Performance Analysis and High-SNR Power Allocation for MIMO ZF Receivers with a Precoder in Transmit-Correlated Rayleigh Channels

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Abstract—In this letter, the multiple-input multiple-output (MIMO) system with a precoder is considered in the transmit-correlated Rayleigh channels. We especially target the MIMO system employing zero forcing receivers. Based on random matrix theory, we first derive the exact probability density function (PDF) of the signal-to-interference-plus-noise ratio (SINR). Using the derived SINR PDF and a close approximation of the Gaussian Q-function, we derive a tight closed-form approximation of the symbol error rate (SER). In the high signal-to-noise ratio (SNR) regime, we also propose a high-SNR power allocation (HPA) by minimizing the global SER approximation under the total power constraint. Our SER analysis suggests that the SER approximation can be used to estimate the error probability. At high SNRs, the computationally efficient HPA converging to the optimal PA achieves noticeable performance gain over the equal PA, particularly in the high transmit correlation scenario. Furthermore, the performance gain of the proposed HPA increases with the diversity order.

Index Terms—Correlated channels, multiple-input multiple-output (MIMO), signal-to-interference-plus-noise ratio (SINR), symbol error rate (SER), zero forcing (ZF).

I. INTRODUCTION

MULTIPLE-input multiple-output (MIMO) communications have emerged as one of the most promising techniques due to the great potential to improve the system reliability and capacity [1]. Diversity gain improves link reliability, while spatial multiplexing (SM) gain increases a data rate. In this letter, we focus on the SM systems with zero forcing (ZF) receiver [2] only.

The transmitted signal of MIMO ZF receivers may be optimized with a precoder [3]–[7]. The design of precoders aims to improve performance by properly allocating resources such as power and bits over multiple antennas. References [3], [4] assumed perfect channel state information (CSI) at both the transmitter and receiver. In practice, however, perfect CSI is rarely available at the transmitter [8]. Hence, it was assumed in [5]–[7] that only the transmit (Tx) and receive (Rx) correlation matrices are available at the transmitter. It was shown in [5]–[7] that, under various criteria (e.g., minimum mean square error and maximum ergodic capacity), the left eigenvectors of the optimal precoder coincide with the eigenvectors of the Tx correlation matrix.

For the uncorrelated Rayleigh channels, [9] provided an exponentially tight upper bound on the error performance of MIMO ZF receivers, and the exact probability density function (PDF) of the signal-to-interference-plus-noise ratio (SINR) and the exact closed-form expression of the bit error rate (BER) for MIMO ZF receivers were derived in [2]. For the transmit-correlated Rayleigh channels, the SINR after ZF receivers was shown as a Gamma distribution [10], which was used in [11] to obtain the BER approximations, and the SINR PDF for the multiuser MIMO interference alignment was shown as an exponential distribution [12]. Some symbol error rate (SER) approximations for MIMO ZF receivers in the correlated Rayleigh channels were derived in [13].

Motivated by the discussions above, this letter considers MIMO ZF receivers with precoding in the transmit-correlated Rayleigh channels. We assume that perfect CSI is known at the receiver, but that only its Tx and Rx correlation matrices are available at the transmitter. We also assume the optimal precoder using the eigenvectors of the Tx correlation matrix [5]–[7]. Based on random matrix theory, we first derive the exact SINR PDF. Using the derived SINR PDF and a close approximation [14] of Gaussian Q-function, we derive the tight closed-form SER approximation.

The power allocation (PA) problem given channel statistics at the transmitter has been widely studied in the literature [5], [7], [15] under a variety of criteria, including maximum ergodic capacity and minimum BER. In this letter, we show that, at high signal-to-noise ratios (SNRs), the global SER approximation (i.e., SER approximation averaged over all streams) is strictly convex with respect to the Tx power ratio. Using a convex optimization framework, we propose the high-SNR PA (HPA) with the aim of minimizing the global SER approximation, assuming the total power constraint. Our analysis indicates that, at high SNRs, the computationally efficient HPA offers almost the same performance gain as the optimal PA (i.e., an exhaustive search at all SNRs) and outperforms the equal PA, especially in the high Tx correlation scenario. Moreover, the performance gain of the proposed HPA increases with the diversity order.

