Observer design of discrete-time impulsive switched nonlinear systems with time-varying delays

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Abstract

This paper investigates the problem of observer design for discrete-time impulsive switched nonlinear systems with time-varying delays. Firstly, based on the average dwell time approach and the Lyapunov–Krasovskii functional technique, a delay-dependent exponential stability criterion for the discrete-time impulsive switched nonlinear systems with time-varying delays is derived in terms of a set of linear matrix inequalities (LMIs). Then, sufficient conditions for the existence of an observer that guarantees the exponential stability of the corresponding error system are proposed. Finally, two numerical examples are given to illustrate the effectiveness of the proposed method.

1. Introduction

Switched systems are a class of important hybrid systems consisting of subsystems and a switching law which defines a specific subsystem being activated during a certain interval of time [1]. Due to the theoretical development as well as practical applications, analysis and synthesis of switched systems have recently gained considerable attention, and a large number of results in this field have appeared (see, e.g., [2–6]). However, in the real world, some switched systems exhibit impulsive dynamical behavior due to sudden changes in the states of these systems at certain instants of switching, this kind of systems are usually called impulsive switched systems [7]. During the last decade, impulsive switched systems have received considerable attention. Some results on stability and stabilization of such systems have been achieved (see, e.g., [8–10]). For instance, stability analysis of systems with impulse effects was investigated in [8]. Analysis and design of impulsive control systems were considered in [10].

In many practical systems, there inevitably exists time delay which plays a crucial role in destroying the system performance. Although, in recent years, many stability conditions of time-delay systems have been obtained (see, e.g., [11–16]), it is worth mentioning that till now, there is only little effort putting on impulsive switched systems with time delays [17–27]. For example, some stability criteria for impulsive switched systems with constant time delays were developed in [17–26]. A delay-dependent stability criterion for a class of impulsive switched discrete systems with time-varying delays was established by using the Lyapunov–Krasovskii functional technique in [27].

On the other hand, it is necessary to design state observers for the systems due to the fact that the states of the systems are not all measurable in practice. Some significant results on this topic have been obtained in [28–33]. For example, the asymptotic stability property of the proposed switching observer was discussed and an LMI-based algorithm was given for a class of impulsive switched systems in [33]. However, to the best of our knowledge, the problem of observer design for discrete-time impulsive switched systems has not been fully investigated, especially for impulsive switched nonlinear systems with time-varying delays, which motivates us for this study.
In this paper, we are interested in investigating the observer design problem for impulsive switched nonlinear systems with time-varying delays. The main contributions of this paper are as follows: (1) based on the dwell time approach, we establish a delay-dependent exponential stability criterion of the underlying system, which is different from those proposed in [33]; and (2) an observer design scheme for the underlying system is proposed.

The remainder of the paper is organized as follows. In Section 2, the problem formulation and some necessary lemmas are presented. In Section 3, the main results are obtained. Section 4 gives two numerical examples to illustrate the effectiveness of the proposed approach. Finally, concluding remarks are given in Section 5.

Notations. Throughout this paper, the superscript “T” denotes the transpose, and the notation $X \geq Y (X > Y)$ means that matrix $X - Y$ is positive semi-definite (positive definite, respectively). $\| \cdot \|$ denotes the Euclidean norm. $I$ represents the identity matrix. $\text{diag} \{ b \}$ denotes a diagonal matrix with the diagonal elements $b_i$, $i = 1, 2, \ldots, n$. $X^{-1}$ denotes the inverse of $X$. The asterisk * in a matrix is used to denote the term that is induced by symmetry. $Z_0^+$ denotes the set of all nonnegative integers. The set of all positive integers is $\mathbb{N}$.

