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Extension of the Olkin and Rubin Characterization to the Wishart distribution on homogeneous cones

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Abstract

The Wishart distribution on an homogeneous cone is a generalization of the Riesz distribution on a symmetric cone which corresponds to a given graph. The paper extends to this distribution, the famous Olkin and Rubin characterization of the ordinary Wishart distribution on symmetric matrices.

Keywords: Vinberg algebra, homogeneous cone, Wishart distribution, orthogonal group.

1 Introduction

In many practical situations, there are manifest inter-relationships among several variables. One important case is when several pair variables are conditionally independent, giving other remaining variables. For multivariate normal distribution, this corresponds to some zeros among the entries of the precision matrix. Due to this there has been an interest in distributions akin to the Wishart but defined more generally on various cones containing the cone Ω of positive definite symmetric matrices as a special case. In particular, Andersson and Wojnar [2] defined the Wishart distribution on homogeneous cones. This distribution is in fact an extension to homogeneous cone of the Riesz distribution on symmetric cones defined by Hassairi and Lajmi in [8]. In the present paper, we give a characterization of the Wishart distribution on homogeneous cones which is parallel to that given in [13] by Olkin and Rubin, or more generally in [6] by Casalis and Letac. In these papers, it is not assumed that densities exist, however distributions are assumed to be invariant by the orthogonal group of the appropriate algebra. Our characterization uses the Laplace transform and a decomposition of a random variable on an homogeneous cone as a sum of random variables concentrated on certain subalgebras. The distribution of each component is then assumed to be invariant by the orthogonal group of the corresponding subalgebra. Let us give, a brief history of this characterization. Luckacs (see [11]) gave the following characterization of the Gamma distribution: Let X and Y be two non Dirac

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and non negative independent random variables such that $X + Y$ is positive almost surely, then $U = X + Y$ is independent of $V = \frac{X}{X+Y}$ if and only if there exist $\sigma > 0$, $p > 0$ and $q > 0$ such that X and Y are distributed as Gamma distributions with parameters (p, σ) and (q, σ) respectively. Olkin and Rubin [13] extended this characterization to the Wishart distribution on the cone Ω of positive definite symmetric matrices. They showed that X and Y in Ω are Wishart if and only if $(X + Y)^{-1/2} X (X + Y)^{-1/2}$ is independent of $X + Y$ and its distribution is invariant by the orthogonal group K . The characterization has then been extended by Carter [5] to the Wishart on the cone of Hermitian matrices with entries in \mathbb{C} , and by Casalis and Letac [6] to any symmetric cone. There are other types of characterizations of the Wishart such as, for example, that given by Letac and Massam in [9], Geiger and Heckerman in [7], and Massam and Wesolowski in [12]. We also mention that more recently Bobecka and Wesolowski gave in [3] a characterization of the Wishart distribution on Ω without any assumption of invariance. However, they had to assume that the densities of both X and Y with respect to the Lebesgue measure exist and are twice differentiable. The extension of this characterization to homogeneous cones has been the object of a paper by Boutouria [4]. The paper is organized as follows. In §2, we recall various definitions and preliminary results relevant to the Wishart on homogeneous cones and we establish some results concerning the Vinberg multiplication and determinant calculation. In §3, we state and prove our main characterization result.

2 The Wishart distribution on homogeneous cones

For the convenience of the reader, we will give here the elements of Vinberg algebras and homogeneous cones essential to working with the family of Wishart distributions on homogeneous cones. These elements are taken from [2] and the reader is referred to this paper for further details. After recalling this development, we give three examples. For the third example, the poset is isomorphic to a rooted tree while this is not so for the second example. The first example corresponds to the ordinary Wishart distribution on symmetric matrices.

Let I be a partially ordered finite set (herewith abbreviated as poset) equipped with a relation denoted \preceq . We will write $i \prec j$ if $i \preceq j$ and $i \neq j$. We assume that I satisfies the following condition

$$(F) : \begin{cases} \text{for any two points } i \text{ and } j \text{ in } I \text{ such that either } i \prec j \text{ or } j \prec i \\ \text{the path on the Hasse diagram of } I \text{ between } i \text{ and } j \text{ is unique.} \end{cases}$$

For all pairs $(i, j) \in I \times I$ with $j \prec i$, let E_{ij} be a finite-dimensional vector space over \mathbb{R} with $n_{ij} = \dim(E_{ij}) > 0$. Set

$$\mathcal{A}_{ij} = \begin{cases} \mathbb{R} & \text{for } i = j \\ E_{ij} & \text{for } j \prec i \\ E_{ji} & \text{for } i \prec j \\ \{0\} & \text{otherwise} \end{cases}$$

and $\mathcal{A} = \prod_{i,j \in I \times I} \mathcal{A}_{ij}$. Define $n_i = \sum_{\mu \prec i} n_{i\mu}$, $n_{.i} = \sum_{i \prec \mu} n_{\mu i}$, $n_i = 1 + \frac{1}{2}(n_{i.} + n_{.i})$, $i \in I$ and $n_{..} = \sum_{i \in I} n_i$.

An element $A \equiv (a_{ij}, i, j \in I)$ of \mathcal{A} may be seen as a matrix and so we define its trace as $\text{tr}A = \sum_{i \in I} a_{ii}$.

Let $f_{ij} : E_{ij} \rightarrow E_{ij}, i \succ j$, be involutorial linear mappings, i.e., $f_{ij}^{-1} = f_{ij}$. They induce an involutorial mapping ($A \mapsto A^*$) of \mathcal{A} given as follows: $A^* = (a_{ij}^* | (i, j) \in I \times I)$, where

$$a_{ij}^* = \begin{cases} a_{ii} & \text{for } i = j \\ f_{ij}(a_{ji}) & \text{for } j \prec i \\ f_{ji}(a_{ji}) & \text{for } i \prec j \\ 0 & \text{otherwise.} \end{cases}$$

We now define the following subspaces of \mathcal{A} :
the upper triangular matrices

$$\mathcal{T}_u = \{A \equiv (a_{ij}) \in \mathcal{A}, \forall i, j \in I : i \not\prec j \Rightarrow a_{ij} = 0\}; \quad (2.1)$$

the lower triangular matrices

$$\mathcal{T}_l = \{A \equiv (a_{ij}) \in \mathcal{A}, \forall i, j \in I : j \not\prec i \Rightarrow a_{ij} = 0\}; \quad (2.2)$$

and the Hermitian matrices $\mathcal{H} = \{A \in \mathcal{A}, A^* = A\}$.

The sets of upper and lower triangular matrices in \mathcal{A} with positive diagonal elements are respectively denoted by \mathcal{T}_u^+ and \mathcal{T}_l^+ . The sets of upper and lower triangular matrices with all diagonal elements equal to 1 are respectively denoted by \mathcal{T}_u^1 and \mathcal{T}_l^1 . The sets of diagonal matrices and of diagonal matrices with positive entries are denoted by \mathcal{D} and \mathcal{D}^+ , respectively.

We are going to equip the vector space \mathcal{A} with a bilinear map called multiplication and denoted by $(A, B) \mapsto AB$. For this purpose we need to define bilinear mappings $\mathcal{A}_{ij} \times \mathcal{A}_{jk} \rightarrow \mathcal{A}_{ik}$, denoted by $(a_{ij}, b_{jk}) \mapsto a_{ij}b_{jk}$, and then define $AB = C \equiv (c_{ij} | (i, j) \in I \times I)$ by $c_{ij} = \sum_{\mu \in I} a_{i\mu}b_{\mu j}$.

The multiplication is required to satisfy the following properties:

$$\begin{aligned} i) & \quad \forall A \in \mathcal{A}; A \neq 0 \Rightarrow \text{tr}(AA^*) > 0 \\ ii) & \quad \forall A, B \in \mathcal{A}; (AB)^* = B^*A^* \\ iii) & \quad \forall A, B \in \mathcal{A}; \text{tr}(AB) = \text{tr}(BA) \\ iv) & \quad \forall A, B, C \in \mathcal{A}; \text{tr}(A(BC)) = \text{tr}((AB)C) \\ v) & \quad \forall U, S, T \in \mathcal{T}_l; (ST)U = S(TU) \\ vi) & \quad \forall U, T \in \mathcal{T}_l; T(UU^*) = (TU)U^*. \end{aligned} \quad (2.3)$$

An algebra \mathcal{A} with the above structure and properties is called a Vinberg algebra (For more details, we can refer to [2]). We choose the element $A \equiv (a_{ij} | (i, j) \in I \times I)$ of \mathcal{D} such that $a_{ii} = 1, \forall i \in I$ as the unit element of \mathcal{A} and we denote it by e . Vinberg proved in [14] that the subset $\mathcal{P} = \{TT^* \in \mathcal{A}, T \in \mathcal{T}_l^+\} \subset \mathcal{H} \subset \mathcal{A}$ forms a homogeneous cone, that is the action of its automorphism group is transitive.