II. SYSTEM MODEL

Let us consider a point-to-point MIMO system with $N_t$ Tx and $N_r$ Rx antennas ($N_r \geq N_t$). We focus on an SM system where the $N_t$ streams are mapped to Gray-coded $M$-ary quadrature amplitude modulation ($M$-QAM) with $M = 4^z$ for some integer $z$. As it was previously mentioned, we assume that the receiver has perfect CSI while the transmitter knows...
the Tx and Rx correlation matrices of the channel. The \( N_r \times 1 \) received signal vector can be modeled as

\[
y = H_s R_s^T F s + n,
\]

where \( s = [s_1, \ldots, s_{N_r}]^T \) is the transmitted vector with \( E[|s|^2] = E_s/N_r \), \( n \) is the noise vector \( \sim C N_{N_r,1}(0_{N_r \times 1}, N_0 I_{N_r}) \), and the channel matrix is decomposed as the \( N_r \times N_t \) Tx correlation matrix \( R_t \) and independent and identically distributed (i.i.d.) complex Gaussian matrix \( H_{s} \sim C N_{N_r,N_t}(0_{N_r \times N_t}, I_{N_r} \otimes I_{N_t}) \). \( F \) is the \( N_t \times N_t \) precoder matrix [5], [7].

\[
F = V \Phi^\dagger, \quad (2)
\]

where the \( N_t \times N_t \) unitary matrix \( V \) is a linear precoder and the \( N_t \times N_t \) diagonal matrix \( \Phi \) determines the fraction of the available power as

\[
\Phi = \text{diag}[p_1, \ldots, p_{N_t}]
\]

with \( \sum_{k=1}^{N_t} p_k = N_t \) and \( p_k > 0 \).

Let us define an effective channel matrix as \( H = H_{s} R_s^T F \). While the analytical results in this letter apply to an arbitrary \( R_t \), throughout this letter, we consider the exponential correlation model [16], which is reasonable in the case of the equally spaced linear array. The \( (i,j) \)th entries of \( R_t \) are given by

\[
R_{t}]_{ij} = \rho^{|i-j|} \quad \text{for } \rho \in [0, 1). \quad (4)
\]

At the receiver, the ZF equalization matrix is given by [2]

\[
W = (H^H H)^{-1} H^H. \quad (5)
\]

Then, the SINR \( \gamma_k \) for the \( k \)th stream after the ZF receiver can be represented as [10]

\[
\gamma_k = \frac{E_s}{N_t N_0} \left[ (H^H H)^{-1} \right]_{kk} \left[ (H^H H)^{-1} \right]_{kk}^{-1} = \frac{1}{\sigma^2} \left[ (H^H H)^{-1} \right]_{kk}^{-1}, \quad (6)
\]

where \( \sigma^2 = N_t/\bar{\eta} \) and \( \bar{\eta} = E_s/N_0 \) is defined as the SNR per Rx antenna.

### III. PERFORMANCE ANALYSIS

#### A. Distribution of SINR

Consider the eigenvalue decomposition of \( R_t \) as \( R_t = U \Lambda U^H \), where \( U \) is the \( N_t \times N_t \) unitary matrix and \( \Lambda \) is the \( N_t \times N_t \) diagonal matrix. The optimal precoding matrix \( F^\text{opt} \) in terms of the maximum ergodic capacity criterion [5], [7] and minimum mean square error criterion [5] is given by

\[
F^\text{opt} = U \Phi^\dagger. \quad (7)
\]

