On the Nonlinear Fuzzy Regulation for under-actuated systems

J. A. Meda-Campaña, B. Castillo-Toledo and Victor Zúñiga

Abstract—In this paper, the Takagi-Sugeno (TS) fuzzy modelling and the nonlinear regulation theory are combined in order to construct a controller capable of taking the output of the fuzzy plant to the reference signal generated by an external system. The fuzzy modelling allows the controller to be designed by means of numerical techniques while the resulting stability region is larger than those obtained when simple linear controllers are used to stabilize nonlinear systems. On the other hand, it is well-known that a fuzzy regulator constructed from linear controllers does not solve the regulation problem in general [4]. Therefore, a fully nonlinear regulator is considered in this work. The results are applied on the pendubot, which is an under-actuated electromechanical system with very complex nonlinear dynamics.

Keywords: Nonlinear systems, regulation theory, fuzzy systems, Takagi-Sugeno modelling, LMIs.

I. INTRODUCTION

The tracking of reference signals is an important problem within the system theory field, that is why in literature, there exist many different approaches to face it. One of these methods is the regulation theory, which provides a very well defined frame of work to achieve asymptotic tracking [5], [6]. The regulation problem consists of finding a state or error feedback controller such that in the absence of external influences the equilibrium point of the closed-loop system is asymptotically stable, and the tracking error goes to zero when the plant is influenced by the reference and/or perturbation signals, both generated by an external system, named the exosystem.

The solution to the linear regulation problem is given by Francis in [5], basically is equivalent to that of finding an algebraic solution of a set of matrix equations (Francis equations). Francis also presents the solution for the linear robust regulator, i.e., a controller capable of keeping the property of regulation despite variations in the parameters of the system. In this case, it has been shown that the linear robust regulation may be achieved by using a dynamical controller containing an internal model, which is generally the model of the exosystem.

In [6] Isidori and Byrnes have extended these results to the nonlinear area, in their work they have shown that the nonlinear regulator can be obtained on the basis of a set of partial differential equations named, henceforth, the Francis-Isidori-Byrnes (FIB) equations. In the nonlinear case, the inclusion of an internal model proved to be a necessary condition to get robustness with respect to parameter variations also. This internal model is obtained as an immersion of the exosystem into a dynamical system which generates all the possible steady state inputs for any admissible parameter variation.

On the other hand, recently, some techniques have been developed in order to characterize nonlinear systems by means of local subsystems [9], [10]. One of this approaches is the well-known TS fuzzy modelling. This technique allows modelling the nonlinear dynamics by means of a suitable “blending” of linear subsystems, each one of them corresponding to different operation points. Basically, the blending is performed by the weighted summation of local subsystems. Then, local controllers can be designed for each subsystems, obtaining the aggregate controller by the same procedure used to define the overall fuzzy system. In [11] is presented how the fuzzy controller known as Parallel Distributed Compensator (PDC) can be designed in a practical way by means of LMI techniques.

So, in this work the regulator for the nonlinear fuzzy system is developed using a combination of the nonlinear regulation theory and the Takagi-Sugeno fuzzy modelling.

It is worth taking notice that there exist some papers related with this approach where the final controller is constructed from linear regulators designed for each subsystem in the TS fuzzy model (see [2] and [13]). However as is shown in [4] and [7] a controller developed in this way only achieves the exact tracking in some particular cases. The reader is referred to [8] where sliding modes are employed to improved the performance of the fuzzy regulator constructed from linear controllers. In the present work the TS fuzzy model is only used to take advantage of the stabilizability results presented in [11], which give a larger attraction region than those obtained by simple linear stabilizers. To satisfy the regulation condition, the component of the controller that achieves the zeroing error will be designed on the nonlinear fuzzy plant and not only on the local subsystems. This is the main difference between the present and the previous works.

This paper is organized as follows. In section II the problem considerations are given and also a brief reminder of Takagi-Sugeno and Nonlinear Regulation Theory. The main result is developed in section III. The real-time application to the pendubot is presented in section IV. Finally, in section V, some conclusions are drawn.