2. Problem formulation and preliminaries

Consider the following discrete-time impulsive switched nonlinear systems with time-varying delays:

\[
\begin{align*}
    x(k+1) &= A_{\sigma(k)}x(k) + A_{\delta(k)}x(k - d(k)) + B_{\sigma(k)}u(k) + D_{\sigma(k)}(f_{\sigma(k)}(x(k))) + f_{\sigma(k)}(\hat{x}(k)), \\
    y(k) &= C_{\sigma(k)}x(k), \\
    \hat{x}(k) &= C_{\sigma(k)}x(k), \\
    \hat{x}(k_0 + \theta) &= \phi(\theta), \quad \theta \in [-d_2, 0],
\end{align*}
\]

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^p$ is the output, $\phi(\theta)$ is a discrete vector-valued initial function defined on the interval $[-d_2, 0]$. $d(k)$ is the discrete time-varying delay which is assumed to satisfy $d_1 \leq d(k) \leq d_2$. $d_1$ and $d_2$ are constant positive scalars representing the minimum and maximum delays, respectively. $k_0$ is the initial time. The function $\sigma(k) : Z_0^+ \rightarrow N := \{1, \ldots, N\}$ is the switching signal with $N$ being the number of subsystems. For each $i \in N$, $f_i(x(k)) \in \mathbb{R}^p$ is a known nonlinear function. $A_i$, $A_{\delta i}$, $B_i$, $D_{\delta i}$, $C_i$, $i, j \in N$, $i \neq j$, are known real constant matrices with appropriate dimensions.

Remark 1. It should be noted that Eq. (1b) has been introduced in the literature [25–26,33] to describe the impulsive dynamical behavior of some practical switched systems due to sudden changes in the state of the system at certain instants of switching.

Remark 2. In the paper, it is assumed that the switching occurs at the instant $k_0$. The switching sequence of the system can be described as

\[ \Sigma = \{(k_0, \sigma(k_0)), (k_1, \sigma(k_1)), (k_2, \sigma(k_2)), \ldots, (k_0, \sigma(k_0)), \ldots\}, \]

where $k_b$ denotes the $b$-th switching instant.

We construct the following discrete-time impulsive switched systems to estimate the state of system (1):

\[
\begin{align*}
    \hat{x}(k+1) &= A_{\sigma(k)}\hat{x}(k) + A_{\delta(k)}\hat{x}(k - d(k)) + B_{\sigma(k)}u(k) + L_{\sigma(k)}(y(k) - \hat{y}(k)) + D_{\sigma(k)}f_{\sigma(k)}(\hat{x}(k)), \\
    \hat{y}(k) &= C_{\sigma(k)}\hat{x}(k), \\
    \hat{x}(k_0 + \theta) &= 0, \quad \theta \in [-d_2, 0],
\end{align*}
\]

where $\hat{x}(k) \in \mathbb{R}^n$ is the estimated state vector of $x(k)$, $\hat{y}(k) \in \mathbb{R}^p$ is the observer output vector. For each $i \in N$, $\hat{f}_i(x(k)) \in \mathbb{R}^p$ is a known nonlinear function. $L_i \in \mathbb{R}^{p \times p}$ and $H_{\delta i} \in \mathbb{R}^{p \times n}$, $\forall i, j \in N$, $i \neq j$, are the matrices to be determined.

Let $\tilde{x}(k) = x(k) - \hat{x}(k)$ be the estimation error, then we can obtain the following error system:

\[
\begin{align*}
    \tilde{x}(k+1) &= (A_{\sigma(k)} - L_{\sigma(k)}C_{\sigma(k)})\tilde{x}(k) + A_{\delta(k)}\tilde{x}(k - d(k)) + D_{\sigma(k)}(f_{\sigma(k)}(x(k)) - f_{\sigma(k)}(\hat{x}(k))), \\
    \tilde{y}(k) &= C_{\sigma(k)}\tilde{x}(k), \\
    \tilde{x}(k_0 + \theta) &= 0, \quad \theta \in [-d_2, 0]
\end{align*}
\]

Before ending this section, we introduce the following definitions and lemmas.
**Definition 1** [4]. For any \( k > k_0 \), let \( N_\sigma(k_0, k) \) denote the switching number of \( \sigma(k) \), during the interval \([k_0, k)\). If there exist \( N_0 > 0 \) and \( \tau_k > 0 \) such that, \( N_\sigma(k_0, k) \leq N_0 + \frac{k-k_0}{\tau_k} \), then \( \tau_k \) and \( N_0 \) are called the average dwell time and the chatter bound, respectively.

