On the non-existence of tight Gaussian 6-designs on two concentric spheres

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Outline

1. Introduction
2. Distance sets
3. Euclidean $t$-designs
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5. Main results

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Introduction


3Ei. Bannai, Et. Bannai, M. Hirao, M. Sawa, On the non-existence of minimal cubature formulas for Gaussian measure on $\mathbb{R}^2$ of degree $t$ supported by $\left\lfloor \frac{t}{4} \right\rfloor + 1$ circles, J. Algebr Comb. 35 (2012) 109–119.
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Introduction

- Gaussian tight 4 and 9-designs are solved by Ei. Bannai and Et. Bannai\textsuperscript{1}.
- Hirao and Sawa\textsuperscript{2} proved that there exists no Gaussian tight $t$-design support by $k + 1$ concentric spheres for $t = 4k + 1 (k \geq 2)$ and $t = 4k + 3 (k \geq 1)$.
- Ei. Bannai, Et. Bannai, Hirao, and Sawa\textsuperscript{3} showed that there exists no Gaussian tight $t$-design of $\mathbb{R}^2$ supported by $\left\lceil \frac{e}{2} \right\rceil + 1$ concentric circles for $t = 2e (e \geq 2)$ except $e = 2$.
- In this talk, we discuss tight Gaussian 6-Designs of $\mathbb{R}^n$, where $n \geq 2$.


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Notation

- $\mathcal{P}(\mathbb{R}^n) = \mathbb{R}[x_1, x_2, \cdots, x_n]$: the vector space of polynomials in $n$ variables $x_1, x_2, \cdots, x_n$.
- $\text{Hom}_l(\mathbb{R}^n)$: the subspace of $\mathcal{P}(\mathbb{R}^n)$ spanned by homogeneous polynomials of degree $l$.
- $\text{Harm}(\mathbb{R}^n)$: the subspace of $\mathcal{P}(\mathbb{R}^n)$ which consists of all the harmonic polynomials.
- $\mathcal{P}_e(\mathbb{R}^n) := \sum_{l=0}^e \text{Hom}_l(\mathbb{R}^n)$, $\mathcal{P}_e^*(\mathbb{R}^n) := \sum_{l=0}^{[\frac{e}{2}]} \text{Hom}_{e-2l}(\mathbb{R}^n)$.
- $\text{Harm}_l(\mathbb{R}^n) := \text{Harm}(\mathbb{R}^n) \cap \text{Hom}_l(\mathbb{R}^n)$.
- For a subset $Y \subseteq \mathbb{R}^n$, $\mathcal{P}(Y) := \{ f|_Y | f \in \mathcal{P}(\mathbb{R}^n) \}$. $\text{Hom}_l(Y)$, $\text{Harm}(Y)$, $\text{Harm}_l(Y)$, $\mathcal{P}_e(Y)$, and $\mathcal{P}_e^*(Y)$ are defined in the same way.
Notation

- $(\ , \ )$: the usual inner product between the vectors in $\mathbb{R}^n$
  $\|x\| = \sqrt{(x, x)}$: the length of the vectors in $\mathbb{R}^n$.  

- $(X, \omega)$: a weighted finite set in $\mathbb{R}^n$ whose weight satisfies $\omega(x) > 0$ for $x \in X$.

- $\{r_1, r_2, \cdots, r_p\} := \{\|x\| | x \in X\}$. Assume $r_1 > r_2 > \cdots > r_p \geq 0$.
- $S_i = \{x \in \mathbb{R}^n | \|x\| = r_i\}$: the sphere of radius $r_i$ centered at the origin for $1 \leq i \leq p$. In this case, we say that $X$ is supported by $p$ concentric spheres.

- $X_i := X \cap S_i$, $R_i := r_i^2$ and $\omega(X_i) := \sum_{x \in X_i} \omega(x), 1 \leq i \leq p$.  