II. PROBLEM FORMULATION

Consider the nonlinear system given by

\[
\dot{x} = f(x, u, u) \quad (1)
\]

\[
\dot{w} = a(w) \quad (2)
\]

\[
e = h(x, w) \quad (3)
\]
where \( x \in \mathbb{R}^n \) is the state vector of the plant, \( w \in W \subset \mathbb{R}^r \) is the state vector of the exosystem, which generates the reference and/or the perturbation signals and \( u \in \mathbb{R}^m \) is the input signal. Equation (3) describes the output tracking error \( e \in \mathbb{R}^m \), which is usually given as the difference between the measurable system outputs of the plant and the reference signals, i.e. \( h(x,w) = c(x) - q(w) \) where \( c(x) \) is the output of the plant and \( q(w) \) is the output of the exosystem. In this work it is considered that the system outputs depend or coincide exactly with the state \( x \). It is assumed that \( f(0,0,0) \), \( s(0) \) and \( h(0,0) \) are analytical functions, with \( s(0) = 0 \), \( f(0,0,0) = 0 \) and \( h(0,0) = 0 \).

It is now well-known that a good approximation for nonlinear systems is provided by the so-called TS fuzzy modelling. A model in this way is defined on the suitable choice of a set of linear subsystems, according to rules associated with some physical knowledge and some linguistic characterization of the properties of the system. These linear subsystems properly describe, at least locally, the behavior of the nonlinear system for a predefined region of the state space. The TS model for the system (1)-(3) is given by [12]:

**Model**

Rule \( i \):

IF \( z_1 \) is \( M^i_1 \) and \( z_2 \) is \( M^i_2 \) and . . . and \( z_p \) is \( M^i_p \)

THEN

\[
\sum_i : \left\{ \begin{array}{l} \dot{x} = A_i x + B_i u + P_i w \\ \dot{w} = s(w) \\ \epsilon_i = C_i x - q(w), \ i = 1 \ldots r \end{array} \right. \tag{4}
\]

where \( r \) is the number of rules in the model. The sets \( M^i_j \) are fuzzy sets defined on the basis of a previous knowledge of the dynamics of the system.

It is assumed that matrices \( A_i, B_i, P_i, C_i, S, Q \) are obtained by linearizing the nonlinear system around some suitable predefined points \((x^i, w^i, u^i)\), i.e.

\[
A_i = \frac{\partial f(x^i, w^i, u^i)}{\partial x} \big|_{(x^i, w^i, u^i)} \quad B_i = \frac{\partial f(x^i, w^i, u^i)}{\partial u} \big|_{(x^i, w^i, u^i)} \quad P_i = \frac{\partial f(x^i, w^i, u^i)}{\partial w} \big|_{(x^i, w^i, u^i)} \quad C_i = \frac{\partial h(x^i, w^i)}{\partial x} \big|_{(x^i, w^i)} \\
S = \frac{\partial s(w)}{\partial w} \big|_{w^i} \quad Q = \frac{\partial q(w)}{\partial w} \big|_{w^i}.
\]

The overall nonlinear TS fuzzy model is then given by

\[
\begin{align*}
\dot{x} &= \sum_{i=1}^{r} h_i(z) [A_i x + B_i u + P_i w] \quad (5) \\
\dot{w} &= s(w) \quad (6) \\
e &= \sum_{i=1}^{r} h_i(z) C_i x - q(w) \quad (7)
\end{align*}
\]

where \( h_i(z) \) is the normalized weight for each rule. These weights depend on the membership function for the premise variable \( z_j \) in \( M^i_j \), namely

\[
h_i(z) = \sum_{j=1}^{r} \frac{\mu_{i,j}(z)}{\sum_{k=1}^{r} \mu_{i,k}(z)} \tag{8}
\]

\[
\mu_{i,j}(z) = \prod_{j=1}^{p} M^i_j(z_j) \tag{9}
\]

\[
\sum_{i=1}^{r} h_i(z) = 1 \tag{10}
\]

\[
h_i(z) \geq 0 \tag{11}
\]

with \( z = [z_1 \ z_2 \ \ldots \ z_p] \), \( i = 1 \ldots r \), \( j = 1 \ldots p \) and \( z_j \) depending on \( x \).