The definition of \mathcal{P} could be changed to the following equivalent definition

$$\mathcal{P} = \{TDT^* \in \mathcal{A}, T \in \mathcal{T}_l^1, D \in \mathcal{D}^+\}.$$

The two decompositions $S = TT^*$, $T \in \mathcal{T}_l^+$ and $S = T_1DT_1^*$, $T_1 \in \mathcal{T}_l^1$, $D \in \mathcal{D}^+$ are unique and their connection is given by $T = T_1\sqrt{D}$ where $\sqrt{D} \equiv \text{diag}(\sqrt{d_i}, i \in I) \in \mathcal{D}^+$ when $D \equiv \text{diag}(d_i, i \in I) \in \mathcal{D}^+$.

For $S = (s_{ij}, i, j \in I) = T_1DT_1^*$, we write $D_{ii} = S_{[i]}$.

If \preceq^{opp} is the opposite ordering on the index set I , i.e., $i \preceq^{\text{opp}} j \Leftrightarrow j \preceq i$. The Vinberg algebra $\mathcal{A}^{\text{opp}} = \prod_{i,j \in I \times I} \mathcal{A}_{ij}^{\text{opp}}$, where

$$\mathcal{A}_{ij}^{\text{opp}} = \begin{cases} \mathbb{R} & \text{for } i = j \\ E_{ji} & \text{for } j \prec^{\text{opp}} i \\ E_{ij} & \text{for } i \prec^{\text{opp}} j \\ \{0\} & \text{otherwise,} \end{cases}$$

differs from the Vinberg algebra \mathcal{A} only in the ordering of the index set I . It is proved (see [12]) that $\mathcal{P}_{\preceq^{\text{opp}}} = \{T^*T \in \mathcal{A}, T \in \mathcal{T}_l^+\}$ is the dual cone of \mathcal{P} which is also denoted \mathcal{P}^* .

Let's explain why condition (F) on the poset I is required for the definition of a Vinberg algebra. In fact, the property vi in (2.3) fails to be verified if the condition (F) is not satisfied. Suppose that (F) is not satisfied, then there exist two elements i and j in I such that either $i \prec j$ or $j \prec i$ and the path on the Hasse diagram of I between i and j is not unique. Without loss of generality, we suppose that $i \prec j$. Then there exist k and s in I such that $k \neq s$, $i \prec k \prec j$ and $i \prec s \prec j$. Consider the elements $T = (t_{nm})_{n,m \in I}$ and $U = (u_{nm})_{n,m \in I}$ of \mathcal{T}_l defined by $t_{nm} \neq 0$ and $u_{nm} \neq 0$ if $n, m \in \{i, j, k\}$ and $t_{nm} = 0$ and $u_{nm} = 0$ otherwise. Then

$$[T(UU^*)]_{jk} = t_{ji}u_{ii}u_{ik} + t_{jk}(u_{ik}^2 + u_{kk}^2) + t_{jj}(u_{ki}u_{ij} + u_{jk}u_{kk})$$

and

$$[(TU)U^*]_{jk} = (t_{ji}u_{ii} + t_{jk}u_{ki} + t_{js}u_{si} + t_{jj}u_{ji})u_{ik} + (t_{jk}u_{kk} + t_{jj}u_{jk})u_{kk}.$$

We see that $T(UU^*) \neq (TU)U^*$ so that vi in (2.3) is not satisfied.

Andersson, Letac and Massam [1] have also shown, without using the theory of Vinberg algebras, that if (F) is satisfied then, the cone \mathcal{P} is homogeneous.

Let G be the connected component of the identity in $\text{Aut}(\mathcal{P})$, the group of linear transformations leaving \mathcal{P} invariant. We recall that $\chi : G \mapsto \mathbb{R}_+$ is said to be a multiplier on the group G if it is continuous, $\chi(e) = 1$ and $\chi(h_1h_2) = \chi(h_1)\chi(h_2)$ for all $h_1, h_2 \in G$. Consider the map $\pi : T \in \mathcal{T}_l^+ \mapsto \pi(T) \in \pi(\mathcal{T}_l^+) \subset G$ such that for $X = WW^* \in \mathcal{P}$, $W \in \mathcal{T}_l^+$

$$\pi(T)(X) = (TW)(W^*T^*). \quad (2.4)$$

Andersson and Wojnar show in [2] that the restriction of a multiplier χ to the (lower) triangular group \mathcal{T}_l^+ , i.e., $\chi \circ \pi : \mathcal{T}_l^+ \rightarrow \mathbb{R}_+$ is in one to one correspondence with the set of $(\lambda_i, i \in I) \in \mathbb{R}^I$. To each χ corresponds a unique (up to a multiplicative constant) equivariant measure ν^χ on C with multiplier χ under the action of G , that is for all $h \in G$, the image measure $h^{-1}\nu^\chi$ of ν^χ by h^{-1} is

$$h^{-1}\nu^\chi = \chi(h)\nu^\chi.$$

For $\theta \in \mathcal{P}^*$, the Laplace transform of ν^χ is

$$L^\chi(\theta) = \int_{\mathcal{P}} \exp\{-\theta(P)\} d\nu^\chi(P). \quad (2.5)$$

We define the χ -inverse of θ by

$$\theta^\chi = -\frac{d}{d\theta} \log L^\chi(\theta).$$

The mapping

$$\theta \in \mathcal{P}^* \mapsto \theta^\chi \in \mathcal{P},$$

is a bijection. Its inverse is denoted by

$$P \in \mathcal{P} \mapsto P^{-\chi} \in \mathcal{P}^*,$$

and we have the properties

$$(h^{-1}P)^{-\chi} = {}^t h P^{-\chi} \quad \text{and} \quad ({}^t h \theta)^\chi = h^{-1}(\theta^\chi), \quad (2.6)$$

where $h \in G$, $P \in \mathcal{P}$ and $\theta \in \mathcal{P}^*$. Also for $\theta \in \mathcal{P}^*$, it is convenient to introduce the notation $\sigma = \theta^{-\chi}$. Then (2.5) may be written as a function of σ as

$$n^\chi(\sigma) = \int_{\mathcal{P}} \exp\{-\sigma^{-\chi}(P)\} d\nu^\chi(P),$$

so that for $h \in G$, $\sigma \in \mathcal{P}$, n^χ has the property

$$n^\chi(h\sigma) = \chi(h)n^\chi(\sigma). \quad (2.7)$$

Andersson and Wojnar (see [2]) consider the set

$$\mathcal{X} = \{\chi \circ \pi : \mathcal{T}_l^+ \rightarrow \mathbb{R}_+; \lambda_i > \frac{n_i}{2}, i \in I\},$$

and show that for $\chi \in \mathcal{X}$, the measure concentrated on \mathcal{P}

$$\nu^\chi(dX) = \prod_{i \in I} x_{[i]}^{\lambda_i - n_i} \mathbf{1}_{\mathcal{P}}(X) dX \quad (2.8)$$

generates the Wishart natural exponential family of distributions on \mathcal{P} absolutely continuous with respect to the Lebesgue measure, parameterized by $\sigma \in \mathcal{P}$

$$HW_{\chi, \sigma}(dX) = \frac{\pi^{\frac{|I|-n..}{2}} \prod_{i \in I} \lambda_i^{\lambda_i} \prod_{i \in I} x_{[i]}^{\lambda_i - n_i}}{\prod_{i \in I} \Gamma(\lambda_i - \frac{n_i}{2}) \prod_{i \in I} \sigma_{[i]}^{\lambda_i}} \exp\{-\text{tr}(\sigma^{-\chi} X)\} \mathbf{1}_{\mathcal{P}}(X) dX. \quad (2.9)$$

It is shown in [2] that if $\sigma = ZZ^*$, where $Z \in \mathcal{T}_l^+$,

$$\sigma^{-\chi} = (Z^*)^{-1} \text{diag}(\lambda_i | i \in I) Z^{-1}. \quad (2.10)$$

We are now going to give three examples of homogeneous cones and their corresponding Wishart distributions.