Definition\textsuperscript{a} (P. Delsarte, M. Goethals and J.J. Seidel, 1977) A subset $X$ in $\mathbb{R}^n$ is called an \textit{s-distance set}, if $|A(X)| = s$, where $A(X) = \{d(x, y) | x, y \in X, x \neq y\}$, $d(x, y)$ is the Euclidean distance of $x$ and $y$ and $|A(X)|$ denotes the cardinality of $A(X)$.

Lemma (P. Delsarte, J.M. Goethals, J.J. Seidel 1977): Let $X \subset S^{n-1}$ be an $s$-distance set. Then

$$|X| \leq \dim(\mathcal{P}_s(S^{n-1})).$$

Theorem\(^a\) (H. Nozaki 2011): Let \(X\) be an \(s\)-distance set in \(\mathbb{R}^n\) with \(s \geq 2\), and \(A(X) = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}\). Let \(N = \binom{n + s - 1}{s - 1} + \binom{n + s - 2}{s - 2}\). If \(|X| \geq 2N\), then

\[
\prod_{j=1,2,\ldots,s, j \neq i} \frac{\alpha_j^2}{\alpha_j^2 - \alpha_i^2}
\]

is an integer for each \(i = 1, 2, \ldots, s\).

In the proof of our main result we will use the following corollary.

**Corollary 1:** Let $X$ be a 3-distance set in $\mathbb{R}^n$ and $A(X) = \{\alpha_1, \alpha_2, \alpha_3\}$. Let $A_1 = \alpha_1^2$, $A_2 = \alpha_2^2$, $A_3 = \alpha_3^2$. Then

$$k_1 = \frac{A_2 A_3}{(A_2 - A_1)(A_3 - A_1)}, \quad k_2 = \frac{A_1 A_3}{(A_1 - A_2)(A_3 - A_2)}, \quad k_3 = \frac{A_1 A_2}{(A_1 - A_3)(A_2 - A_3)}$$

are integers. Moreover,

$$k_1 k_2 + k_1 k_3 + k_2 k_3 = -\frac{A_1 A_2 A_3^2}{(A_1 - A_2)^2 (A_1 - A_3)(A_2 - A_3)} + \frac{A_1 A_2^2 A_3}{(A_1 - A_2)(A_1 - A_3)^2 (A_2 - A_3)} - \frac{A_1^2 A_2 A_3}{(A_1 - A_2)(A_1 - A_3)(A_2 - A_3)^2}$$

is an integer.
The concept of Euclidean $t$-designs was defined by Neumaier and Seidel.

**Euclidean $t$-designs**

**Definition** (Neumaier and Seidel 1988): Let $X$ be a finite set in $\mathbb{R}^n$ supported by $p$ concentric spheres. Let $\omega$ be a positive weight function on $X$. $X$ is called a *Euclidean $t$-design* if

$$\sum_{i=1}^{p} \frac{\omega(X_i)}{|S_i|} \int_{S_i} f(x) d\sigma_i(x) = \sum_{x \in X} \omega(x) f(x)$$

holds for any polynomial $f(x) \in \mathcal{P}_t(\mathbb{R}^n)$.

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Euclidean t-designs

Equivalent condition for Euclidean t-designs

Lemma 1 (Neumaier and Seidel 1988): Let $X$ be a finite set in $\mathbb{R}^n$. Let $\omega$ be a positive weight function defined on $X$. Then the following conditions are equivalent:

(i) $X$ is a Euclidean $t$-design with weight $\omega$.