So, the Nonlinear Fuzzy Regulator Problem (NFRP) is defined as the problem of finding, if possible, a controller

\[
u = c(x, w) \tag{12}
\]

such that,

**FS**  The equilibrium point \( x = 0 \) of the closed-loop system with no external signal

\[
\dot{x} = \sum_{i=1}^{r} h_i(z) [A_i x + B_i c(x, 0)]
\]

is asymptotically stable.

**FR**  The solution of the closed-loop system (5), (6), (7) and (12) satisfies

\[
\lim_{t \rightarrow \infty} e = 0,
\]

when the plant is under the effects of the exosystem.

Before proceeding with the main result, it is recalled that, according to Isidori’s regulation theory, for the nonlinear system (1)-(3) whose linearization around \( x = 0 \) is given by

\[
\begin{align*}
\dot{x} &= Ax + Bu + Pw \quad (13) \\
\dot{w} &= s(w) \quad (14) \\
e &= Cx - q(w) \quad (15)
\end{align*}
\]

the Nonlinear Regulator Problem (NRP) consist in finding a controller

\[
u = c(x, w) \tag{16}
\]

such that the closed loop system

\[
\dot{x} = Ax + Bc(x, 0)
\]

has an asymptotically stable equilibrium point, and the solution of closed-loop system (13), (14), (15) and (16) satisfies

\[
\lim_{t \rightarrow \infty} e = 0.
\]

In [6] is presented the following theorem which gives the existence conditions for the solution of the NRP.

**Theorem 1:** Suppose the following assumptions hold:

**H1**  the exosystem \( \dot{w} = s(w) \) is Poisson stable,
H2) there exist a gain $K$ such that the matrix $A + BK$ is stable,

H3) there exist mappings $x_{ss} = \pi(w)$, $u_{ss} = \gamma(w)$ with $\pi(0) = 0$, and $\gamma(0) = 0$ satisfying

$$\frac{\partial \pi(w)}{\partial w} s(w) = f(\pi(w), w, \gamma(w)) \quad (17)$$

$$0 = h(\pi(w), w) \quad (18)$$

then, the Nonlinear Regulation Problem is solvable and the controller is given by

$$u = K(x - \pi(w)) + \gamma(w). \quad (19)$$

It can be easily seen that, for the linear case the mappings $x_{ss} = \pi(w)$ and $u_{ss} = \gamma(w)$ turn into $x_{ss} = \Pi w$ and $u_{ss} = \Gamma w$, respectively. Therefore the conditions (17)–(18) reduce to the linear matrix equations:

$$\Pi S = A\Pi + B\Gamma + P \quad (20)$$

$$0 = C\Pi - Q \quad (21)$$

At this point it is stressed that the control signal for (5)–(7) is not simply given by the weighted summation of local regulators, i.e.

$$u = \sum_{i=1}^{r} h_i(z) [K_i x + L_i w]$$

where $L = \sum_{i=1}^{r} h_i(z) \left[ \Gamma_i x - \sum_{j=1}^{r} h_j(z) \Pi_j w \right]$, but by a controller in the form of (19) instead, where gain $K$ will be replaced by a fuzzy stabilizer and the mappings $\pi(w)$ and $\gamma(w)$ are those that solve the Nonlinear Fuzzy Regulation Problem for the overall fuzzy model (5)–(7). That is, in the next section will not be designed several local regulators but only a fully nonlinear one.

### III. THE NONLINEAR FUZZY REGULATION PROBLEM (NFRP)

Consider again the fuzzy model defined by (5)–(7). To solve the NFRP problem, it is proposed to design a fuzzy controller based on the Tanaka’s PDC approach [11], and then solving the nonlinear regulation problem for (5)–(7).

In that way, as is mentioned before, the resulting overall controller is given by:

$$u = \sum_{i=1}^{r} h_i(z) K_i [x - \pi(w)] + \gamma(w). \quad (22)$$

The following theorem gives conditions for the existence of a solution to the NFRP.