From the property of the multivariate Normal distribution, \( H \) is then described by [17, Theorem 2.3.10]

\[
H \sim C N_{N_r,N_t} (0_{N_r \times N_t}, I_{N_r} \otimes (\Phi^\dagger \Lambda \Phi^\dagger)). \quad (8)
\]

Applying [10, Lemma 1] to (6) and (8), \( \gamma_k \) follows the scaled version of chi-square distribution with \( 2(N_r - N_t + 1) \) degrees of freedom as

\[
f_{\gamma_k}(\gamma_k) = \frac{c e^{-\frac{\gamma_k}{\lambda_{kk}}}}{\lambda_{kk}^n (n-1)!} (\frac{\gamma_k}{\lambda_{kk}})^{n-1}, \quad (9)
\]

where \( n = N_r - N_t + 1, \lambda_{kk} = |\Lambda|_{kk}, \) and \( c = \sigma^2/\eta_k \).

#### B. Error Probability Analysis

Using a closely approximated Q-function given by [14]

\[
Q(x) \simeq \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \frac{1}{6} e^{-\frac{x^2}{3}} e^{-\frac{2x^2}{3}}, \quad (10)
\]

the probability of symbol error in the \( M \)-QAM system can be closely approximated as [14]

\[
\bar{P}_{N_t \times N_r}(\gamma_k) \simeq \left( \frac{1}{\sqrt{M}} \right) \left( \frac{1}{3} e^{-\frac{\eta_k}{M}} + \frac{2}{3} e^{-\frac{2\eta_k}{M}} \right). \quad (11)
\]

Using (11), the average SER for the \( k \)th stream can then be tightly approximated as

\[
P_{N_t \times N_r}^k(\gamma_k) = \int_0^\infty \bar{P}_{N_t \times N_r}(\gamma_k) f_{\gamma_k}(\gamma_k) d\gamma_k \simeq \left( \frac{1}{\sqrt{M}} \right) \left( \frac{1}{3} M_{N_t \times N_r}(\gamma_k) \left( \frac{3}{2(M-1)} \right) \right. \left. + \frac{2}{3} M_{N_t \times N_r}(\gamma_k) \left( \frac{2}{M-1} \right) \right), \quad (12)
\]

where \( M_{N_t \times N_r}(\cdot) \) is the moment generating function (MGF) of \( \gamma_k \) defined as

\[
M_{N_t \times N_r}(m) = \int_0^\infty \exp (-m \gamma_k) f_{\gamma_k}(\gamma_k) d\gamma_k. \quad (13)
\]

Applying (9) to (13), we obtain the closed-form MGF as

\[
M_{N_t \times N_r}^k(m) = \frac{1}{(n-1)!} \left( \frac{e}{\lambda_{kk}} \right)^n \int_0^\infty \gamma_k^{-n} e^{-\left( \frac{\lambda_{kk} m}{\lambda_{kk}} \right) \gamma_k} \gamma_k d\gamma_k = \left( \frac{1}{\lambda_{kk} m + 1} \right)^n, \quad (14)
\]

where the last line followed by applying the identity [18, pp. 336, 3.351.3]. Substituting (14) into (12) yields the tight closed-form SER approximation

\[
P_{N_t \times N_r}^k(\gamma_k) \simeq A_1 \left( \frac{\lambda_{kk} A_2 + 1}{3} + 2 \left( \frac{\lambda_{kk} A_3 + 1}{3} \right)^n \right), \quad (15)
\]

where \( A_1 = (1 - \frac{1}{\sqrt{M}}), A_2 = \frac{3}{2(M-1)}, \) and \( A_3 = \frac{2}{(M-1)} \).

