times \Omega\} = 1$; ii) $\text{Ch}\{\Theta\} + \text{Ch}\{\Theta^c\} = 1$ for any event $\Theta$; iii) $\text{Ch}\{\Theta_1\} \leq \text{Ch}\{\Theta_2\}$ for any events $\Theta_1$ and $\Theta_2$ with $\Theta_1 \subset \Theta_2$. Besides, Hou [2] proved the subadditivity of chance measure, that is,

$$\text{Ch}\left\{\bigcup_{i=1}^\infty \Theta_i\right\} \leq \sum_{i=1}^\infty \text{Ch}\{\Theta_i\}$$

for a sequence of events $\Theta_1, \Theta_2, \cdots$

Definition 5. (Liu [12]) An uncertain random variable $\xi$ is a measurable function from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers, i.e.,

$$\{\xi \in B\} = \{(\gamma, \omega) | \xi(\gamma, \omega) \in B\}$$

is an uncertain random event for any Borel set $B$.  

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4
Random variable and uncertain variable can be regarded as special cases of uncertain random variable. Let \( \eta \) be a random variable, \( \tau \) be an uncertain variable. Then \( \eta + \tau \) and \( \eta \times \tau \) are both uncertain random variables.

**Definition 6.** (Liu [12]) Let \( \xi \) be an uncertain random variable. Then its chance distribution is defined by

\[
\Phi(x) = \text{Ch}\{\xi \leq x\}, \quad \forall x \in \mathbb{R}.
\]

Liu [12] proved that a function \( \Phi(x) \) is a chance distribution of an uncertain random variable if and only if it is a monotone increasing function with respect to \( x \) except \( \Phi(x) \equiv 0 \) and \( \Phi(x) \equiv 1 \). The uncertain random variables are said to be identically distributed if they have a common chance distribution. The chance distribution of a random variable is just its probability distribution, and the chance distribution of an uncertain variable is just its uncertainty distribution. As to a random variable \( \eta \) with a probability distribution \( \Psi \), an uncertain variable \( \tau \), and a measurable function \( f \), Liu [13] proved that \( f(\eta, \tau) \) has a chance distribution

\[
\Phi(y) = \int_{-\infty}^{+\infty} F(x, y) d\Psi(x)
\]

where \( F(x, y) \) is the uncertainty distribution of \( f(x, \tau) \).

**Theorem 2.** (Yao and Gao [19], Law of Large Numbers) Let \( \eta_1, \eta_2, \cdots \) be a sequence of iid random variables with a common probability distribution \( \Psi \), \( \tau_1, \tau_2, \cdots \) be a sequence of iid uncertain variables, and \( f(x, y) \) be a strictly monotone function. Define \( S_0 = 0 \), and \( S_n = f(\eta_1, \tau_1) + \cdots + f(\eta_n, \tau_n) \), \( \forall n \geq 1 \).

Then

\[
\frac{S_n}{n} \to \int_{-\infty}^{+\infty} f(x, \tau_1) d\Psi(x)
\]

in the sense of convergence in distribution as \( n \to \infty \).

**Definition 7.** (Liu [12]) Let \( \xi \) be an uncertain random variable. Then its expected value is defined by

\[
E[\xi] = \int_0^{+\infty} \text{Ch}\{\xi \geq r\} dr - \int_{-\infty}^0 \text{Ch}\{\xi \leq r\} dr
\]

provided that at least one of the two integrals is finite.

Let \( \Phi \) denote the chance distribution of \( \xi \). Liu [13] proved that if \( E[\xi] \) exists, then

\[
E[\xi] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx.
\]

For a random variable \( \eta \) and an uncertain variable \( \tau \), Liu [13] proved that \( E[\eta + \tau] = E[\eta] + E[\tau] \) and \( E[\eta \times \tau] = E[\eta] \times E[\tau] \). In fact, we have the following theorem about the expected value of an uncertain random variable.
Theorem 3. Let \( \xi \) be an uncertain random variable with a finite expected value. Then

\[
E[\xi] = \int_{\Omega} E[\xi(\cdot, \omega)] \Pr(d\omega)
\]

where \( \xi(\cdot, \omega) \) denotes the uncertain variable \( \xi(\gamma, \omega) \) with a fixed \( \omega \).