**Theorem 2:** Assume the following conditions hold:

FH1) the exosystem $w = s(w)$ is Poisson stable,

FH2) there exist matrices $K_i$ and $P$ such that

$$\left( \frac{A_{ij} + A_{ji}}{2} \right)^T P + P \left( \frac{A_{ij} + A_{ji}}{2} \right) \leq 0 \quad (24)$$

for all $i = 1 \ldots r$ and

$$\left( \frac{A_{ij} + A_{ji}}{2} \right)^T P + P \left( \frac{A_{ij} + A_{ji}}{2} \right) \leq 0 \quad (24)$$

for $i < j \leq r$, are asymptotically stable with

$$\dot{A}_{ij} = (A_i + B_i K_j), \quad (25)$$

FH3) there exist mappings $x_{ss} = \pi(w)$, $u_{ss} = \gamma(w)$ with $\pi(0) = 0$, and $\gamma(0) = 0$ satisfying

$$\frac{\partial \pi(w)}{\partial w} s(w) = \sum_{i=1}^{r} h_i(z) [A_i \pi(w) + B_i \gamma(w) + P_i w] \quad (26)$$

$$0 = \sum_{i=1}^{r} h_i(z) C_i \pi(w) - q(w), \quad (27)$$

then the NFRP is solvable.$\Box$

**Proof:** Condition FH1) is included to ensure that the reference signal will not decay to zero as time increases, and therefore the regulation problem do not turn into a stabilization one. Assumption FH2) guarantees the existence of a fuzzy controller $\alpha(x, 0)$ defined as

$$\alpha(x, 0) = \sum_{i=1}^{r} h_i(z) K_i x$$

such that

$$\dot{e} = \sum_{i=1}^{r} h_i(z) [A_i x + B_i \alpha(x, 0)]$$

is asymptotically stable.

To analyze the regulation property the steady state error is defined as $e_{ss} = x - \pi(w)$ and its derivative as

$$\dot{e}_{ss} = \dot{x} - \frac{\partial \pi(w)}{\partial w} s(w)$$

$$= \sum_{i=1}^{r} h_i(z) \left[ A_i + B_i \sum_{j=1}^{r} h_j(z) K_j \right] e_{ss}$$

$$+ \sum_{i=1}^{r} h_i(z) [A_i \pi(w) + B_i \gamma(w) + P_i w]$$

$$- \frac{\partial \pi(w)}{\partial w} s(w),$$

while, the tracking error is given by

$$e = \sum_{i=1}^{r} h_i(z) C_i x - q(w),$$

which in terms of the steady state error becomes in

$$e = \sum_{i=1}^{r} h_i(z) C_i (e_{ss} + \pi(w)) - q(w).$$

In that way, it results clear that the asymptotic tracking of the reference, i.e., $\lim_{t \to \infty} e = 0$, is achieved when FH2) and FH3) are satisfied.

In the latter result, the stability condition rests in designing local controllers and then finding a matrix $P$ such that
condition FH2) is satisfied. However, by means of Linear Matrix Inequalities (LMIs) techniques these two tasks can be performed at the same time. This simplify the design in general. The reader is referred to [1], where a complete analysis of LMIs technique in control theory is presented.

The next result involve LMIs and it shows how this numerical approach can be applied to obtain the solution of the NFRP.

Theorem 3: Assume the following conditions hold:
FLH1) the exosystem \( \dot{w} = s(w) \) is Poisson stable,
FLH2) the following LMIs are feasible

\[
Q A_i^T + M_i^T B_i^T + A_i \mathbf{Q} + B_i M_i < 0
\] (28)

for all \( i = 1 \ldots r \) and

\[
0 > \left( \frac{Q A_i^T + M_i^T B_i^T + Q A_j^T + M_j^T B_j^T}{2} \right) + \left( \frac{A_i \mathbf{Q} + B_i M_j + A_j \mathbf{Q} + B_j M_i}{2} \right)
\] (29)

for \( i < j \leq r \),
FLH3) there exist mappings \( x_s = \pi(w) \), \( u_s = \gamma(w) \) with \( \pi(0) = 0 \), and \( \gamma(0) = 0 \) satisfying the conditions

\[
\frac{\partial \pi(w)}{\partial w} s(w) = \sum_{i=1}^{r} h_i(z) [A_i \pi(w) + B_i \gamma(w)] + P_i w
\]

\[0 = \sum_{i=1}^{r} h_i(z) [C_i \pi(w)] - q(w),
\]

then the NFRP is solvable. Moreover, the matrices \( \mathbf{P} \) and \( K_i \) are given by \( \mathbf{P} = \mathbf{Q}^{-1} \) and \( K_i = M_i \mathbf{P} \) respectively.