(ii) $\sum_{x \in X} \omega(x)f(x) = 0$ for any polynomial $f \in \|x\|^{2j}\text{Harm}_l(\mathbb{R}^n)$ with $1 \leq l \leq t$, $0 \leq j \leq \left\lfloor \frac{t-l}{2} \right\rfloor$. 
Euclidean $t$-designs

**Euclidean tight 2e-design**

**Definition** (Ei. Bannai, Et. Bannai 2005) Let $X$ be a Euclidean $2e$-design supported by $S$, the union of $p$ concentric spheres. If $|X| = \dim(\mathcal{P}_e(S))$, then $X$ is called a *Euclidean tight 2e-design on $p$ concentric spheres*. If $|X| = \dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$, then $X$ is called a *Euclidean tight 2e-design of $\mathbb{R}^n$*.

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Some properties of Euclidean tight $2e$-design

Lemma 2 (Ei. Bannai, Et. Bannai 2005): Let $X$ be a Euclidean tight $2e$-design of $\mathbb{R}^n$. If $0 \in X$, then $e$ is even and $p = \frac{e}{2} + 1$.

Lemma 3$^a$ (Et. Bannai 2006): Let $X$ be a Euclidean tight $2e$-design on $p$ concentric spheres. If $e - p + \varepsilon_S \geq 0$, then each $X_i$ is similar to a spherical $(2e - 2p + 2\varepsilon_S + 2)$-design.

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Gaussian $t$-designs

**Definition**$^a$ (Ei. Bannai, Et. Bannai 2005): Let $X \subset \mathbb{R}^n$ be a finite set. We say $X$ is a *Gaussian $t$-design* if the following condition holds for any polynomial $f(x)$ in $n$ variables $x_1, x_2, \ldots, x_n$ of degree at most $t$:

$$
\frac{1}{V(\mathbb{R}^n)} \int_{\mathbb{R}^n} f(x) e^{-\alpha^2 \|x\|^2} \, dx = \sum_{x \in X} \omega(x) f(x)
$$

where $\alpha$ is a positive real number, $V(\mathbb{R}^n) = \int_{\mathbb{R}^n} e^{-\alpha^2 \|x\|^2} \, dx$, and $\omega$ is a weight function on $X$ satisfying $\omega(x) > 0$ for any $x \in X$ and $\sum_{x \in X} \omega(x) = 1$.

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The relationship between Gaussian and Euclidean $t$-designs


Tight Gaussian $2e$-designs

Definition (Ei. Bannai, Et. Bannai 2005): Gaussian $2e$-design $X$ is said tight if $|X| = \binom{n+e}{e}$ holds.

The structure of tight Gaussian $t$-designs

Lemma 5 (Ei. Bannai, Et. Bannai 2005): Let $X$ be a tight Gaussian $2e$-design. Suppose that $X$ is supported by $p$ concentric spheres. Then the followings hold:

(i) $p \geq \left\lceil \frac{e}{2} \right\rceil + 1$.

(ii) $\omega(x)$ is constant on each $X_i$, for $i = 1, \cdots, p$.

(iii) Each $X_i$ is an at most $e$-distance set, for $i = 1, \cdots, p$. 

On the non-existence of tight Gaussian 6-designs on two concentric spheres
Two important equations

Assume that $X = \bigcup_{i=1}^{p} X_i$ is a tight Gaussian $2e$-design with a weight function $\omega$. Let $u, v \in X_i$. Write $R_i = r_i^2 = \|u\|^2 = \|v\|^2$.

Ei. Bannai and Et. Bannai\textsuperscript{a} obtained the following important equations:

$$\omega(u) \sum_{l+2j \leq e} R_i^l g_{l,j}(R_i)^2 Q_l(1) = 1,$$  \hspace{1cm} (2)

$$\sum_{l+2j \leq e} R_i^l g_{l,j}(R_i)^2 Q_l\left(\frac{(u,v)}{R_i}\right) = 0.$$ \hspace{1cm} (3)

where $g_{l,j}(R)$ is a polynomial in one variable $R$ of degree $j$ and

$$\frac{1}{\int_0^{\infty} r^{n-1} e^{-\alpha^2 r^2} dr} \int_0^{\infty} g_{l,j_1}(r^2) g_{l,j_2}(r^2) r^{n-1+2l} e^{-\alpha^2 r^2} dr = \delta_{j_1,j_2},$$ \hspace{1cm} (4)

holds.