**Definition 2.** System (1) is said to be exponentially stable, if its solution satisfies
\[
\|x(k)\| \leq K\|x(k_0)\|_h \rho^{k-k_0}, \forall k \geq k_0
\]
for any initial conditions \( \phi(\theta) \), where \( \|x(k_0)\|_h = \sup_{\theta \in [0, \infty)} \{\|\phi(\theta)\|, \|\phi(\theta+1) - \phi(\theta)\|\} \). \( K > 0 \) is the decay coefficient, and \( 0 < \rho < 1 \) is the decay rate.

**Lemma 1** [34]. For a given matrix \( S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \), where \( S_{11} \) and \( S_{22} \) are square matrices, the following conditions are equivalent:

(i) \( S < 0 \);
(ii) \( S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0 \);
(iii) \( S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0 \).

**Lemma 2** [35]. For a matrix \( G \) of full rank and a positive definite symmetric matrix \( S \), we have
\[
(S-G)^T S^{-1} (S-G) \geq 0,
\]
which is equivalent to
\[
G^T S^{-1} G \geq G^T + G - S.
\]

Without loss of generality, we make the following assumption.

**Assumption 1.** For each \( i \in \mathbb{N} \), \( f_i(x(k)) \) satisfies
\[
f_i^T (x(k))(f_i(x(k)) - a_i(x(k))) \leq 0,
\]
and \( f_i(x(k)) - \hat{f}_i(x(k)) \) satisfies
\[
(f_i(x(k)) - \hat{f}_i(x(k)))^T (f_i(x(k)) - \hat{f}_i(x(k)) - a_i(x(k) - \hat{x}(k))) \leq 0,
\]
where \( a_i \) is a known real constant.

The object of the paper is to determine the matrices \( L_i \) and \( H_\sigma(i, j \in \mathbb{N}, i \neq j) \) in (3) such that error system (4) is exponentially stable.

**3. Main results**

**3.1. Stability analysis**

To obtain the main results, we first consider the exponential stability of the following system
\[
x(k+1) = A_{\sigma(k)} x(k) + B_{\sigma(k)}(x(k) - d(k)) + D_{\sigma(k)} f_{\sigma(k)}(x(k)), \quad k \neq k_0 - 1, \ b \in \mathbb{Z}^+,
\]
\[
x(k+1) = E_{\sigma(k+1)} x(k), \quad k = k_0 - 1, \ b \in \mathbb{Z}^+,
\]
\[
x(k_0 + \theta) = \phi(\theta), \quad \theta \in [-d_\sigma, 0].
\]

Before giving the stability result, we present the following lemma which is essential for our later development.

**Lemma 3.** Suppose that there exists a \( C^1 \) function \( V : \mathbb{R}^n \to \mathbb{R} \), and two positive scalars \( \lambda_1 \) and \( \lambda_2 \) such that the following inequality holds
\[
\lambda_1 \|x(k)\|^2 \leq V(x(k)) \leq \lambda_2 \|x(k)\|^2, \forall k \geq k_0
\]
where \( \|x(k)\|^2_h = \sup_{\theta \in [0, \infty)} \{\|x(k+\theta)\|^2, \|x(k+\theta+1) - x(k+\theta)\|^2\} \), along the solution of system (8), we have the following inequalities
\[
V(x(k+1)) \leq \beta V(x(k)), \quad \forall k \in [k_0-1, k_0-1), \ b \in \mathbb{Z}^+.
\]
\[ V(x(k + 1)) \leq \mu V(x(k)), \quad k = k_0 - 1, \quad b \in Z^+, \]  
\[ \text{where } 0 < \beta < 1 \text{ and } \mu \geq 1, \text{ then system (8) is exponentially stable for any switching signals } \sigma(k) \text{ with the average dwell time satisfying} \]
\[ \tau_\alpha > \tau_\alpha' = 1 - \frac{\ln \mu}{\ln \beta}. \]  
\[ \text{Proof. Similar to the proof line of Lemma 2 in [33], it can be deduced from (10) and (11) that} \]
\[ V(x(k)) \leq (\mu \beta^{-1})^{\frac{\beta}{1-\beta}} \beta^k V(x(k_0)) = (\mu \beta^{-1})^{\frac{\beta}{1-\beta}} \beta^k V(x(k_0)). \]  
\[ \text{where } \rho = \mu \beta^{-1} \beta^k. \]
\[ \text{It follows from (13) that} \]
\[ \|x(k)\| \leq \sqrt[n]{\frac{1}{\lambda_1} \frac{1}{\lambda_2} (\mu \beta^{-1})^{\frac{\beta}{1-\beta}} \beta^k ||x(k_0)||}, \forall k \geq k_0. \]
\[ \text{One can verify that } 0 < \rho < 1. \text{ Hence, system (8) is exponentially stable under the average dwell time scheme (12). This completes the proof.} \]