Example 2.1 In this example, we show how the ordinary Wishart distribution on symmetric matrices is a particular case of the general Wishart distribution $HW_{\chi, \sigma}$ on homogeneous cones. Take $I = \{1, \dots, I\}$ equipped with the usual total ordering \preceq on integers, denote by I its cardinality, and set $E_{ij} = \mathbb{R}$, $j \prec i$. Then the vector space \mathcal{A} is the space $\mathcal{M}(I, \mathbb{R})$ of all $I \times I$ matrices. With the standard multiplication and inner product the vector space $\mathcal{M}(I, \mathbb{R})$ is a Vinberg algebra. The homogeneous cone \mathcal{P} in this Vinberg algebra is the usual cone of $I \times I$ positive definite symmetric matrices usually denoted Ω . In this case, we have

$$n_{\cdot i} = I - i, \quad n_{i \cdot} = i - 1, \quad n_i = \frac{I+1}{2}, \quad n_{..} = \frac{I(I+1)}{2}$$

and every multiplier $\chi : G \rightarrow \mathbb{R}_+$ has the form $\chi(h) = |\det(h)|^\alpha$, $\alpha \in \mathbb{R}$. Thus for all $i \in I$, λ_i doesn't depend on i , it is equal to $\lambda = \frac{I+1}{2}\alpha$, so that χ can be replaced by λ . From (2.10), it follows that $\sigma^{-\chi} = \lambda \sigma^{-1}$, where σ^{-1} is the usual inverse of σ in \mathcal{P} . Since for $S = (s_{ij}, \quad i, j \in I) \in \Omega$, $\det(S) = \prod_{i \in I} s_{[i]}$, the density of the Wishart distribution (2.9) becomes

$$W_{\lambda, \sigma}(dS) = \frac{\lambda^{I\lambda} \det(S)^{\lambda - \frac{I+1}{2}}}{\pi^{\frac{I(I-1)}{4}} \prod_{i \in I} \Gamma(\lambda - \frac{i-1}{2}) \det(\sigma)^\lambda} \exp\{-\lambda \text{tr}(\sigma^{-1})\} \mathbf{1}_\Omega(S) dS, \quad (2.11)$$

which is the usual Wishart distribution on Ω with parameters $\lambda > \frac{I-1}{2}$ and $\sigma \in \Omega$.

Example 2.2 Let $I = \{1, 2, 3, 4\}$ be an index set and consider the poset defined by

$$1 \prec 3, 2 \prec 3, 2 \prec 4.$$

The homogeneous cone corresponding to this poset cannot be of the type Q_G of incomplete matrices with submatrices corresponding to the cliques of G being positive definite. We will not give the details of the argument for this assertion here. However, an accurate, albeit short, argument is that according to Theorem 2.2 in [10], this can only happen if the undirected graph obtained by dropping the directions on the graphical representation of the poset is a homogeneous graph. The graphical representation of the poset in this example is the directed graph with directed edges

$$\{(1, 3), (2, 3), (2, 4)\}.$$

The undirected graph obtained by dropping the directions is the three-link chain with undirected edges $\{(1, 3), (3, 2), (2, 4)\}$, which, by definition, is not a homogeneous graph.

Let us illustrate this in the case where $E_{ij} = \mathbb{R}$ for all appropriate (i, j) . The Vinberg algebra \mathcal{A} is thus the vector space of all $I \times I$ real matrices with zeros at the $(1, 2)$, $(2, 1)$, $(3, 4)$ and $(4, 3)$ entries. An element $X = (x_{ij})_{i, j \in I}$ of \mathcal{P} is decomposed into $X = TDT^*$ with $T \in \mathcal{T}_1^1$, $D \in \mathcal{D}$, such that

$$T^* = \begin{pmatrix} 1 & 0 & t_{13} & 0 \\ 0 & 1 & t_{23} & t_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix}.$$

Then the equation

$$\begin{aligned}
X &= \begin{pmatrix} x_1 & 0 & x_{13} & 0 \\ 0 & x_2 & x_{23} & x_{24} \\ x_{31} & x_{32} & x_3 & 0 \\ 0 & x_{42} & 0 & x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t_{31} & t_{32} & 1 & 0 \\ 0 & t_{42} & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & t_{13} & 0 \\ 0 & 1 & t_{23} & t_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} d_1 & 0 & d_1 t_{13} & 0 \\ 0 & d_2 & d_2 t_{23} & d_2 t_{24} \\ t_{31} d_1 & t_{32} d_2 & t_{32} d_2 t_{23} + d_3 & t_{32} d_2 t_{24} \\ 0 & t_{42} d_2 & t_{32} d_2 t_{23} + d_3 & t_{32} d_2 t_{24} \\ 0 & t_{42} d_2 & t_{42} d_2 t_{23} & t_{42} d_2 t_{24} + d_4 \end{pmatrix}
\end{aligned}$$

has a unique solution

$$\begin{aligned}
d_1 &= x_1 = x_{[1].}, & t_{13} &= \frac{x_{13}}{x_1}, & d_2 &= x_2 = x_{[2].}, & t_{23} &= \frac{x_{23}}{x_2}, & t_{24} &= \frac{x_{24}}{x_2}, \\
d_3 &= x_3 - x_{32} x_2^{-1} x_{23} = x_{[3].}, & d_4 &= x_4 - x_{42} x_2^{-1} x_{24} = x_{[4].}
\end{aligned} \tag{2.12}$$

The cone \mathcal{P} is then the set of symmetric matrices $X = (x_{ij})$ such that

$$x_1 > 0, \quad x_2 > 0, \quad x_3 - x_2^{-1} x_{23}^2 > 0 \quad \text{and} \quad x_4 - x_2^{-1} x_{24}^2 > 0.$$

Since $E_{ij} = \mathbb{R}$, we have that $n_{ij} = 1$, $n_{1.} = 0$, $n_{2.} = 0$, $n_{3.} = 2$ and $n_{4.} = 1$. Therefore

$$\mathcal{X} = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4) : \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 1, \lambda_4 > \frac{1}{2}\}.$$

and the Wishart distribution on this homogeneous cone is given for $\chi \in \mathcal{X}$ and $\sigma \in \mathcal{P}$ by

$$\begin{aligned}
HW_{\chi, \sigma}(dX) &= \frac{\pi^{\frac{4-8}{2}} \prod_{i=1}^4 \lambda_i^{\lambda_i} x_{[1].}^{\lambda_1 - \frac{3}{2}} x_{[2].}^{\lambda_2 - 2} x_{[3].}^{\lambda_3 - 2} x_{[4].}^{\lambda_4 - \frac{3}{2}}}{\Gamma(\lambda_1) \Gamma(\lambda_2) \Gamma(\lambda_3 - 1) \Gamma(\lambda_4 - \frac{1}{2}) \prod_{i=1}^4 \sigma_{[i].}^{\lambda_i}} \exp\{-\text{tr}(\sigma^{-\chi} X)\} \mathbf{1}_{\mathcal{P}}(X) dX.
\end{aligned}$$

Example 2.3 In this example, the poset is isomorphic to a rooted tree and therefore the homogeneous cone \mathcal{P} is of the Q_G type, that is, it corresponds to a homogeneous graph (see [10], Theorem 2.2).

Let the index set be $I = \{1, \dots, k\}$ and the poset be defined by

$$1 \prec i, \quad i = 2, \dots, k.$$

We can immediately see here that the undirected graph obtained by dropping the directions on the directed graphical representation of the poset is the star-shaped graph with k vertices and undirected edges $\{(1, i), i = 2, \dots, k\}$, which is a homogeneous graph. According to Theorem 2.2 in [10], the cone \mathcal{P} is therefore of the Q_G type. Again, we take $E_{ij} = \mathbb{R}$ for all appropriate (i, j) . Then $n_{.1} = k - 1$, $n_{1.} = 0$, $n_{1.} = \frac{k+1}{2}$ and $n_{.i} = 0$, $n_{i.} = 1$, $n_i = \frac{3}{2}$, $\forall i \neq 1$. The Vinberg algebra \mathcal{A} is the vector space of all $A = (a_{ij} | (i, j) \in I \times I) \in \mathcal{M}(I, \mathbb{R})$ with $a_{ij} = 0$ when i and j are not related. We have

$$\mathcal{X} = \{(\lambda_i, i = 1, \dots, k) : \lambda_1 > 0, \lambda_i > \frac{1}{2}, i = 2, \dots, k\},$$

and the Wishart distribution is given by

$$HW_{\chi, \sigma}(dX) = \frac{\pi^{\frac{1}{2}} \prod_{i=1}^k \lambda_i^{\lambda_i} x_{[1]}^{\lambda_1 - \frac{k+1}{2}} \prod_{i=2}^k x_{[i]}^{\lambda_i - \frac{3}{2}}}{\Gamma(\lambda_1) \prod_{i=2}^k \Gamma(\lambda_i - \frac{1}{2}) \prod_{i=1}^k \sigma_{[i]}^{\lambda_i}} \exp - \{ \text{tr} (\sigma^{-\chi} X) \} \mathbf{1}_{\mathcal{P}}(X) dX,$$

where $\chi = \{\lambda_i, i \in I\} \in \mathcal{X}$ and $\sigma^{-\chi}$ is defined as in (2.10), for $\sigma \in \mathcal{P}$.