Main result

**Theorem**

There exists no tight Gaussian 6-design supported by two concentric spheres in $\mathbb{R}^n$ for $n \geq 2$. 
In order to prove the main result we need calculate Equations (2) and (3) for the case $e = 3$ and the values of $R_i, \omega_i, \ (i = 1, 2)$.

**Lemma 6:** Assume that $X = X_1 \cup X_2$ is a tight Gaussian 6-design on two concentric spheres in $\mathbb{R}^n$. Pick $i \ (i = 1, 2)$ and let $u, v \in X_i$ with $u \neq v$. Write $R_i = \|u\|^2 = r_i^2$ and $\|u - v\|^2 = A$. Then

\[
\frac{4}{3} \alpha^6 R_i^3 - 2\alpha^4 R_i^2 + \alpha^2 R_i n + 2\alpha^2 R_i + \frac{n}{2} + 1 = \frac{1}{\omega(u)}, \tag{5}
\]

and

\[
-\frac{\alpha^6 A^3}{6} + (\alpha^6 R_i + \frac{\alpha^4}{2}) A^2 - (2\alpha^6 R_i^2 + \frac{\alpha^2 n}{2} + 2\alpha^2) A
\]

\[
+ \left( \frac{4\alpha^6 R_i^3}{3} - 2\alpha^4 R_i^2 + \alpha^2 n R_i + 2\alpha^2 R_i + \frac{n}{2} + 1 \right) = 0. \tag{6}
\]
Proof. To show the assertion we need compute the polynomials $g_{l,j}(R)$ of degree $j$ satisfying equation (4) for $l + 2j \leq 3$ and $l, j \geq 0$. We note that

$$
\int_0^\infty r^{l+1} e^{-\alpha^2 r^2} dr = \frac{l}{2\alpha^2} \int_0^\infty r^{l-1} e^{-\alpha^2 r^2} dr
$$

(7)

holds for all $l > 0$. Combining equations (4) and (7), we obtain
\begin{align*}
g_{0,0}^2 &= 1, \quad (8) \\
g_{0,1}^2(R) &= \frac{2\alpha^4}{n}(R - \frac{n}{2\alpha^2})^2, \quad (9) \\
g_{1,0}^2 &= \frac{2\alpha^2}{n}, \quad (10) \\
g_{1,1}^2(R) &= \frac{\alpha^2}{n}\left(\frac{2\alpha^2}{\sqrt{n+2}}R - \sqrt{n+2}\right)^2, \quad (11) \\
g_{2,0}^2 &= \frac{4\alpha^4}{n(n+2)}, \quad (12) \\
g_{3,0}^2 &= \frac{8\alpha^6}{n(n+4)(n+2)}. \quad (13)
\end{align*}
Evaluating equations (2) and (3) by using equations (8)-(13), and $Q_0 \equiv 1$, $Q_1(y) = ny$, $Q_2(y) = \frac{n+2}{2}(ny^2 - 1)$, and $Q_3(y) = \frac{n(n+4)}{6}((n + 2)y^3 - 3y)$, we get Lemma 6.
Next, we give the values of $R_i, \omega_i$.

**Lemma 7:** Let $X = X_1 \cup X_2$ be a Gaussian 6-design on two concentric spheres in $\mathbb{R}^n$. Let $\omega(u) = \omega_1$ and $R_1 = r_1^2 = \|u\|^2$ for $u \in X_1$ and let $\omega(u) = \omega_2$ and $R_2 = r_2^2 = \|u\|^2$ for $u \in X_2$. Then

\[
R_1 = \frac{(n + 2) + \sqrt{2n + 4}}{2\alpha^2},
\]

(14)

\[
R_2 = \frac{(n + 2) - \sqrt{2n + 4}}{2\alpha^2},
\]