The following theorem provides a stability criterion for system (8). \[ \square \]

**Theorem 1.** For a given positive scalar \(0 < \beta < 1\), if there exist positive definite symmetric matrices \(P_i, Q_i, R_i, Z_i\), and any matrices \(X_i = [X_{i11} X_{i12} \ldots X_{i22}] > 0, \quad N_i = [N_{i11} N_{i12}], \quad N_i = [N_{i11} N_{i12}], \quad (i \in \mathbb{N}) \) with appropriate dimensions, such that \[
\begin{bmatrix}
\Gamma_{i11} & \Gamma_{i12} & -\beta^d N_{i21} & \frac{\beta}{2} I & A_i^T P_i & (A_i - I)^T Z_i \\
* & \Gamma_{i22} & -\beta^d N_{i22} & 0 & A_i^T P_i & A_i^T Z_i \\
* & * & -\beta^d R_i & 0 & 0 & 0 \\
* & * & * & -I & D_i^T P_i & D_i^T Z_i \\
* & * & * & * & -P_i & 0 \\
* & * & * & * & * & -d_i^{-1} Z_i \\
\end{bmatrix}
\begin{bmatrix}
X_i \\
N_i \\
Z_i \\
\end{bmatrix}
\geq 0, \quad \begin{bmatrix}
X_i \\
N_i \\
Z_i \\
\end{bmatrix}
\geq 0, \tag{15}
\]

where
\[
\begin{align*}
\Gamma_{i11} &= -\beta P_i + (d_i - d_i + 1) Q_i + R_i + \beta^d (N_{i11}^T + N_{i11}) + d_i \beta^d X_{i11}, \\
\Gamma_{i12} &= \beta^d (-N_{i11} + N_{i12}^T + N_{i21}) + d_i \beta^d X_{i12}, \\
\Gamma_{i22} &= -\beta^d Q_i - \beta^d (N_{i12}^T + N_{i12} - N_{i22} - N_{i22}^T) + d_i \beta^d X_{i22},
\end{align*}
\]
then system (8) is exponentially stable for any switching signals \(\sigma(k)\) with the average dwell time satisfying \[
\tau_\alpha > \tau_\alpha' = 1 - \frac{\ln \mu}{\ln \beta}. \]  
\[ \text{Proof. Denote} \]
\[ \eta(k) = x(k + 1) - x(k) = A_{\sigma(k)} x(k) + A_{\sigma(k)} x(k - d(k)) + D_{\sigma(k)} f_{\sigma(k)}(x(k)) - x(k) \]
\[ = (A_{\sigma(k)} - I) x(k) + A_{\sigma(k)} x(k - d(k)) + D_{\sigma(k)} f_{\sigma(k)}(x(k)) \tag{19} \]
\[ \text{For system (8), choose the following piece-wise Lyapunov functional candidate} \]
\[ V(x(k)) = V_{\sigma(k)}(x(k)), \]
\[ \text{where} \quad \beta Q_i \leq \mu Q_i, \quad \beta R_i \leq \mu R_i, \quad \beta Z_i \leq \mu Z_i, \quad i, j \in \mathbb{N}, i \neq j. \]
the form of each $V_{\sigma(k)}(x(k))$ is given by

$$V_{\sigma(k)}(x(k)) = V_{\sigma(k)1}(x(k)) + V_{\sigma(k)2}(x(k)) + V_{\sigma(k)3}(x(k)) + V_{\sigma(k)4}(x(k)) + V_{\sigma(k)5}(x(k)),$$