We now analyze the SER performance in the high-SNR regime \( (\bar{\eta} \rightarrow \infty) \) in order to evaluate the diversity order of the system. Consider the MGF expression given by (14). Since \( c = \frac{\lambda_{kk}}{\eta_k}, \) we simply obtain the \( M_{N_t \times N_r}^k(m) \) at high SNRs, denoted by \( M_{N_t \times N_r}^{k,\infty}(m) \), as

\[
M_{N_t \times N_r}^{k,\infty}(m) = \frac{1}{(\lambda_{kk} \eta_k m + 1)^n} \left( \frac{\lambda_{kk} m}{\lambda_{kk} \eta_k m} \right)^n \left( \frac{\lambda_{kk} \eta_k m}{\lambda_{kk} m} \right)^{-n}, \quad (16)
\]
Substituting (16) into (12) gives \( P_{N_t \times N_r}^k \) at high SNRs, denoted by \( P_{N_t \times N_r}^{k, \infty} \), as

\[
P_{N_t \times N_r}^{k, \infty} = A_1 \left( \frac{N_t}{\lambda_{kk} p_k} \right)^n \left( \frac{1}{3 A_2^\infty} \right) p_k^{-n}. \tag{17}
\]

Therefore, clearly the MIMO ZF receivers with a precoder achieve the asymptotical diversity order of \( n = N_r - N_t + 1 \) in the transmit-correlated Rayleigh channels.

Fig. 1 shows the SER of each stream \( k \) in the \( 2 \times 6 \) MIMO system with correlated MIMO channels (Set 1, \( \rho = 0.4 \) and \( \Phi = \text{diag} \{1,1\} \) and Set 2 (i.e., \( \rho = 0.99 \) and \( \Phi = \text{diag} \{1.9,0.1\} \) when 16-QAM modulation is used. The approximation curves are generated from (15). The approximation (high SNR) curves are generated from (17). All investigated cases generally show good agreement with the Monte Carlo (MC) simulations. Clearly the approximation curves asymptotically approach the approximation (high SNR) curves, confirming that the diversity order is achieved.

IV. HIGH-SNR POWER ALLOCATION

In this section, we provide a PA strategy in the high-SNR regime by minimizing the global SER subject to a constraint of \( \sum_{k=1}^{N_t} p_k = N_t \).

The global SER can be estimated as

\[
P_{N_t \times N_r} = \frac{1}{N_t} \sum_{k=1}^{N_t} P_{N_t \times N_r}^k. \tag{18}
\]

Let \( P_{N_t \times N_r}^{k, \infty} \) be the global SER in the high-SNR regime. From (17), \( P_{N_t \times N_r}^{k, \infty} \) can be written as

\[
P_{N_t \times N_r}^{k, \infty} = A_1 \left( \frac{N_t}{\lambda_{kk} \bar{q}} \right)^n \left( \frac{1}{3 A_2^\infty} \right) p_k^{-n}. \tag{19}
\]

If \( T_{N_t \times N_r}^k \) is given, one can find that \( T_{N_t \times N_r}^k > 0 \). Then, we have

\[
\frac{\partial^2}{\partial p_k^2} P_{N_t \times N_r}^{k, \infty} = n (n + 1) T_{N_t \times N_r}^k p_k^{-n-2} \tag{20}
\]

so that \( \frac{\partial^2}{\partial p_k^2} P_{N_t \times N_r}^{k, \infty} > 0 \) for \( p_k > 0 \). Therefore, \( P_{N_t \times N_r}^{k, \infty} \) is strictly convex [19, pp. 71]. Since a nonnegative, nonzero weighted sum of strictly convex functions is strictly convex [19, pp. 79], \( P_{N_t \times N_r}^{k, \infty} \) is also strictly convex. The PA optimization problem at high SNRs can then be formulated as follows:

\[
\begin{aligned}
\text{minimize} & \quad P_{N_t \times N_r}^{\infty} \\
\text{subject to} & \quad \sum_{k=1}^{N_t} p_k = N_t, \tag{21}
\end{aligned}
\]