(15)

\[
\omega_1 = \frac{\sqrt{2n + 4} - 2}{2|X_1|\sqrt{2n + 4}},
\]

(16)

and

\[
\omega_2 = \frac{\sqrt{2n + 4} + 2}{2|X_2|\sqrt{2n + 4}}.
\]

(17)
Proof. Taking $f(x) = \|x\|^{2j} (0 \leq j \leq 3)$ in the definition of Gaussian 6-design we get

$$\frac{1}{V(\mathbb{R}^n)} \int_{\mathbb{R}^n} \|x\|^{2j} e^{-\alpha^2 \|x\|^2} \, dx = |X_1| \omega_1 R_1^j + |X_2| \omega_2 R_2^j$$

for $0 \leq j \leq 3$. Thus, we obtain

$$|X_1| \omega_1 + |X_2| \omega_2 = 1,$$

$$|X_1| \omega_1 R_1 + |X_2| \omega_2 R_2 = \frac{n}{2\alpha^2},$$

$$|X_1| \omega_1 R_1^2 + |X_2| \omega_2 R_2^2 = \frac{n(n+2)}{4\alpha^4},$$

and

$$|X_1| \omega_1 R_1^3 + |X_2| \omega_2 R_2^3 = \frac{n(n+2)(n+4)}{8\alpha^6}. $$

Solving the system of equations, we obtain Lemma 7.
Sketch of proof. Suppose $X = X_1 \cup X_2$ is a tight Gaussian 6-design on two concentric spheres in $\mathbb{R}^n$. Note that the following facts:

- $0 \notin X$ by Lemma 2.
- If $n = 2$, there is nonexistence Gaussian tight 6-designs on two concentric spheres$^4$.
- If $3 \leq n \leq 8$, there is nonexistence tight Gaussian 6-designs on two concentric spheres$^5$.

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Proof of main results

So, we may assume $0 \notin X$, $n \geq 9$ and $|X_1| \geq |X_2|$. Then

- Each $X_i(i = 1, 2)$ is similar to a spherical 4-design by Lemma 3.
- Each $X_i(i = 1, 2)$ is an at most 3-distance set by Lemma 5.

Moreover, by calculating the maximum cardinality of the 2-distance sets in $\mathbb{R}^n$ and since $|X_1| > \frac{n^2 + 3n}{2}$ for $n \geq 3$, we obtain $X_1$ is a 3-distance set.

Taking $i = 1$ and replacing $R_1$ and $\omega_1$ in equation (5) by (14) and (16), respectively, we have

$$|X_1| = \frac{(n - 1)(2n + 3)(n + 2)}{6\sqrt{2n + 4}} + \frac{(n + 3)(n + 2)(n + 1)}{12}.$$ 

Since $|X_1|$ is an integer, obviously, $\sqrt{2n + 4}$ is an integer by the above equation, namely, $n = 2m^2 - 2$ for some integer $m$. 
Proof of main results

Clearly, there exists no Gaussian tight 6-design supported by two concentric spheres in $\mathbb{R}^n$ when $n \neq 2m^2 - 2$. 
**Proof of main results**

**Next we discuss** \( n = 2m^2 - 2 \). Taking \( i = 1 \) and replacing \( R_1 \) in equation (6) by (14), we obtain

\[
-\frac{\alpha^6 A^3}{6} + (\alpha^4 m^2 + \alpha^4 m + \frac{\alpha^4}{2})A^2 - (2\alpha^2 m^4 + 4\alpha^2 m^3 + 3\alpha^2 m^2 + \alpha^2)A
\]
\[
+ \left( \frac{4m^6}{3} + 4m^5 + 4m^4 - \frac{2m^3}{3} - m^2 \right) = 0.
\]

(18)