To illustrate the decomposition of an element of \mathcal{P} in this example, we consider the case

where $k = 4$. An element $X = \begin{pmatrix} x_1 & x_{12} & x_{13} & x_{14} \\ x_{21} & x_2 & 0 & 0 \\ x_{31} & 0 & x_3 & 0 \\ x_{41} & 0 & 0 & x_4 \end{pmatrix}$ of \mathcal{P} can then be written as an

incomplete matrix

$$X^* = \begin{pmatrix} x_1 & x_{12} & x_{13} & x_{14} \\ x_{21} & x_2 & * & * \\ x_{31} & * & x_3 & * \\ x_{41} & * & * & x_4 \end{pmatrix}.$$

It can also be decomposed as $X = TDT^*$ with $T \in \mathcal{T}_1^1, D \in \mathcal{D}$, where

$$T^* = \begin{pmatrix} 1 & t_{12} & t_{13} & t_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix}.$$

The equation

$$\begin{aligned} X &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_{21} & 1 & 0 & 0 \\ t_{31} & 0 & 1 & 0 \\ t_{41} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \begin{pmatrix} 1 & t_{12} & t_{13} & t_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} d_1 & t_{12}d_1 & t_{13}d_1 & t_{14}d_1 \\ t_{21}d_1 & t_{21}d_1t_{12} + d_2 & 0 & 0 \\ t_{31}d_1 & 0 & t_{31}d_1t_{13} + d_3 & 0 \\ t_{41}d_1 & 0 & 0 & t_{41}d_1t_{14} + d_4 \end{pmatrix} \end{aligned}$$

has a unique solution

$$\begin{aligned} d_1 &= x_1 = x_{[1]}, \quad t_{12} = \frac{x_{12}}{x_1}, \quad t_{13} = \frac{x_{13}}{x_1}, \quad t_{14} = \frac{x_{14}}{x_1} \\ d_2 &= x_2 - x_{21}x_1^{-1}x_{12} = x_{[2]}, \quad d_3 = x_3 - x_{31}x_1^{-1}x_{13} = x_{[3]}, \quad d_4 = x_4 - x_{41}x_1^{-1}x_{14} = x_{[4]}. \end{aligned}$$

Hence, in this example, the cone \mathcal{P} is the set of symmetric matrices X such that

$$x_1 > 0, x_2 - x_1^{-1}x_{12}^2 > 0, x_3 - x_1^{-1}x_{13}^2 > 0 \text{ and } x_4 - x_1^{-1}x_{14}^2 > 0.$$

Let us note that the distribution of $X \in Q_G$ can be obtained from the distributions of $d_i, i = 1, \dots, k$ and $t_{1i}, i = 2, \dots, k$. This transformation is given explicitly in [10], Formula (3.15) and Theorem 4.5.

Now, we recall that an edge is written as an arrow from its origin to its destination, and we define the in-degree of a vertex to be the number of edges having this vertex as their destination. A vertex is considered a source in a graph if its in-degree is 0 (no vertices have a source as their destination) and if it is the parent of at least two vertices. This notion is needed in the following result.

Proposition 2.1 *The Vinberg multiplication TT^* is the same as the standard matrix multiplication $T.T^*$ if and only if there is no source in I .*

Proof (\Rightarrow) Suppose that there exists a source a in I . Let b and c be in I such that $a \prec b$, $a \prec c$, and consider $T = (t_{ij})_{i,j \in I} \in \mathcal{T}_l^+$ such that $t_{bc} = 0$. Then $T.T^* \neq TT^*$, which contradicts our assumption and therefore the source does not exist.

(\Leftarrow) Suppose that there exists T such that $T.T^* \neq TT^*$. Then there exist in I two different and non connected element i and j such that $(TT^*)_{ij} = 0$ and $(T.T^*)_{ij} \neq 0$. As $(TT^*)_{ij} = \sum_{k \preceq i, j} t_{ik}t_{jk}$, there exists k in I such that $k \prec i$ and $k \prec j$. This implies that

k is a source which is a contradiction. \square

In what follows, we suppose that I has no source and we call m a maximal element in I , if it is not less than any element of I .

Proposition 2.2 *Let m be a maximal element in I and let $I_{\preceq m} = \{i \in I, i \preceq m\}$. Then $I_{\preceq m}$ has no source.*

Proof Suppose that there exists a source $a \in I_{\preceq m}$ and let b and c in $I_{\preceq m}$ be two children of a with $a \prec b$, $a \prec c$. As m is a maximal, we have that $a \neq m$. Also $m \neq b$, because, if $m = b$, then $c \preceq b$. Thus $b \preceq m$, $c \preceq m$ and $a \preceq m$. Hence we obtain $a \prec b$, $a \prec c$, $b \prec m$, $c \prec m$ and $a \prec m$ which is impossible by Condition (F) of this section, and the lemma is proved. \square

For $T = (t_{ij})_{i,j \in I}$ in \mathcal{T}_l^+ , we define the element $T_{i \preceq}$ of \mathcal{T}_l by $T_{i \preceq} = (t'_{ij})_{i,j \in I}$, with $t'_{jk} = t_{jk}$ if $i \preceq j, k$ and $t'_{jk} = 0$ otherwise. It follows from Proposition 2.2 that if m is maximal, then for any $Z \in \mathcal{P}$, we have

$$\det Z_{\preceq m} = \prod_{i \preceq m} Z_{[i]},$$

Indeed, m maximal implies that $I_{\preceq m}$ has no source, and in this case, the Vinberg multiplication is nothing but the standard multiplication of matrices.

Also, we have that for any $i \in I$,

$$\det Z_{\preceq i} = \prod_{i' \preceq i} Z_{[i']}, \text{ and } \det Z_{\prec i} = \prod_{i' \prec i} Z_{[i']}.$$

because i is maximal in $I_{\preceq i}$. Therefore, for all $v \in I$,

$$Z_{[v]} = \frac{\det Z_{\prec v}}{\det Z_{\prec v}}. \quad (2.13)$$

We now introduce the notion of a division algorithm in a homogeneous cone needed for our characterization of the Wishart distribution on the cone \mathcal{P} . This notion is defined in [6] in the case of a symmetric cone. A division algorithm is a measurable map

$$\begin{aligned} g : \mathcal{P} &\rightarrow G \\ U &\mapsto g(U) \end{aligned}$$

such that $g(U)(U) = e$. In particular, for $U = TT^*$, we define $g(U) = \pi(T^{-1})$, where π is defined in (2.4), then g is a division algorithm, so that if $X = WW^* \in \mathcal{P}$, then

$$g(U)(X) = (T^{-1}W)(W^*(T^{-1})^*).$$

This algorithm is the one that we will use in all what follows.

3 Main characterization result

In this section, we state and prove our main characterization result concerning the Wishart distribution on homogeneous cones in the line of the characterizations given for the ordinary Wishart on symmetric matrices by Olkin and Rubin [13] and by Casalis and Letac [6]. Our considerations here will be restricted to the case of homogeneous cones with $E_{ij} = \mathbb{R}$, $(i, j) \in I \times I$.

It is easy to see that the Laplace transform of the Wishart distribution $HW_{\chi, \sigma}$ on the homogeneous cone \mathcal{P} is given for $\theta \in \mathcal{P}^*$ by

$$L_{HW_{\chi, \sigma}}(\theta) = \frac{n^\chi [(\theta + \sigma^{-\chi})^\chi]}{n^\chi(\sigma)}.$$

As $n^\chi(\sigma) = \prod_{i \in I} \sigma_{[i]}^{\lambda_i} n^\chi(e)$, this can be written as

$$L_{HW_{\chi, \sigma}}(\theta) = \frac{\prod_{i \in I} [(\theta + \sigma^{-\chi})^\chi]_{[i]}^{\lambda_i}}{\prod_{i \in I} \sigma_{[i]}^{\lambda_i}}.$$

Recall that for $i \in I$, we denote $I_{\prec i} = \{j \in I; j \prec i\}$ and $I_{i \preceq} = \{j \in I; i \preceq j\}$. Let i_1 and i_2 in I , such that $i_1 \not\prec i_2$ and $i_2 \not\prec i_1$, we say that j separates i_1 and i_2 if $j \in I_{i_1 \preceq} \cap I_{i_2 \preceq}$ and $j \notin \{i_1, i_2\}$. In this case, j is called a separator. We also denote $S_i = \{j \in I_{i \preceq}; j \text{ is a separator and } \forall k \neq j, k \not\prec j\}$, $\mathfrak{S} = \bigcup_{i \in I} S_i$ and $S = \{i \in \mathfrak{S}, \forall j \neq i, j \not\prec i\}$. This leads to the following decomposition of an element of \mathcal{P} which will serve in our characterization result.