Proof: It follows from the definition of expected value that

\[
E[\xi] = \int_{0}^{+\infty} \text{Ch}\{\xi \geq x\} dx - \int_{-\infty}^{0} \text{Ch}\{\xi \leq x\} dx
\]

\[
= \int_{0}^{+\infty} \int_{0}^{1} \Pr(\omega \in \Omega | M\{\xi(\cdot, \omega) \geq r\}) dr dx - \int_{-\infty}^{0} \int_{0}^{1} \Pr(\omega \in \Omega | M\{\xi(\cdot, \omega) \leq r\}) dr dx
\]

\[
= \int_{0}^{+\infty} \int_{\Omega} M\{\xi(\cdot, \omega) \geq x\} dx \Pr(d\omega) dx - \int_{-\infty}^{0} \int_{\Omega} M\{\xi(\cdot, \omega) \leq x\} dx \Pr(d\omega) dx.
\]

Then by Fubini theorem, we have

\[
E[\xi] = \int_{\Omega} \left( \int_{0}^{+\infty} M\{\xi(\cdot, \omega) \geq x\} dx \Pr(d\omega) - \int_{-\infty}^{0} M\{\xi(\cdot, \omega) \leq x\} dx \Pr(d\omega) \right) \Pr(d\omega)
\]

\[
= \int_{\Omega} E[\xi(\cdot, \omega)] \Pr(d\omega).
\]

The theorem is thus verified.

3 Uncertain Random Process

This section aims at proposing an uncertain random process to model a dynamic uncertain random system. Generally speaking, an uncertain random process is in fact a sequence of uncertain random variables indexed by time or space.

Definition 8. Let \( T \) be a totally ordered set, and \( (\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, M \times \Pr) \) be a chance space. An uncertain random process is a function \( X_t(\gamma, \omega) \) from \( T \times (\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, M \times \Pr) \) to the set of real numbers such that for any Borel set \( B \) of real numbers, the set

\[
\{ X_t \in B \} = \{ (\gamma, \omega) \in \Gamma \times \Omega | X_t(\gamma, \omega) \in B \}
\]

is an uncertain random event, i.e., for each \( t \in T \), the function \( X_t \) is an uncertain random variable.

Remark 1. For each fixed \( \gamma^* \in \Gamma \), the function \( X_t(\gamma^*, \cdot) \) is a stochastic process. For each fixed \( \omega^* \in \Omega \), the function \( X_t(\cdot, \omega^*) \) is an uncertain process.

Example 1. A stochastic process is a special type of uncertain random process.

Example 2. An uncertain process is a special type of uncertain random process.
Example 3. Let \( Y_t \) be a stochastic process, and \( Z_t \) be an uncertain process. If \( f \) is a measurable function, then
\[
X_t = f(Y_t, Z_t)
\]
is an uncertain random process.

Definition 9. An uncertain random process \( X_t \) is said to have a chance distribution \( \Phi_t(x) \) if at each time \( t \), the uncertain variable \( X_t \) has the uncertainty distribution \( \Phi_t(x) \).

Example 4. Assume that \( Y_t \) is a stochastic process with a probability distribution \( \Psi_t \), \( Z_t \) is an uncertain process and \( f \) is a measurable function. Then the uncertain random process
\[
X_t = f(Y_t, Z_t)
\]
has a chance distribution
\[
\Phi_t(y) = \int_{-\infty}^{+\infty} F_t(x,y) d\Psi_t(x)
\]
where \( F_t(x,y) \) is the uncertainty distribution of \( f(x,Y_t) \).

Theorem 4. (Sufficient and Necessary Condition) A function \( \Phi_t(x) : T \times \mathbb{R} \to [0,1] \) is a chance distribution of an uncertain random process if and only if at each time \( t \), it is a monotone increasing function with respect to \( x \) except \( \Phi_t(x) \equiv 0 \) and \( \Phi_t(x) \equiv 1 \).