Proof: If one considers matrices \( K_i \) and \( \mathbf{P} \) as unknowns, then it can be easily seen that inequalities (23) and (24) are nonlinear. However they can be transformed into linear matrix inequalities. This can be done by pre-multiplying and post-multiplying them by \( \mathbf{P}^{-1} \). The resulting expressions (28) and (29) are linear matrix inequalities that can be treated by numerical methods widely known as LMI techniques. The new unknowns are \( Q = \mathbf{P}^{-1} \) and \( M_i = K_i \mathbf{P}^{-1} \). The rest of the proof follows similar to theorem 2.

IV. THE PENDUBOT

In this section we apply the latter result to an under-actuated nonlinear system named pendubot.

The schematic of the pendubot (Pendulum Robot) is shown in Figure 1. Basically, it can be described as an electromechanical system composed by two rigid links. The first link (Link 1) is directly under the influence of a dc motor, which is the only actuator within the system. The second link (Link 2) is stills under-actuated, therefore its behavior is similar to that of the inverted pendulum on a car. So, the interesting problem is to control the link 2. To do this, the system has two outputs given by two encoders placed in each one of the joints. In that way, the dynamics of the resulting system are much more richer than those given by the inverted pendulum.

The parametric values of the pendubot are:

\begin{align*}
l_1 & \quad \text{Length of link 1} \quad 0.2032 \, \text{m} \\
l_2 & \quad \text{Length of link 2} \quad 0.3817 \, \text{m} \\
l_{c1} & \quad \text{Distance of the center of mass for link 1} \quad 0.1551 \, \text{m} \\
l_{c2} & \quad \text{Distance of the center of mass for link 2} \quad 0.1635 \, \text{m} \\
m_1 & \quad \text{Mass of link 1} \quad 0.8293 \, \text{kg} \\
m_2 & \quad \text{Mass of link 2} \quad 0.3402 \, \text{kg} \\
I_{zz1} & \quad \text{Moment of inertia for link 1, referred to its center of mass} \quad 59 \times 10^{-3} \, \text{kg} \cdot \text{m}^2 \\
I_{zz2} & \quad \text{Moment of inertia for link 2, referred to its center of mass} \quad 43 \times 10^{-3} \, \text{kg} \cdot \text{m}^2 \\
m_1 & \quad \text{Friction constant for link 1} \quad 0.00545 \\
m_2 & \quad \text{Friction constant for link 2} \quad 0.00047 \\
g & \quad \text{Gravitational constant} \quad 9.81 \, \text{m/s}^2 \\
q_1 & \quad \text{Angular position of link 1 (referred to x-axis)} \quad \text{rad} \\
q_2 & \quad \text{Angular position of link 2 (referred to link 1)} \quad \text{rad} \\
\tau & \quad \text{Torque applied by the actuator to link 1} \quad \text{N} \cdot \text{m}.
\end{align*}

In order to obtain the state space model, it is considered the following change of variable: \( x_1 = q_1, \quad x_2 = q_2, \quad x_3 = \dot{q}_1, \quad x_4 = \dot{q}_2, \quad y = x_2 \) and \( u = \tau \).

Then, the dynamics of the plant can be described by a nonlinear equation of the form