Proposition 3.1 *Let $Z = TT^*$ be an element of \mathcal{P} with $T \in \mathcal{T}_I^+$. Denote $\wp = \{i \in I, I_{\prec i} = \emptyset\}$ and define, for $i \in I$,*

$$Z_{i \preceq} = T_{i \preceq} T_{i \preceq}^* \quad \text{and} \quad Z_i = \begin{cases} Z_{i \preceq} - \sum_{s \in S_i} Z_{s \preceq} & \text{if } i \in \wp \\ Z_{i \preceq} & \text{if } i \in S \\ 0 & \text{otherwise,} \end{cases}$$

Then we have that

$$Z = \sum_{i \in I} Z_i = \sum_{i \in \varphi \cup S} Z_i. \quad (3.14)$$

Proof: In order to prove the equality (3.14), we compare the blocks of Z and $\sum_{i \in I} Z_i$ on each subalgebra $\mathcal{A}_j = \prod_{k, l \in I_{j \preceq}} \mathcal{A}_{kl}$, $j \in I$ which we denote respectively by $(Z)_j$ and $(\sum_{i \in \varphi \cup S} Z_i)_j$. From the definition of $(Z_{j \preceq})_j$, we first observe that

$$(Z)_j = (Z_{j \preceq})_j, \quad \forall j \in I. \quad (3.15)$$

Now we discuss according to the position of j .

If $j \notin \varphi \cup S$, then there exists a unique $i_0 \in \varphi \cup S$ such that $j \in I_{i_0 \preceq}$. The fact that $j \in I_{i_0 \preceq}$ implies that $(Z_{i_0})_j = (Z_{j \preceq})_j$. From this and (3.15), we get

$$(Z)_j = (Z_{j \preceq})_j = (Z_{i_0})_j = \left(\sum_{i \in \varphi \cup S} Z_i \right)_j.$$

If $j \in S$, we have $(Z_i)_j = 0$, for all $i \in \varphi$. Therefore

$$\left(\sum_{i \in \varphi \cup S} Z_i \right)_j = \left(\sum_{i \in S} Z_i \right)_j = (Z_j)_j = (Z_{j \preceq})_j.$$

For $j \in \varphi$, we have that $\left(\sum_{i \in \varphi} Z_i \right)_j = Z_j = Z_{j \preceq} - \sum_{s \in S_j} Z_{s \preceq}$, and we consider separately the

cases $S_j = \emptyset$ and $S_j \neq \emptyset$. If $S_j = \emptyset$, then $\left(\sum_{i \in S} Z_i \right)_j = 0$ and $(Z)_j = (Z_{j \preceq})_j = Z_j = \left(\sum_{i \in \varphi} Z_i \right)_j$.

If $S_j \neq \emptyset$, $\left(\sum_{i \in S} Z_i \right)_j = \left(\sum_{i \in S_j} Z_i \right)_j = \left(\sum_{i \in S_j} Z_{i \preceq} \right)_j$. Therefore $(Z)_j = \left(\sum_{i \in \varphi \cup S} Z_i \right)_j$. \square

Example 3.1 Consider the following poset on $I = \{1, 2, 3, 4\}$ where $1 \prec 3$, $1 \prec 4$, $2 \prec 4$. Then $S = S_1 = S_2 = \{4\}$, $I_{3 \preceq} = \{3\}$. Hence $Z_3 = (0)$,

$$Z_4 = Z_{4 \preceq} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix},$$

$$Z_1 = \begin{pmatrix} t_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ t_{13} & 0 & t_{33} & 0 \\ t_{14} & 0 & 0 & t_{44} \end{pmatrix} \begin{pmatrix} t_{11} & 0 & t_{13} & t_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & t_{33} & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix} - Z_4,$$

and

$$Z_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & t_{24} & 0 & t_{44} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & t_{22} & 0 & t_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{44} \end{pmatrix} - Z_4.$$

Consider the Vinberg subalgebra of \mathcal{A} defined by $\mathcal{A}_i = \prod_{k,l \in I_{i \prec}} \mathcal{A}_{kl}$, and denote \mathcal{P}_i , G_i and e_i respectively, the corresponding homogeneous cone, connected component of the identity in $\text{Aut}(\mathcal{P}_i)$, and unit element. Also denote $K = \{g \in G, g(e) = e\}$ the orthogonal group of \mathcal{A} , and $K_i = \{k \in K, k(e_i) = e_i\}$. Finally let

$$\begin{aligned} g_i : \quad \mathcal{P}_i &\rightarrow G_i \\ U_{i \prec} = T_{i \prec} T_{i \prec}^* &\mapsto g_i(U_{i \prec}) \end{aligned} \quad (3.16)$$

such that $g_i(U_{i \prec})(X_{i \prec}) = (T_{i \prec}^{-1} W_{i \prec})(W_{i \prec}^* (T_{i \prec}^{-1})^*)$, where $X_{i \prec} = W_{i \prec} W_{i \prec}^*$. Then it is easy to see that

$$(g(X + Y)X)_{i \prec} = g_i(X_{i \prec} + Y_{i \prec})X_{i \prec}.$$

From now on, a Wishart distribution HW_{χ, e^X} will be called a standard Wishart distribution. Next, we verify that any Wishart distribution on the homogeneous cone \mathcal{P} may be standardized by a linear transformation.

Proposition 3.2 *Let X be a random variable valued in \mathcal{P} . Then X is $HW_{\chi, \sigma}$ if and only if there exist ρ in G such that $\rho(X)$ is HW_{χ, e^X} .*

Proof (\Rightarrow) Suppose that X is $HW_{\chi, \sigma}$ and write $\sigma = TT^*$ with $T \in \mathcal{T}_l^+$, $\chi = \{\lambda_i, i \in I\}$. We have that $\sqrt{\text{diag}(\lambda_i, i \in I)}T^{-1} \in \mathcal{T}_l^+$, we then consider the element of G , $\rho = \pi(\sqrt{\text{diag}(\lambda_i, i \in I)}T^{-1})$, where π is defined by (2.4). Using (2.6) and (2.7), we have for $\theta \in \mathcal{P}^*$,

$$\begin{aligned} L_{\rho(X)}(\theta) &= E(e^{-\text{tr}(\theta\rho(X))}) \\ &= E(e^{-\text{tr}(\rho^*(\theta)X)}) \\ &= \frac{\prod_{i \in I} [(\rho^*(\theta) + \sigma^{-X})^X]_{[i]}^{\lambda_i}}{\prod_{i \in I} \sigma_{[i]}^{\lambda_i}} \\ &= \frac{\prod_{i \in I} [\rho^{-1}(\theta + \rho^{*-1}(\sigma^{-X}))^X]_{[i]}^{\lambda_i}}{\prod_{i \in I} \sigma_{[i]}^{\lambda_i}}, \end{aligned}$$

and using the fact that $n^X(g^{-1}(u)(\sigma)) = \chi(g^{-1}(u))n^X(\sigma)$ and $\chi(g(u)) = \prod_{i \in I} u_{[i]}^{-\lambda_i}$ (see [2]),

we easily verify that

$$\prod_{i \in I} (g^{-1}(u)\sigma)_{[i]}^{\lambda_i} = \prod_{i \in I} u_{[i]}^{\lambda_i} \sigma_{[i]}^{\lambda_i}.$$

This applied to $g^{-1}(u) = \rho$ gives

$$L_{\rho(X)}(\theta) = \frac{\prod_{i \in I} [(\theta + \rho^{*-1}(\sigma^{-X}))^X]_{[i]}^{\lambda_i}}{\prod_{i \in I} [\rho(\sigma)]_{[i]}^{\lambda_i}}.$$

As $\rho^{*-1}(\sigma^{-\chi}) = (\rho(\sigma))^{-\chi} = (e^{\chi})^{-\chi} = e$, we obtain

$$L_{\rho(X)}(\theta) = \frac{\prod_{i \in I} [(\theta + e)^{\chi}]_{[i]}^{\lambda_i}}{\prod_{i \in I} (e^{\chi})_{[i]}^{\lambda_i}},$$

which is the Laplace transform of a $HW_{\chi, e^{\chi}}$ distribution.