Proof: On the one hand, assume that \( \Phi_t(x) \) is a chance distribution of an uncertain random process. Then according to Definition 9, the function \( \Phi_t(x) \) is a chance distribution of an uncertain random variable at each time \( t \). As a result, \( \Phi_t(x) \) is a monotone increasing function with respect to \( x \) except \( \Phi_t(x) \equiv 0 \) and \( \Phi_t(x) \equiv 1 \) by Liu [12]. On the other hand, assume \( \Phi_t(x) \) is a monotone increasing function with respect to \( x \) except \( \Phi_t(x) \equiv 0 \) and \( \Phi_t(x) \equiv 1 \). Then it follows from Liu [10] that there exists an uncertain process \( X_t \) whose uncertainty distribution is just \( \Phi_t(x) \). Since an uncertain process is a special case of an uncertain random process, the theorem follows immediately.

Definition 10. Assume \( X_t \) is an uncertain random process on a chance space \((\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\). Then for each fixed \( \gamma^* \in \Gamma \) and \( \omega^* \in \Omega \), the function \( X_t(\gamma^*, \omega^*) \) is called a sample path of the uncertain random process of \( X_t \).

Note that each sample path is a real-valued function with respect to \( t \). An uncertain random process can also be regarded as a function from a chance space to the collection of all sample paths. An uncertain random process is said to be sample-continuous if almost all the sample paths are continuous.

Definition 11. Assume \( X_t \) is an uncertain random process and \( z \) is a given level. Then the uncertain random variable
\[
\zeta_z = \inf\{ t \geq 0 \mid X_t = z \}
\]
is called the first hitting time that \( X_t \) reaches the level \( z \).
Theorem 5. Let $X_t$ be a sample-continuous uncertain random process and $z$ be a given level. Then the first hitting time $\zeta_z$ that $X_t$ reaches the level $z$ has a chance distribution

$$
\Phi(s) = \begin{cases} 
\text{Ch}\left\{\sup_{0 \leq t \leq s} X_t \geq z\right\}, & \text{if } X_0 < z \\
\text{Ch}\left\{\inf_{0 \leq t \leq s} X_t \leq z\right\}, & \text{if } X_0 > z.
\end{cases}
$$

Proof: The argument breaks down into two cases. Case 1: Assume $X_0 < z$. Since $X_t$ is sample-continuous, it follows from the definition of first hitting time that

$$
\{\zeta_z \leq s\} = \sup_{0 \leq t \leq s} X_t \geq z.
$$

Thus the chance distribution of $\zeta_z$ is

$$
\Phi(s) = \text{Ch}\{\zeta_z \leq s\} = \text{Ch}\left\{\sup_{0 \leq t \leq s} X_t \geq z\right\}.
$$

Case 2: Assume $X_0 > z$. Since $X_t$ is sample-continuous, it follows from the definition of first hitting time that

$$
\{\zeta_z \leq s\} = \inf_{0 \leq t \leq s} X_t \leq z.
$$

Thus the chance distribution of $\zeta_z$ is

$$
\Phi(s) = \text{Ch}\{\zeta_z \leq s\} = \text{Ch}\left\{\inf_{0 \leq t \leq s} X_t \leq z\right\}.
$$

The theorem is verified.

4 Stationary Increment Process

Definition 12. An uncertain random process $X_t$ is called a stationary increment uncertain random process if, for any given $t > 0$, the increments $X_{t+s} - X_s$ are identically distributed uncertain random variables for all $s > 0$.

Example 5. Let $\eta_1, \eta_2, \cdots$ be a sequence of iid random variables, and $\tau_1, \tau_2, \cdots$ be a sequence of iid uncertain variables. If $f$ is a measurable function, then

$$
X_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n)
$$

is a stationary increment uncertain random process.