\[
\dot{x} = f(x) + g(x)u, \\
y = h(x) = x_2
\]
with
\[
f(x) = \begin{bmatrix}
x_3 \\
x_4 \\
f_{31}(x) f_{32}(x) \\
f_{41}(x) f_{42}(x) + f_{44}(x)
\end{bmatrix},
\]
and
\[
g(x) = \begin{bmatrix}
0 \\
0 \\
f_{31}(x) \\
f_{41}(x)
\end{bmatrix},
\]
where
\[
d_{11}(x) = m_1 l_1^2 + m_2 (l_1^2 + l_2^2 + 2 l_1 l_2 \cos(x_2)) + l_3 + I_{z1} + I_{z2},
\]
\[
d_{12}(x) = m_2 (l_1^2 + l_2^2 \cos(x_2)) + I_{z2},
\]
\[
d_{22}(x) = m_2 l_2^2 + I_{z2},
\]
\[
c_1(x) = -2 m_2 l_2 l_2 x_4 \sin(x_2) x_3 - m_2 l_2 l_2 x_4 \sin(x_3) x_4,
\]
\[
c_2(x) = m_2 l_2 l_2 x_3 \sin(x_2),
\]
\[
g_1(x) = m_1 g_1 \cos(x_1) + m_2 g l_1 \cos(x_1) + l_2 \cos(x_1 + x_2),
\]
\[
g_2(x) = m_2 g_2 \cos(x_1 + x_2),
\]
\[
f_1(x) = \mu_1 x_3,
\]
\[
f_2(x) = \mu_2 x_3,
\]
\[
f_{31}(x) = \frac{d_{22}(x)}{d_{11}(x) d_{22}(x) - d_{12}(x)^2},
\]
\[
f_{32}(x) = \frac{d_{22}(x) c_2(x) + d_{12}(x) g_2(x)}{d_{22}(x)} - \frac{d_{12}(x) f_2(x)}{d_{22}(x)} + c_1(x) + g_1(x) + f_1(x),
\]
\[
f_{41}(x) = -\frac{d_{12}(x)}{d_{11}(x) d_{22}(x) - d_{12}(x)^2},
\]
\[
f_{42}(x) = -\frac{c_2(x) - g_2(x)}{d_{22}(x) - d_{22}(x)} \frac{f_2(x)}{d_{22}(x)}
\]
and
\[
x = [x_1, x_2, x_3, x_4]^T.
\]

In this example, the latter nonlinear behavior is approximated by a nine-ruled TS fuzzy model. The linear subsystems are obtained by linearizing the nonlinear plant in different operation points. For this case those points are:

<table>
<thead>
<tr>
<th>$x^1$</th>
<th>$x^2$</th>
<th>$x^3$</th>
<th>$x^4$</th>
<th>$x^5$</th>
<th>$x^6$</th>
<th>$x^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>45°</td>
<td>60°</td>
<td>75°</td>
<td>90°</td>
<td>105°</td>
<td>120°</td>
</tr>
<tr>
<td>60°</td>
<td>45°</td>
<td>30°</td>
<td>15°</td>
<td>0°</td>
<td>-15°</td>
<td>-30°</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x^8$</td>
<td>$x^9$</td>
<td>$x^{10}$</td>
<td>$x^{11}$</td>
<td>$x^{12}$</td>
<td>$x^{13}$</td>
<td>$x^{14}$</td>
</tr>
<tr>
<td>135°</td>
<td>150°</td>
<td>0°</td>
<td>0°</td>
<td>0°</td>
<td>0°</td>
<td></td>
</tr>
<tr>
<td>-45°</td>
<td>-60°</td>
<td>0°</td>
<td>0°</td>
<td>0°</td>
<td>0°</td>
<td></td>
</tr>
<tr>
<td>0°</td>
<td>0°</td>
<td>0°</td>
<td>0°</td>
<td>0°</td>
<td>0°</td>
<td></td>
</tr>
</tbody>
</table>

Notice that the operation points accomplish the condition:
\[x_1 + x_2 = 90°.\] This is a pendubot restriction that must be satisfied during all the control process in order to keep it stable. Due to the lack of space the subsystems are not shown in this paper, but it is worth to say that they can be easily computed using mathematical software as Matlab or Maple.

For the blending of the subsystems, there are considered seven triangular membership functions between two pseudo-trapezoid membership functions, all of them depending on $x_2$ exclusively. Figure 2 shows the fuzzy sets used.

![Fig. 2. Fuzzy sets.](image)

The exosystem employed to generate the reference signal is
\[
\dot{w} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},
\]
\[
y_r = w_1.
\]
So, the tracking error is given by \( e = x_2 - w_1 \).