(20)

where

$$V_{11}(x(k)) = x^T(k)P_1x(k),$$
$$V_{12}(x(k)) = \sum_{r=k-d(k)}^{k-1} \beta^{k-r-1}x^T(r)Q_1x(r),$$
$$V_{13}(x(k)) = \sum_{s=d_2+1-r-k+1}^{d_1} \sum_{r=k-d_2+1}^{k-1} \beta^{k-r-1}x^T(r)Q_2x(r),$$
$$V_{14}(x(k)) = \sum_{r=k-d_2+1}^{k-1} \beta^{k-r-1}x^T(r)R_3x(r),$$
$$V_{15}(x(k)) = \sum_{s=d_2+1-r-k+1}^{d_1} \sum_{r=k-d_2+1}^{k-1} \beta^{k-r-1}\eta^T(r)Z_3\eta(r),$$

where $\eta(r) = x(r+1) - x(r)$, and $P_i$, $Q_i$, $R_i$, and $Z_i$ ($i \in \mathbb{N}$) are positive definite matrices to be determined.

By (20), we can easily find two positive scalars $\lambda_1$ and $\lambda_2$ such that

$$\lambda_1 \|x(k)\|^2 \leq V(x(k)) \leq \lambda_2 \|x(k)\|^2,$$

(21)

where

$$\lambda_1 = \min_{i \in \mathbb{N}}(\lambda_{\text{min}}(P_i)), \quad \lambda_2 = \max_{i \in \mathbb{N}}(\lambda_{\text{max}}(P_i) + d_1 \lambda_{\text{max}}(Q_1) + (d_2 - d_1)^2 \lambda_{\text{max}}(Q_2) + d_2 \lambda_{\text{max}}(R_3) + d_2^2 \lambda_{\text{max}}(Z_3))$$

When $k \in [k_0, k_0+1)$, let $\sigma(k) = i$ ($i \in \mathbb{N}$), along the trajectory of system (8), we have

$$V_{12}(x(k+1)) - \beta V_{12}(x(k)) = \sum_{r=k-d_2+1}^{k-1} \beta^{k-r}x^T(r)Q_1x(r) - \sum_{r=k-d_2+1}^{k-1} \beta^{k-r}x^T(r)Q_2x(r),$$

$$\leq x^T(k)Q_1x(k) + \sum_{r=k-d_2+1}^{k-d_1} \beta^{k-r}x^T(r)Q_2x(r) - \beta^{d_2}x^T(k-d_2)Q_2x(k-d_2),$$

(23)

$$V_{13}(x(k+1)) - \beta V_{13}(x(k)) = (d_2 - d_1)x^T(k)Q_3x(k) - \sum_{r=k-d_2+1}^{k-d_1} \beta^{k-r}x^T(r)Q_3x(r),$$

(24)

$$V_{14}(x(k+1)) - \beta V_{14}(x(k)) = x^T(k)R_3x(k) - \beta^{d_2}x^T(k-d_2)R_3x(k-d_2),$$

(25)

$$V_{15}(x(k+1)) - \beta V_{15}(x(k+1)) = \sum_{s=d_2+1-r-k+1}^{d_1} \sum_{r=k-d_2+1}^{k-1} \beta^{k-r}x^T(r)Q_2x(r) - \beta^{d_2}x^T(k-d_2)Q_2x(k-d_2),$$

(26)

From (6) in Assumption 1, we can obtain that

$$\phi_1^T(k) \begin{bmatrix} 0 & -\phi_1 \\ \phi_2 & I \end{bmatrix} \phi_1(k) \leq 0,$$

(27)

where $\phi_1^T(k) = [x^T(k) \quad f_1^T(x(k))].$

Denoting

$$\zeta_1^T(k) = [x^T(k) \quad x^T(k-d(k)) \quad x^T(k-d_2) \quad f_1^T(x(k))],$$

we obtain from (22)–(27) that

$$V_{1}(x(k+1)) - \beta V_{1}(x(k)) \leq \zeta_1^T(k)(\psi_1 + \psi_2)\zeta_1(k) - \sum_{r=k-d_2}^{k-1} \beta^{k-r}\eta^T(r)Z_3\eta(r),$$