Theorem 6. Let $X_t$ be a stationary increment uncertain random process. Then for any real numbers $a$ and $b$, the uncertain process

$$
Y_t = aX_t + b
$$

is also a stationary increment uncertain random process.
Proof: It follows from the definition of stationary increment uncertain random process that the increments $X_{t+s} - X_s$ are identically distributed uncertain random variables for all $s > 0$. Since

$$Y_{t+s} - Y_s = a(X_{t+s} - X_t),$$

the uncertain random increments $Y_{t+s} - Y_s$ are also identically distributed. Thus $Y_t$ is also a stationary increment uncertain random process.

5 Renewal Process

**Definition 13.** Let $\eta_1, \eta_2, \cdots$ be a sequence of iid random variables, $\tau_1, \tau_2, \cdots$ be a sequence of iid uncertain variables, and $f(x, y)$ be a positive measurable function. Define $S_0 = 0$ and

$$S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n)$$

for $n \geq 1$. Then

$$N_t = \max_{n \geq 0} \{ n | S_n \leq t \}$$

is called an uncertain random renewal process.

**Example 6.** A stochastic renewal process is an uncertain random renewal process, because a random variable is a special uncertain random variable.

**Example 7.** An uncertain renewal process is an uncertain random renewal process, because an uncertain variable is a special uncertain random variable.

Note that the uncertain random process $N_t$ is the total renewal times before $t$. So $N_t/t$ is the average renewal rate during the interval $[0, t]$. The next theorem aims to give the chance distribution of the average renewal rate.

**Theorem 7.** Let $\eta_1, \eta_2, \cdots$ be a sequence of iid random variables with a common probability distribution $\Psi$, $\tau_1, \tau_2, \cdots$ be a sequence of iid uncertain variables, and $f(x, y)$ be a positive and strictly monotone function. Assume that $N_t$ is an uncertain random renewal process with interarrival times $f(\xi_1, \eta_1), f(\xi_2, \eta_2), \cdots$. Then

$$\frac{N_t}{t} \to \left( \int_{-\infty}^{+\infty} f(x, \tau_1) \text{d}\Psi(x) \right)^{-1}$$

in the sense of convergence in distribution as $t \to \infty$.

**Proof:** Define $S_0 = 0$ and $S_n = f(\eta_1, \tau_1) + \cdots + f(\eta_n, \tau_n)$ for $n \geq 1$. Then it follows from the definition of renewal process that

$$\text{Ch} \left\{ \frac{N_t}{t} \leq y \right\} = \text{Ch}\{N_t \leq ty\} = \text{Ch}\{N_t \leq \lfloor ty \rfloor\} = \text{Ch}\{S_{\lfloor ty \rfloor + 1} > t\} = \text{Ch}\left\{ \frac{S_{\lfloor ty \rfloor} + 1}{\lfloor ty \rfloor + 1} > \frac{t}{\lfloor ty \rfloor + 1} \right\}$$
where \(\lfloor ty \rfloor\) represents the maximal integer less than or equal to \(ty\). Writing \(n = \lfloor ty \rfloor + 1\), we have \(n \to \infty\) as \(t \to +\infty\) for any given \(y > 0\). Besides, we also have \(ty \in [n-1, n)\), which results in

\[
\left(1 - \frac{1}{n}\right) \frac{1}{y} \leq \frac{t}{\lfloor ty \rfloor + 1} < \frac{1}{y}.
\]

As a result,

\[
\text{Ch} \left\{ \frac{S_n}{n} > \frac{1}{y} \right\} \leq \text{Ch} \left\{ \frac{S_{\lfloor ty \rfloor + 1}}{\lfloor ty \rfloor + 1} > \frac{t}{\lfloor ty \rfloor + 1} \right\} \leq \text{Ch} \left\{ \frac{S_n}{n} > \left(1 - \frac{1}{n}\right) \frac{1}{y} \right\}.
\]