With the fuzzy model, the exosystem and the tracking error defined, the next step is to apply Theorem 3 in order to be sure that the NFRP can be solved. First of all, the existence of the fuzzy stabilizer must be checked. Again, this can be done by means of computational tools as the LMI Toolbox of Matlab. As is mentioned before, the fuzzy stabilizer for this example is given by
\[
\alpha(x,0) = \sum_{i=1}^{r} h_i(x) K_i x,
\]
with \( r = 1.9 \) and
\[
K_1 = \begin{bmatrix} 50.0198 & 47.3289 & 9.2681 & 7.3587 \end{bmatrix},
\]
\[
K_2 = \begin{bmatrix} 33.4527 & 31.3957 & 3.6315 & 4.8541 \end{bmatrix},
\]
\[
K_3 = \begin{bmatrix} 25.5238 & 24.0076 & 5.3410 & 3.6882 \end{bmatrix},
\]
\[
K_4 = \begin{bmatrix} 21.6217 & 20.4694 & 4.6913 & 3.1281 \end{bmatrix},
\]
\[
K_5 = \begin{bmatrix} 20.4253 & 19.4011 & 4.4894 & 2.9587 \end{bmatrix},
\]
\[
K_6 = \begin{bmatrix} 21.6217 & 20.4694 & 4.6913 & 3.1281 \end{bmatrix},
\]
\[
K_7 = \begin{bmatrix} 25.5238 & 24.0076 & 5.3410 & 3.6882 \end{bmatrix},
\]
\[
K_8 = \begin{bmatrix} 33.4527 & 31.3957 & 3.6315 & 4.8541 \end{bmatrix},
\]
\[
K_9 = \begin{bmatrix} 50.0198 & 47.3289 & 9.2681 & 7.3587 \end{bmatrix}.
\]

At this point, one has to solve the partial-derivative equations to design the nonlinear mappings that assure the tracking
of the reference. The steady state functions can be obtained as the solution of
\[
\begin{align*}
\frac{\partial \pi_1(w)}{\partial w} s(w) &= \pi_3(w) \\
\frac{\partial \pi_2(w)}{\partial w} s(w) &= \pi_4(w) \\
\frac{\partial \pi_3(w)}{\partial w} s(w) &= f_{31}(\pi(w)) [f_{32}(\pi(w)) + \gamma(w)] \\
\frac{\partial \pi_4(w)}{\partial w} s(w) &= f_{41}(\pi(w)) [f_{42}(\pi(w)) + \gamma(w)] + f_{42}(\pi(w)).
\end{align*}
\]
Considering \( x_s = \pi(w) \) and \( e = x_1 - w \), the following preliminary result is evident
\[
\pi_2(w) = w_1. \tag{34}
\]
The first and the second derivative of (34) result in
\[
\begin{align*}
\frac{\partial \pi_2(w)}{\partial w} s(w) &= w_1 = \beta w_2 = \pi_4(w) \\
\frac{\partial \pi_4(w)}{\partial w} s(w) &= -\beta^2 w_1_1 = f_{41}(\pi(w)) [f_{42}(\pi(w)) + \gamma(w)] + f_{42}(\pi(w)). \tag{36}
\end{align*}
\]
From the latter, it can be seen that the steady state input is
\[
\gamma(w) = -\frac{\beta^2 w_1 + f_{42}(\pi(w))}{f_{41}(\pi(w))} f_{32}(\pi(w)). \tag{37}
\]
The following equation is obtained by substituting (37) in (32)
\[
\begin{align*}
\frac{\partial \pi_3(w)}{\partial w} s(w) &= -f_{31}(\pi(w)) \left[ \frac{\beta^2 w_1 - f_{42}(\pi(w))}{f_{41}(\pi(w))} \right] \\
&= \frac{d_{21}(\pi(w))}{d_{11}(\pi(w))} \left[ \frac{\beta^2 w_1 - c_{21}(\pi(w))}{d_{22}(\pi(w))} \right] - \frac{e_{21}(\pi(w))}{d_{22}(\pi(w))} \frac{c_{21}(\pi(w))}{d_{22}(\pi(w))}. \tag{38}
\end{align*}
\]
It is obvious, that the solution of equations (36) and (38) cannot be easily computed, that is why in this work it is applied the approach proposed in [3] to approximate the nonlinear functions \( \pi(w) \) and \( \gamma(w) \). Basically, this approach shows that the nonlinear mappings can be arbitrarily approximated by Taylor series. The approximation to \( \pi_1(w) \) is taken as
\[
\pi_1(w) = a_1 w_1 + a_2 w_2 + a_3 w_1^2 + a_4 w_1 w_2 + a_5 w_2^2 + a_6 w_3^2 + a_7 w_1 w_2 + a_8 w_1 w_2^2 + a_9 w_2^3 + \theta (|w|^4). \tag{39}
\]
For this example, the exponents of the series higher than three are ignored. The first and second derivative of (39) are
\[
\begin{align*}
\frac{\partial \pi_1(w)}{\partial w} s(w) &= -a_2 w_1 - a_4 w_1^2 - a_7 w_1^3 + a_1 w_2 \\
&+ 2 (a_3 - a_5) w_1 w_2 + a_4 w_2^2 \\
&+ (3a_6 - 2a_8 w_1^2 w_2 + a_6 w_2^2) + (2a_7 - 3a_8 w_1 w_2^2 + \theta (|w|^4)) \\
&= \pi_3(w).
\end{align*}
\]
Now, is necessary to expand (38) in Taylor series around \( x = [1 0 0 0]^T \) and \( w = [0 0 0]^T \), this can be done using some mathematical software like Maple. In this case the result is given in equation (42) which is shown in next page.