(\Leftarrow) Suppose that there exist ρ in G such that $X' = \rho(X)$ is $HW_{\chi, e^{\chi}}$. Then using again (2.6) and (2.7), we have for $\theta \in \mathcal{P}^*$,

$$\begin{aligned} L_X(\theta) &= E(e^{-\text{tr}(\theta X)}) \\ &= E(e^{-\text{tr}(\rho^{*-1}(\theta) X')}) \\ &= \frac{\prod_{i \in I} [(\rho^{*-1}(\theta) + e)^{\chi}]_{[i]}^{\lambda_i}}{\prod_{i \in I} (e^{\chi})_{[i]}^{\lambda_i}} \\ &= \frac{\prod_{i \in I} [(\theta + \rho^*(e))^{\chi}]_{[i]}^{\lambda_i}}{\prod_{i \in I} [(\rho^*(e))^{\chi}]_{[i]}^{\lambda_i}}. \end{aligned}$$

Hence X is $HW_{\chi, \sigma}$, with $\sigma = (\rho^*(e))^{\chi} \in \mathcal{P}$. \square

We are now in a position to give our characterization results. According to Proposition 3.2, the statements will concern the Wishart distribution $HW_{\chi, e^{\chi}}$.

Theorem 3.1 *i) Let $X = TT^*$, with $T = (t_{ij})_{i,j \in I}$ in \mathcal{T}_l^+ , be a random variable with Wishart distribution $HW_{\chi, \sigma}$. Then $\sigma = e^{\chi}$ if and only if the t_{ij} , $i, j \in I$ are independent.
ii) If X and Y are two independent random variables with respective Wishart distribution, $HW_{\chi, e^{\chi}}$ and $HW_{\chi', e^{\chi'}}$, then for $i \in I$, the distribution of $V_{i \preceq} = g_i(X_{i \preceq} + Y_{i \preceq})X_{i \preceq}$ is K_i invariant, where g_i is defined by (3.16).*

Next, we give the reciprocal of this theorem.

Theorem 3.2 *Let X and Y be independent random variables valued in \mathcal{P} . Write $X = TT^*$ and $Y = MM^*$, with $T = (t_{ij})_{i,j \in I}$ and $M = (m_{ij})_{i,j \in I}$ in \mathcal{T}_l^+ . Consider the divisions algorithms g defined by (2.14) and g_i defined by (3.16) and suppose that
(i) the t_{ij} , $i, j \in I$ are independent and the m_{ij} , $i, j \in I$ are independent,
(ii) $X + Y$ is independent of $g(X + Y)(X)$,
(iii) for $i \in \varphi \cup S$, the distribution of $V_{i \preceq} = g_i(X_{i \preceq} + Y_{i \preceq})X_{i \preceq}$ is K_i invariant.
Then there exist $\chi, \chi' \in \mathcal{X}$ such that $X \sim HW_{\chi, e^{\chi}}$ and $Y \sim HW_{\chi', e^{\chi'}}$.*

Before embarking on the proofs of these theorems, it is worth mentioning that in the particular case where the Vinberg algebra is the algebra $\mathcal{M}(I, \mathbb{R})$ of $I \times I$ matrices (see Example 2.1), Theorem 3.1 and Theorem 3.2 give together the famous Olkin and Rubin characterization of the ordinary Wishart distribution on symmetric matrices. In fact in this case we have, $\varphi = \{1\}$, $S = \emptyset$, so that $\varphi \cup S = \{1\}$, and it follows that only

$K_1 = K = \{g \in G, g(e) = e\}$ appears in Theorem the point (iii) of 3.2. We also have $e^X = \lambda e$. Hence Theorem 3.1 becomes:

- i) Let $X = TT^*$, with $T = (t_{ij})_{i,j \in I}$ in \mathcal{T}_I^+ , be a random variable with Wishart distribution $W_{\lambda, \sigma}$. Then $\sigma = \lambda e$ if and only if the t_{ij} , $i, j \in I$ are independent
- ii) If X and Y are two independent random variables with respective Wishart distribution, $W_{\lambda, \lambda e}$ and $W_{\lambda', \lambda' e}$, then for $i \in I$, the distribution of $g_i(X_{i\prec} + Y_{i\prec})X_{i\prec}$ is K_i invariant, where g_i is defined by (3.16), in particular $g(X + Y)(X)$ is K invariant.

For the proof of the theorems, we need to establish the following result.

Lemma 3.1 *Let X and Y be two independent random variables with respective Wishart distribution, HW_{χ, e^X} and $HW_{\chi', e^{X'}}$. For $i \in I$, let $\alpha_{U_{i\prec}}$ denote the distribution of $X_{i\prec}$ conditional on $X_{i\prec} + Y_{i\prec} = U_{i\prec}$. Then for f in G_i , the image measure $f\alpha_{U_{i\prec}}$ of $\alpha_{U_{i\prec}}$ by f is such that $f\alpha_{U_{i\prec}} = \alpha_{f(U_{i\prec})}$.*

Proof Let $H : \mathcal{A} \rightarrow \mathbb{R}$ and $F : \mathcal{A} \rightarrow \mathbb{R}$ be any continuous functions with compact support. Let χ_i and χ'_i be two multipliers on G_i and set $\chi_i^1 = \chi_i \chi'_i$. Denote ν^{χ_i} , $\nu^{\chi'_i}$ and $\nu^{\chi_i^1}$ the corresponding equivariant measures concentrated on \mathcal{P}_i as defined in (2.8) and set $X_{i\prec}^1 = f(X_{i\prec})$ and $Y_{i\prec}^1 = f(Y_{i\prec})$. We consider the following equalities obtained by using the decomposition

$$\nu^{\chi_i}(dX_{i\prec})\nu^{\chi'_i}(dY_{i\prec}) = \alpha_{U_{i\prec}}(dY_{i\prec})\nu^{\chi_i^1}(dU_{i\prec})$$

and using the change of variable formula for integrals. For $U_{i\prec} = X_{i\prec} + Y_{i\prec}$, we have

$$\begin{aligned} J &= \int_{\mathcal{A}^2} H(U_{i\prec})F(X_{i\prec}^1)f(\alpha_{U_{i\prec}})(dX_{i\prec}^1)\nu^{\chi_i^1}(dU_{i\prec}) \\ &= \int_{\mathcal{A}^2} H(U_{i\prec})(F \circ f)(X_{i\prec})\alpha_{U_{i\prec}}(dX_{i\prec})\nu^{\chi_i^1}(dU_{i\prec}) \\ &= \int_{\mathcal{A}^2} H(X_{i\prec} + Y_{i\prec})F \circ f(X_{i\prec})\nu^{\chi_i}(dX_{i\prec})\nu^{\chi'_i}(dY_{i\prec}). \end{aligned}$$

Thus

$$\begin{aligned} J &= \int_{\mathcal{A}^2} H(f^{-1}(X_{i\prec}^1 + Y_{i\prec}^1))F(X_{i\prec}^1)(f\nu^{\chi_i})(dX_{i\prec}^1)(f\nu^{\chi'_i})(dY_{i\prec}^1) \\ &= \chi_i(f^{-1}) \int_{\mathcal{A}^2} H(f^{-1}(X_{i\prec}^1 + Y_{i\prec}^1))F(X_{i\prec}^1)\nu^{\chi_i}(dX_{i\prec}^1)\nu^{\chi'_i}(dY_{i\prec}^1) \\ &= \chi_i(f^{-1}) \int_{\mathcal{A}^2} H(f^{-1}(S_{i\prec}))F(X_{i\prec}^1)\alpha_{S_{i\prec}}(dX_{i\prec}^1)\nu^{\chi_i^1}(dS_{i\prec}), \end{aligned}$$

where $S_{i\prec} = X_{i\prec}^1 + Y_{i\prec}^1$. Since $f^{-1}(S_{i\prec}) = U_{i\prec}$, we have

$$\begin{aligned} J &= \chi_i^1(f^{-1}) \int_{\mathcal{A}^2} H(U_{i\prec})F(X_{i\prec}^1)\alpha_{f(U_{i\prec})}(dX_{i\prec}^1)f^{-1}\nu^{\chi_i^1}(dU_{i\prec}) \\ &= \int_{\mathcal{A}^2} H(U_{i\prec})F(X_{i\prec}^1)\alpha_{f(U_{i\prec})}(dX_{i\prec}^1)\nu^{\chi_i^1}(dU_{i\prec}). \end{aligned}$$

Comparing this expression with the definition of J , the lemma is proved. \square

Proof of Theorem 3.1 i) (\Leftarrow) See [2] where it is proved that if $X \sim W_{\chi, e^X}$, then t_{ij} , $i, j \in I$ are independent.