Note that the convex optimization problem (21) has the classical Lagrange multiplier optimality condition and a unique solution [19, Ch. 4]. By introducing the Lagrange multiplier \( \mu \) for the constraint in (21), the Lagrangian function can be written as

\[
\mathcal{L}(p, \mu) = P_{N_t \times N_r}^{\infty} + \mu \left( \sum_{k=1}^{N_t} p_k - N_t \right) \tag{22}
\]

with \( p = (p_1, \cdots, p_{N_t}) \). The HPA solution can be obtained from the conditions as follows:

\[
\begin{align*}
\frac{\partial}{\partial \mu} \mathcal{L}(p, \mu) & = \sum_{k=1}^{N_t} p_k - N_t = 0 \tag{23} \\
\frac{\partial}{\partial p_k} \mathcal{L}(p, \mu) & = -n A_1 \left( \frac{N_t}{\lambda_{kk} \bar{q}} \right)^n \left( \frac{1}{3 A_2^\infty} \right) p_k^{-n-1} + \mu = 0. \tag{24}
\end{align*}
\]

From (24), we have

\[
S_{N_t \times N_r, p_k}^{k-1} = S_{N_t \times N_r, p_k}^{k-1} = \cdots = S_{N_t \times N_r, p_1}^{k-1}, \tag{25}
\]

yielding

\[
p_k = \left( \frac{S_{N_t \times N_r}^{1}}{S_{N_t \times N_r}^{k}} \right)^{-\frac{1}{n+1}} p_1. \tag{26}
\]

Applying (26) to (23), we have

\[
p_1 = N_t \left( \sum_{k=1}^{N_t} \left( \frac{S_{N_t \times N_r}^{1}}{S_{N_t \times N_r}^{k}} \right)^{-\frac{1}{n+1}} \right)^{-1}. \tag{27}
\]

Using (26) and (27), we obtain the HPA solution \( p^* \) as follows:

\[
p^*_k = N_t \left( \frac{S_{N_t \times N_r}^{1}}{S_{N_t \times N_r}^{k}} \right)^{-\frac{1}{n+1}} \left( \sum_{k=1}^{N_t} \left( \frac{S_{N_t \times N_r}^{1}}{S_{N_t \times N_r}^{k}} \right)^{-\frac{1}{n+1}} \right)^{-1}. \tag{28}
\]
Fig. 2 shows the global SER of the proposed HPA in the various MIMO configurations with 16-QAM modulation, comparing various correlation scenarios. The global SERs using the optimal PA and equal PA are also plotted. Note that the optimal PA solution is obtained through an exhaustive search over the $N_t$-dimensional space of $(p_1, \ldots, p_{N_t})$, assuming that each $p_k$ takes discrete values [20]. We see that, at high SNRs, extra correlation increases the performance gain compared to the optimal PA. In particular, for all SERs of practical interest (i.e., $P_{N_t} < 0.01$), the proposed HPA curves match almost exactly with the optimal PA curves. It can be seen that the global SER curves increase monotonically with the correlation. It is also observed that, at high SNRs, extra correlation increases the performance gain of the proposed HPA as compared to the equal PA. The performance gain of the proposed HPA can be increased even more when the diversity order is increased. At $\rho = 0.9$, the high-SNR performance gains in $2 \times 2$ and $2 \times 6$ MIMO systems are about 1.4 and 2.5 dB, respectively. It is noted that, when $\rho \approx 0$, the optimal PA is the equal PA.

V. CONCLUSION

We investigated the performance of the MIMO ZF receivers with precoding in the transmit-correlated Rayleigh channels. Based on the derived transmit-covariance PDF and the close approximation of the Gaussian Q-function, we derived the tight closed-form SER approximation. Our SER analysis suggests that the SER approximation can be used to estimate the error probability. In the high-SNR regime, we presented a computationally efficient HPA by minimizing the global SER approximation subject to the total power constraint. At high SNRs, the HPA converging to the optimal PA outperforms the equal PA, particularly in the high Tx correlation scenario. Moreover, the performance gain of the proposed HPA increases with the diversity order.

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REFERENCES