(28)
where

$$
\psi_i = \begin{bmatrix}
-\beta P_i + (d_2 - d_1 + 1)Q_i + R_i & 0 & 0 & \frac{1}{C} I
* & -\beta^2 d_i Q_i & 0 & 0
* & * & -\beta^2 d_i R_i & 0 \\
* & * & * & -I
\end{bmatrix},
$$

$$
\psi_{i1} = \begin{bmatrix}
A_i^T \\
A_{di}^T \\
0 \\
D_{fi}^T
\end{bmatrix} P_i [A_i A_{di} 0 D_{fi}] ,
$$

$$
\psi_{i2} = d_2 \begin{bmatrix}
(A_i - I)^T \\
A_{di}^T \\
0 \\
D_{fi}^T
\end{bmatrix} Z_i [A_i A_{di} 0 D_{fi}] .
$$

In addition, the following equations hold for any matrices $N_{i1} = \begin{bmatrix} N_{i11} & \end{bmatrix}$ and $N_{i2} = \begin{bmatrix} N_{i21} & \end{bmatrix}$ with appropriate dimensions:

$$
0 = 2\beta^2 \left[ x^T(k) N_{i11} + x^T(k - d(k)) N_{i12} \right] \times \left[ x(k) - x(k - d(k)) - \sum_{r=k-d(k)}^{k-1} \eta(r) \right],
$$

(29)

$$
0 = 2\beta^2 \left[ x^T(k) N_{i21} + x^T(k - d(k)) N_{i22} \right] \times \left[ x(k - d(k)) - x(k - d_2) - \sum_{r=k-d_2}^{k-d(k)-1} \eta(r) \right]
$$

(30)

On the other hand, for any matrices $X_i = \begin{bmatrix} X_{i11} & X_{i12} \end{bmatrix} > 0$, the following equation also holds:

$$
0 = d_2 \beta^2 \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right] X_i \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right] - \beta^2 \sum_{r=k-d(k)}^{k-1} \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right] X_i \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right] - \beta^2 \sum_{r=k-d_2}^{k-d(k)-1} \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right] X_i \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right].
$$

(31)

From (28)-(31), we have

$$
V_i(x(k+1)) - \beta V_i(x(k)) \leq z_i^T(k)(\psi_i' + \psi_{i1} + \psi_{i2})z_i(k) - \beta^2 \sum_{r=k-d(k)}^{k-1} \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right] X_i \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right] - \beta^2 \sum_{r=k-d_2}^{k-d(k)-1} \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right] X_i \left[ \begin{array}{c}
x(k) \\
x(k - d(k))
\end{array} \right],
$$

(32)

where

$$
\psi_i' = \begin{bmatrix}
\Gamma_{i11} & \Gamma_{i12} & -\beta^2 N_{i21} & \frac{1}{C} I \\
* & \Gamma_{i22} & -\beta^2 N_{i22} & 0 \\
* & * & -\beta^2 R_i & 0 \\
* & * & * & -I
\end{bmatrix}.
$$

By Lemma 1, we can obtain that $\psi_i' + \psi_{i1} + \psi_{i2} < 0$ is equivalent to

$$
\begin{bmatrix}
\Gamma_{i11} & \Gamma_{i12} & -\beta^2 N_{i21} & \frac{1}{C} I \\
* & \Gamma_{i22} & -\beta^2 N_{i22} & 0 \\
* & * & -\beta^2 R_i & 0 \\
* & * & * & -I \\
* & * & * & -P_i^{-1} \\
* & * & * & -d_2^{-1} Z_i^{-1}
\end{bmatrix} < 0
$$

(33)

Using $\text{diag}(I, I, I, P_i, I, Z_i)$ to pre- and post-multiply both sides of (33), it is easy to get that (14) is equivalent to (33). Thus it can be obtained from (14) and (15) that