(\Rightarrow) Suppose that $X = TT^*$, with $T = (t_{ij})_{i,j \in I}$ in \mathcal{T}_l^+ , is $HW_{\chi, \sigma}$ with $\sigma \neq e^\chi$. We will show that in this case, the t_{ij} are not all independent. As $\sigma \neq e^\chi$, there exist $i \neq j$ such that $\sigma_{ij} \neq 0$. Writing $\sigma = WW^*$, with $W = (w_{ij})_{i,j \in I} \in \mathcal{T}_l^+$, we have that $\sigma_{ij} = \sum_{k \in I} w_{ik}w_{jk}$, and as $\sigma_{ij} \neq 0$, there exists $k \in I$ such that $w_{ik}w_{jk} \neq 0$. From Proposition 3.2, we have that $X' = \pi(\sqrt{\text{diag}(\lambda_i, i \in I)}W^{-1})(X)$ is HW_{χ, e^χ} . If we set $X' = SS^*$, with $S = (s_{ij})_{i,j \in I} \in \mathcal{T}_l^+$ and $Z = W\sqrt{\text{diag}(\lambda_i^{-1}, i \in I)} = (z_{ij})_{i,j \in I}$, we can write

$$X = \pi^{-1}(\sqrt{\text{diag}(\lambda_i, i \in I)}W^{-1})(X') = (ZS)(S^*Z^*).$$

This implies that $t_{jk} = \sum_{\mu \in I} z_{j\mu}s_{\mu k}$ for $j \neq k$. Hence

$$\begin{aligned} E(t_{jk}t_{kk}) - E(t_{jk})E(t_{kk}) &= \sum_{\mu \in I} z_{j\mu}z_{kk}E(s_{\mu k}s_{kk}) - \sum_{\mu \in I} z_{j\mu}z_{kk}E(s_{\mu k})E(s_{kk}) \\ &= \sum_{\mu \neq k} z_{kk}z_{j\mu}E(s_{\mu k}s_{kk}) + z_{jk}z_{kk}E(s_{kk}^2) - \sum_{\mu \neq k} z_{j\mu}z_{kk}E(s_{\mu k})E(s_{kk}) - z_{jk}z_{kk}E(s_{kk})^2. \end{aligned}$$

We use the fact that the s_{ij} are independent, because X' is HW_{χ, e^χ} , to obtain

$$\begin{aligned} E(t_{jk}t_{kk}) - E(t_{jk})E(t_{kk}) &= z_{jk}z_{kk}(E(s_{kk}^2) - E(s_{kk})^2) \\ &= z_{jk}z_{kk}\text{var}(s_{kk}). \end{aligned}$$

This is different from 0 because $z_{kk} = w_{kk}\sqrt{\lambda_{kk}^{-1}} \neq 0$, $z_{jk} = w_{jk}\sqrt{\lambda_{kk}^{-1}} \neq 0$ and s_{kk} is not degenerate, since s_{kk}^2 is gamma distributed (see [2]).

ii) Using the notation of Lemma 3.1 and the fact that U and V are independent, for two arbitrary continuous functions with compact support, $H : \mathcal{A} \rightarrow \mathbb{R}$ and $F : \mathcal{A} \rightarrow \mathbb{R}$, we have that

$$E(H(U_{i_{\prec}})F(V_{i_{\prec}})) = E(H(U_{i_{\prec}}) \int_{\mathcal{A}} F(g_i(U_{i_{\prec}})X_{i_{\prec}})\alpha_{U_{i_{\prec}}}(dX_{i_{\prec}})).$$

Writing $v_{i_{\prec}} = g_i(U_{i_{\prec}})X_{i_{\prec}}$ in the last integral and using Lemma 4.1, we get

$$E(H(U_{i_{\prec}})F(V_{i_{\prec}})) = E(H(U_{i_{\prec}}) \int_{\mathcal{A}} F(V_{i_{\prec}})\alpha_{e_i}(dV_{i_{\prec}}))$$

This proves (ii). \square

Proof of Theorem 3.2 Let $i \in I$ such that $X_i \neq 0$, where X_i is defined as in (3.14) and let $U = X + Y = WW^*$, $V = g(X + Y)(X)$. Without loss of generality, we can suppose that S_i has just one element; $S_i = \{s\}$. We will consider first the case where $i \notin S$. In this case, from the hypotheses of independence, we have, for A_1, A_2, B_1, B_2 and C_1 in \mathcal{P}^* ,

$$\begin{aligned} &E(\exp \text{tr}(A_1W_{i_{\prec}} - A_2W_{s_{\prec}} + B_1W_{i_{\prec}}W_{i_{\prec}}^* - B_2W_{s_{\prec}}W_{s_{\prec}}^* + C_1V_{i_{\prec}})) \\ &= E(\exp \text{tr}(A_1W_{i_{\prec}} - A_2W_{s_{\prec}} + B_1W_{i_{\prec}}W_{i_{\prec}}^* - B_2W_{s_{\prec}}W_{s_{\prec}}^*))E(\exp \text{tr}(C_1V_{i_{\prec}})) \\ &= f_i(A_1, A_2, B_1, B_2)h_i(C_1), \end{aligned} \tag{3.17}$$

where $f_i(A_1, A_2, B_1, B_2) = E(\exp \text{tr}(A_1 W_{i\leq} - A_2 W_{s\leq} + B_1 W_{i\leq} W_{i\leq}^* - B_2 W_{s\leq} W_{s\leq}^*))$
and $h_i(C_1) = E(\exp \text{tr} C_1 V_{i\leq})$.

We adopt the notations:

$$(f_i)^{jk} = \frac{\partial f_i}{\partial (A_1)_{jk}}, \quad (f_i)_{jk} = \frac{\partial f_i}{\partial (B_1)_{jk}} \quad \text{and} \quad (h_i)_{jk} = \frac{\partial h_i}{\partial (C_1)_{jk}},$$

and we define, for $B \in \mathcal{P}^*$, $\tilde{f}_i(B) = E(\exp \text{tr} B(X + Y)_i)$.

It is clear that if $A_1 = A_2 = 0$ and $B_1 = B_2 = B$, then

$$(\tilde{f}_i)_{jk} = (f_i)_{jk} \quad \text{when} \quad \{j, k\} \not\subset S_i \quad (3.18)$$

and

$$(\tilde{f}_i)_{jk,ln} = (f_i)_{jk,ln} \quad \text{when} \quad \{j, k\} \not\subset S_i \quad \text{and} \quad \{l, n\} \not\subset S_i. \quad (3.19)$$

Since we can differentiate under the expectation, there is a relation between the second partial derivatives with respect to A_1 and the first partial derivatives with respect to B_1 , namely we have

$$\sum_{\alpha} \frac{\partial^2}{\partial (A_1)_{j\alpha} \partial (A_1)_{k\alpha}} \frac{\partial^q f_i}{\partial t_1 \cdots \partial t_q} = \frac{\partial}{\partial (B_1)_{jk}} \frac{\partial^q f_i}{\partial t_1 \cdots \partial t_q}, \quad (3.20)$$

where t_1, \dots, t_q are any arguments of f_i .

Differentiating (3.17) successively with respect to $(A_1)_{j\lambda}$, $(C_1)_{\lambda\mu}$, and $(A_1)_{k\mu}$, for $\{j, k\} \not\subset S_i$ and summing over λ and μ , give the basic differential equations

$$\begin{aligned} & E((X_i)_{jk} \exp \text{tr}(A_1 W_{i\leq} - A_2 W_{s\leq} + B_1 W_{i\leq} W_{i\leq}^* - B_2 W_{s\leq} W_{s\leq}^* + C_1 V_{i\leq})) \\ &= \sum_{\lambda, \mu \in I_{i\leq}} f_i^{j\lambda, k\mu} (h_i)_{\lambda\mu}. \end{aligned} \quad (3.21)$$

And differentiating (3.21) successively with respect to $(A_1)_{l\nu}$, $(C_1)_{\nu\sigma}$ and $(A_1)_{n\sigma}$, for $\{l, n\} \not\subset S_i$ and summing, we obtain

$$\begin{aligned} & E((X_i)_{jk} (X_i)_{ln} \exp \text{tr}(A_1 W_{i\leq} - A_2 W_{s\leq} + B_1 W_{i\leq} W_{i\leq}^* - B_2 W_{s\leq} W_{s\leq}^* + C_1 V_{i\leq})) \\ &= \sum_{\lambda, \mu, \nu, \sigma \in I_{i\leq}} f_i^{j\lambda, k\mu, l\nu, n\sigma} (h_i)_{\lambda\mu, \nu\sigma}. \end{aligned} \quad (3.22)$$

Now, we use the hypothesis of invariance of the distribution of $V_{i\leq}$ by the orthogonal group K_i . In terms of Laplace transform, we have that for any $k \in K_i$,

$$h_i(C_1) = E(\exp \text{tr} C_1 V_{i\leq}) = E(\exp \text{tr} C_1 k(V_{i\leq})). \quad (3.23)$$

By the choice of suitable k in K_i and differentiation of (3.23), we first establish that there exists a real constant θ_i such that

$$(h_i)_{jm}(0) = \left. \frac{\partial h_i(C_1)}{\partial C_{jm}} \right|_{C_1=0} = \theta_i \delta_{jm}, \quad \text{for } j, m \in I_{i\leq}, \quad (3.24)$$

where δ_{jm} is the Kronecker delta, then that

$(h_i)_{jj,jm}(0,0) = (h_i)_{jj,ml}(0,0) = (h_i)_{jm,jl}(0,0) = (h_i)_{jm,ln}(0,0) = 0$, for all different j, m, l and n in $I_{i\prec}$. Now, we set $k = (t_{jm})_{j,m \in I_{i\prec}}$ in (3.23) and we differentiate to get

$$(h_i)_{jm,ln}(0,0) = \sum_{\alpha,\beta,\eta,\delta} t_{j\alpha} t_{m\beta} t_{l\eta} t_{n\delta} (h_i)_{\alpha\beta,\eta\delta},$$

which yields

$$(h_i)_{jm,ln}(0,0) = \eta_i \delta_{jm} \delta_{ln} + \xi_i [\delta_{jl} \delta_{mn} + \delta_{jn} \delta_{ml}], \quad j, m, l, n \in I_{i\prec}, \quad (3.25)$$

where η_i and ξ_i are real constants.