Finally, the solution is obtained from matching coefficients of (41) and (42), and is given by
\[
\begin{align*}
\alpha_1 &= -0.98018337, \quad \alpha_2 = 0.00082308, \quad \alpha_3 = 0, \\
\alpha_4 &= 0, \quad \alpha_5 = 0, \quad \alpha_6 = -0.0078, \\
\alpha_7 &= 0.00003254, \quad \alpha_8 = 0.01349134, \quad \alpha_9 = 0.000002
\end{align*}
\]
After some tedious but simple calculations, one can check that the approximation to the solution of the regulator equations \( (\pi(w), \gamma(w)) \), with \( \beta = 1 \), is the one that appears in the following page.

Before applying the controller on the real time system we test its behavior in a simulated environment using Simulink for this effect. Figures 3 and 4 present the simulated behavior when the Nonlinear Fuzzy Regulator is applied to the original nonlinear plant. On the other hand, figures 5 and 6 show the simulation results when the fuzzy controller is obtained from linear regulators. It can be seen that the tracking error is notably smaller when the approach proposed in this paper is used to design the regulator. This difference is due to the fact that when the fuzzy regulator is designed by the simple blending of local regulators, the regulation conditions are not satisfied in general, as is mention in [4] and [8].

Afterwards, we also apply this controller to the pendubot in real time and the results are given in Figures 7, 8, 9 and 10. The behavior under the effects of the Nonlinear Fuzzy regulator is shown in Figure 7 and Figure 8, while the response of the pendubot to the controller computed combining local regulators is presented in figures 9 and 10. As expected, the behavior obtained under the action of the Nonlinear Fuzzy Regulator is better than the observed result when the controller is designed on linear regulators.

V. CONCLUSIONS

We present a nonlinear controller for continuous-time systems based on the combination of the nonlinear regulation
theory and the Takagi-Sugeno fuzzy modelling. We define in this way the Nonlinear Fuzzy Regulation Problem and give conditions for the existence of a solution. The relative advantage of this approach is that a global convergence of the
Figure 7. Output versus reference in real time (Nonlinear Fuzzy Regulator).

Figure 8. Tracking error in real time (Nonlinear Fuzzy Regulator).

Figure 9. Output versus reference in real time (Controller computed by blending local regulators).

Figure 10. Tracking error in real time (Controller computed by blending local regulators).

Output tracking error in the region of interest can be obtained, contrasting with the local properties of the classical nonlinear regulator method. On the other hand, one can easily notice that the result presented in this paper allows the tracking error to be arbitrarily reduced by considering Taylor series of higher orders. Finally, a real time application on a complex underactuated system, namely the pendubot, is successfully carried out, showing the effectiveness of this method.

REFERENCES


2202