For any continuous point \(y\) of the uncertainty distribution of

\[
\left( \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \right)^{-1},
\]

i.e., for any continuous point \(1/y\) of the uncertainty distribution of

\[
\int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x),
\]

since

\[
\frac{S_n}{n} \to \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x)
\]

in the sense of convergence in distribution as \(n \to \infty\) by the law of large numbers (Theorem 2), we have

\[
\lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} > \frac{1}{y} \right\} = 1 - \lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \frac{1}{y} \right\} = 1 - \mathcal{M} \left\{ \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \leq \frac{1}{y} \right\}
\]

\[
= \mathcal{M} \left\{ \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) > \frac{1}{y} \right\} = \mathcal{M} \left\{ \left( \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \right)^{-1} < y \right\}
\]

\[
= \mathcal{M} \left\{ \left( \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \right)^{-1} \leq y \right\}
\]

and

\[
\lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} > \left(1 - \frac{1}{n}\right) \frac{1}{y} \right\} = \lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \frac{n}{n - 1} > \frac{1}{y} \right\} = 1 - \lim_{n \to \infty} \text{Ch} \left\{ \frac{S_n}{n} \leq \frac{n}{n - 1} \leq \frac{1}{y} \right\}
\]

\[
= 1 - \mathcal{M} \left\{ \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \leq \frac{1}{y} \right\} = \mathcal{M} \left\{ \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) > \frac{1}{y} \right\}
\]

\[
= \mathcal{M} \left\{ \left( \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \right)^{-1} < y \right\}
\]

\[
= \mathcal{M} \left\{ \left( \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \right)^{-1} \leq y \right\}.
\]

Thus

\[
\lim_{t \to +\infty} \text{Ch} \left\{ \frac{N_t}{t} \leq y \right\} = \lim_{t \to +\infty} \text{Ch} \left\{ \frac{S_{\lfloor ty \rfloor + 1}}{\lfloor ty \rfloor + 1} > \frac{t}{\lfloor ty \rfloor + 1} \right\} = \mathcal{M} \left\{ \left( \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \right)^{-1} \leq y \right\}
\]

for any continuous point \(y\) of the uncertainty distribution of

\[
\left( \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \right)^{-1}.
\]
In other words, the average renewal rate

\[
\frac{N_t}{t} \to \left( \int_{-\infty}^{+\infty} f(x, \tau_1) \, d\Psi(x) \right)^{-1}
\]

in the sense of convergence in distribution as \( t \to +\infty \). The theorem is proved.

**Remark 2.** Let \( N_t \) be a renewal process with iid positive random interarrival times \( \eta_1, \eta_2, \cdots \). Then the average renewal rate

\[
\frac{N_t}{t} \to \frac{1}{E[\eta_1]}
\]

in the sense of convergence in distribution as \( t \to \infty \).

**Remark 3.** Let \( N_t \) be a renewal process with iid positive uncertain interarrival times \( \tau_1, \tau_2, \cdots \). Then the average renewal rate

\[
\frac{N_t}{t} \to \frac{1}{\tau_1}
\]

in the sense of convergence in distribution as \( t \to \infty \).

**Example 8.** Let \( N_t \) be a renewal process with uncertain random interarrival times \( \eta_1 + \tau_1, \eta_2 + \tau_2, \cdots \) where \( \{\eta_i\} \) is a sequence of iid positive random variables, and \( \{\tau_i\} \) is a sequence of iid positive uncertain variables. Then the average renewal rate

\[
\frac{N_t}{t} \to \frac{1}{E[\eta_1] + \tau_1}
\]

in the sense of convergence in distribution as \( t \to +\infty \).

**Example 9.** Let \( N_t \) be a renewal process with uncertain random interarrival times \( \eta_1 \times \tau_1, \eta_2 \times \tau_2, \cdots \) where \( \{\eta_i\} \) is a sequence of iid positive random variables, and \( \{\tau_i\} \) is a sequence of iid positive uncertain variables. Then the average renewal rate

\[
\frac{N_t}{t} \to \frac{1}{E[\eta_1] \times \tau_1}
\]

in the sense of convergence in distribution as \( t \to +\infty \).

### 6 Conclusion

This paper first proposed a concept of uncertain random process. As special types of uncertain random processes, this paper also proposed stationary increment uncertain random process and uncertain random renewal process. Besides, this paper studied the average renewal rate of the renewal process, and gave the chance distribution of its limit.
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References