Let \( A_1, A_2, A_3 \) be the roots of equation (18). By Vieta’s theorem, we have

\[
A_1 + A_2 + A_3 = -\frac{b}{a}, \quad A_1 A_2 + A_1 A_3 + A_2 A_3 = \frac{c}{a}, \quad A_1 A_2 A_3 = -\frac{d}{a}.
\]

(19)

Here,
\[
a = -\frac{\alpha^6}{6}, \quad b = \alpha^4 m^2 + \alpha^4 m + \frac{\alpha^4}{2}, \quad c = -(2\alpha^2 m^4 + 4\alpha^2 m^3 + 3\alpha^2 m^2 + \alpha^2)
\]
and \( d = \frac{4m^6}{3} + 4m^5 + 4m^4 - \frac{2m^3}{3} - m^2 \).
Proof of main results

Combining (1) and (19), we get

\[ k_1k_2 + k_2k_3 + k_1k_3 = \frac{9a^2d^2 + b^3d - 4abcd}{27a^2d^2 + 4b^3d - 18abcd + 4ac^3 - b^2c^2}. \tag{20} \]

Evaluate \( a, b, c, d \) in (20) using

\[ a = -\frac{\alpha^6}{6}, \quad b = \alpha^4m^2 + \alpha^4m + \frac{\alpha^4}{2}, \quad c = -(2\alpha^2m^4 + 4\alpha^2m^3 + 3\alpha^2m^2 + \alpha^2) \]

and \( d = \frac{4m^6}{3} + 4m^5 + 4m^4 - \frac{2m^3}{3} - m^2 \). Then

\[ k_1k_2 + k_2k_3 + k_1k_3 = \frac{m^2(4m^4 + 12m^3 + 12m^2 - 2m - 3)(4m^4 + 12m^3 + 8m^2 + 2m - 5)}{-6(8m^6 + 48m^5 + 84m^4 + 16m^3 - 42m^2 + 12m - 5)}. \tag{21} \]

is an integer by Corollary 1. That is

\[ -6k_1k_2 + k_2k_3 + k_1k_3 = 2m^4 + 7m^2 - 16m + 41 \]

\[ -\frac{848m^5 + 2972m^4 + 1408m^3 - 1964m^2 + 572m - 205}{8m^6 + 48m^5 + 84m^4 + 16m^3 - 42m^2 + 12m - 5}. \]
For notational convenience, let

$$f(m) = \frac{848m^5 + 2972m^4 + 1408m^3 - 1964m^2 + 572m - 205}{8m^6 + 48m^5 + 84m^4 + 16m^3 - 42m^2 + 12m - 5}.$$ 

Then $f(m)$ is an integer. Now, we divide the argument into three cases.
Proof of main results

Case 1: When $m > 10^3$, since the numerator and the denominator of $f(m)$ are positive integers and the numerator of $f(m)$ is less than the denominator, $-6(k_1k_2 + k_2k_3 + k_1k_3)$ is not an integer, a contradiction.

Case 2: When $m \leq 10^3$ and $m$ is an odd integer, the numerator of the right side of formula (21) is an odd integer, and the denominator is an even integer, so $k_1k_2 + k_2k_3 + k_1k_3$ is not an integer, a contradiction.

Case 3: When $m \leq 10^3$ and $m$ is an even integer, we compute $f(m)$ explicitly for each $m$ and find out $f(m)$ is not an integer (see the following table), and hence $-6(k_1k_2 + k_2k_3 + k_1k_3)$ is not an integer, a contradiction.
Proof of main results

- Case 1: When $m > 103$, since the numerator and the denominator of $f(m)$ are positive integers and the numerator of $f(m)$ is less than the denominator, $-6(k_1k_2 + k_2k_3 + k_1k_3)$ is not an integer, a contradiction.

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(When \( m \leq 103 \) and \( m \) is an even integer, we give the values of \( f(m) \) in the table, which shows all \( f(m) \) are not integers.)
Acknowledgement

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Thank you for your attention!