We set in (3.21) and (3.22) $A_1 = A_2 = C_1 = 0$ and $B_1 = B_2 = B$. Taking into account (3.20), (3.24) and (3.25), we obtain, for $\{j, k\} \not\subset S_i$ and $\{l, n\} \not\subset S_i$,

$$\begin{aligned} E((X_i)_{jk} \exp \operatorname{tr} B(X+Y)_i) &= \sum_{\lambda,\mu \in I_{i\prec}} (\tilde{f}_i)^{j\lambda,k\mu} (h_i)_{\lambda\mu} \\ &= \theta_i \sum_{\lambda,\mu \in I_{i\prec}} (\tilde{f}_i)^{j\lambda,k\mu} \delta_{\lambda\mu} \\ &= \theta_i (\tilde{f}_i)_{jk}, \end{aligned} \quad (3.26)$$

and

$$\begin{aligned} E((X_i)_{jk} (X_i)_{ln} \exp \operatorname{tr} B(X+Y)_i) &= \sum_{\lambda,\mu,\nu,\sigma \in I_{i\prec}} f_i^{j\lambda,k\mu,l\nu,n\sigma} [\eta_i \delta_{\lambda\mu} \delta_{\nu\sigma} + \xi_i (\delta_{\lambda\nu} \delta_{\mu\sigma} + \delta_{\lambda\sigma} \delta_{\mu\nu})] \\ &= \eta_i (\tilde{f}_i)_{jk,ln} + \xi_i [(\tilde{f}_i)_{jl,kn} + (\tilde{f}_i)_{jn,lk}]. \end{aligned} \quad (3.27)$$

In the case where i is a separator, the system (3.17) is replaced by

$$\begin{aligned} E(\exp \operatorname{tr} (A_1 W_{i\prec} + B_1 W_{i\prec} W_{i\prec}^* + C_1 V_{i\prec})) &= E(\exp \operatorname{tr} (A_1 W_{i\prec} + B_1 W_{i\prec} W_{i\prec}^*)) E(\exp \operatorname{tr} C_1 V_{i\prec}) \\ &= f_i(A_1, B_1) h_i(C_1). \end{aligned}$$

and by the same reasoning, we also obtain equations (3.26) and (3.27), but for j, k, l, n in $I_{i\prec}$.

Now, to solve (3.26) and (3.27), we introduce the functions

$$\varphi_i(B) = E(\exp \operatorname{tr} B X_i), \quad \psi_i(B) = E(\exp \operatorname{tr} B Y_i). \quad (3.28)$$

Then (3.26) and (3.27) can be written as

$$(\varphi_i)_{jk} \psi_i = \theta_i (\varphi_i \psi_i)_{jk}, \quad (3.29)$$

$$(\varphi_i)_{jk,ln} \psi_i = \eta_i (\varphi_i \psi_i)_{jk,ln} + \xi_i [(\varphi_i \psi_i)_{jl,kn} + (\varphi_i \psi_i)_{jn,lk}]. \quad (3.30)$$

Equation (3.29) implies in particular that

$$\varphi_i = (\varphi_i \psi_i)^{\theta_i}. \quad (3.31)$$

Let ϕ_i be such that, for B sufficiently close to zero,

$$\varphi_i \psi_i = \exp \phi_i. \quad (3.32)$$

Then (3.31) and (3.32) lead to the differential equation system defined for $j, k, l, n \in I_{i \preceq}$ by

$$\begin{aligned} \theta_i(\phi_i)_{jk,ln} + \theta_i^2(\phi_i)_{jk}(\phi_i)_{ln} &= \xi_i[(\phi_i)_{jl,kn} + (\phi_i)_{jl}(\phi_i)_{kn} + (\phi_i)_{jn,lk} + (\phi_i)_{jn}(\phi_i)_{lk}] \\ &+ \eta_i[(\phi_i)_{jk,ln} + (\phi_i)_{jk}(\phi_i)_{ln}]. \end{aligned} \quad (3.33)$$

The general solution of this system is

$$\phi_i(B) = \beta_i \ln(B + D)_{[i]}^{\chi_1} + a_i,$$

where $D \in \mathcal{P}^*$, $\chi_1 \in \mathcal{X}$, and $a_i \in \mathbb{R}$ are arbitrary constants and β_i is a constant depending on θ_i, ξ_i, η_i . This is in particular, justified by the fact that

$$(B + D)_{[i]}^{\chi_1} = (B + D_i)_{[i]}^{\chi_1},$$

where $D_i \in \mathcal{P}_i^*$. Now, if $A = R^*R \in \mathcal{P}^*$, we denote $A_{\preceq \text{opp}} = R^{-1}(R^{-1})^* \in \mathcal{P}$. Therefore, we use (2.13) and a result in [2] due to Andersson and Wojnar which says that, for $\theta = Z^*Z$, where $Z \in \mathcal{T}_I^+$, $\theta^X = Z^{-1}e^X(Z^*)^{-1}$, to write

$$\phi_i(B) = \beta_i \ln((B + D)_{\preceq \text{opp}})_{[i]} + a'_i = \beta_i [\ln \det(B + D)_{\preceq \text{opp}} i - \ln \det(B + D)_{\prec \text{opp}} i] + a'_i,$$

where a'_i is a constant.

Thus

$$\varphi_i(B) = \frac{((B + D)_{\preceq \text{opp}} i)_{[i]}^{\beta_i \theta_i}}{(D_{\preceq \text{opp}} i)_{[i]}^{\beta_i \theta_i}} \quad \text{and} \quad \psi_i(B) = \frac{(B + D)_{\preceq \text{opp}} i)_{[i]}^{\beta_i(1-\theta_i)}}{(D_{\preceq \text{opp}} i)_{[i]}^{\beta_i(1-\theta_i)}}.$$

We now observe that the fact that the t_{ij} , $i, j \in I$ are independent implies that the X_i , $i \in I$ are independent. This is important for the calculation of the Laplace transform of $X = \sum_{i \in I} X_i$. In fact, for each $i \in I$, by its very definition, X_i depends only on the t_{ij}

such that $i \preceq j$ and on the t_{kj} such that $j, k \notin S_i$ and $i \preceq k, j$, which are different from the ones on which any other component $X_{i'}, i' \neq i$ depends. In other words, there exists a partition $(\tau_i)_{i \in I}$ of the set $\{t_{jk}, j, k \in I\}$ such that each X_i depends only on the t_{kj} in τ_i . Similarly the independence of the a_{ij} , $i, j \in I$ implies the independence of the Y_i , $i \in I$. Finally denoting $\chi = \{\beta_i \theta_i, i \in I\}$ and $\chi' = \{\beta_i(1 - \theta_i), i \in I\}$, we obtain the Laplace transforms φ of $X = \sum_{i \in I} X_i$ and ψ of $Y = \sum_{i \in I} Y_i$ as

$$\varphi(B) = \prod_{i \in I} \varphi_i(B) = \frac{\prod_{i \in I} [(B + D)^X]_{[i]}^{\beta_i \theta_i}}{\prod_{i \in I} [D^X]_{[i]}^{\beta_i \theta_i}}, \quad \psi(B) = \prod_{i \in I} \psi_i(B) = \frac{\prod_{i \in I} [(B + D)^{\chi'}]_{[i]}^{\beta_i(1-\theta_i)}}{\prod_{i \in I} [D^{\chi'}]_{[i]}^{\beta_i(1-\theta_i)}}.$$

Thus $X \sim HW_{\chi, \sigma}$ and $Y \sim HW_{\chi', \sigma'}$, where $\sigma = D^X$ and $\sigma' = D^{\chi'}$. Invoking Theorem 3.1 i), we have necessarily $D = e$. This concludes the proof of the Theorem 3.2. \square

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