$$
V_i(x(k+1)) \leq \beta V_i(x(k)),
$$

(34)
When \( k = k_0 - 1 \), denote \( \sigma(k_0 - 1) = j, (j \in \mathbb{N}) \), then we have
\[
V_{i3}(x(k_0)) - \mu V_{i3}(x(k_0 - 1)) = (E_j x(k_0 - 1))^T P_i E_j x(k_0 - 1) - \mu x^T(k_0 - 1) P_j x(k_0 - 1), \tag{35}
\]
\[
V_{i2}(x(k_0)) - \mu V_{i2}(x(k_0 - 1)) = \sum_{r-k_0-d_1}^{k_0-1} \beta^{k_0-r-1} x^T(r) Q_1 x(r) - \mu \sum_{r-k_0-d_1}^{k_0-2} \beta^{k_0-r-2} x^T(r) Q_1 x(r)
\leq x^T(k_0 - 1) Q_1 x(k_0 - 1) + \sum_{r-k_0-d_1}^{k_0-2} \beta^{k_0-r-2} x^T(k_0 - 1) Q_1 x(k_0 - d_1 - 1), \tag{36}
\]
\[
V_{i3}(x(k_0)) - \mu V_{i3}(x(k_0 - 1)) \leq (d_2 - d_1) x^T(k_0 - 1) Q_2 x(k_0 - 1) + \sum_{r-k_0-d_2}^{k_0-1} \beta^{k_0-r-2} x^T(r) Q_2 x(r) - \mu \sum_{r-k_0-d_2}^{k_0-2} \beta^{k_0-r-2} x^T(r) Q_2 x(r), \tag{37}
\]
\[
V_{i4}(x(k_0)) - \mu V_{i4}(x(k_0 - 1)) = x^T(k_0 - 1) R_i x(k_0 - 1) + \sum_{r-k_0-d_2}^{k_0-1} \beta^{k_0-r-2} x^T(r) (R_i - \mu R_j) x(r) - \mu \beta^{k_0-1} x^T(k_0 - 1 - d_2) R_j x(k_0 - 1 - d_2), \tag{38}
\]
\[
V_{i5}(x(k_0)) - \mu V_{i5}(x(k_0 - 1)) \leq d_2 x^T(k_0 - 1) Z_i x(k_0 - 1) + \sum_{r-k_0-d_2}^{k_0-1} \beta^{k_0-r-2} x^T(r) (Z_i - \mu Z_j) x(r) - \mu \sum_{r-k_0-d_2}^{k_0-2} \beta^{k_0-r-2} x^T(r) Z_j x(r), \tag{39}
\]
Combining (35)-(39), we obtain from (17)-(18) that
\[
V_{\sigma(k_3)}(x(k_0)) \leq \mu V_{\sigma(k_3)}(x(k_0 - 1)). \tag{40}
\]
Thus according to Lemma 3, system (8) is exponentially stable. The proof is completed. \( \square \)

**Remark 3.** Compared with the existing result in the literature [33], the proposed stability condition is delay-dependent. Also, the system considered in this section contains nonlinearity and time-varying delay, which are universal in the application. On the other hand, our result is different from those presented in [25–26], where the delay is constant and the nonlinearity is not considered.

**Remark 4.** When \( D_i = 0 \), the condition (14) will degenerate to
\[
\begin{bmatrix}
\Gamma_{i11} & \Gamma_{i12} & -\beta^2 N_{i21} A_i^T P_i & (A_i - I)^T Z_i \\
* & \Gamma_{i22} & -\beta^2 N_{i22} A_i^T P_i & A_i^T Z_i \\
* & * & -\beta^2 R_i & 0 \\
* & * & * & -P_i \\
* & * & * & -d_2^{-1} Z_i
\end{bmatrix} < 0
\]

### 3.2. Observer design

In this subsection, we will focus on the design of observer (3) for system (1). The following theorem presents sufficient conditions for the existence of the observer.

**Theorem 2.** Consider system (1), for a given positive scalar \( 0 < \beta < 1 \), if there exist positive definite symmetric matrices \( P_i, Q_i, R_i, Z_i \), any matrices \( W_i \) of full rank, and any matrices \( Y_i \), such that
\[
\begin{bmatrix}
\Gamma_{i11} & \Gamma_{i12} & -\beta^2 N_{i21} A_i^T W_i^T & (A_i W_i - Y_i)^T \\
* & \Gamma_{i22} & -\beta^2 N_{i22} & 0 \\
* & * & -\beta^2 R_i & 0 \\
* & * & * & -I \\
* & * & * & -W_i^T - W_i + P_i \\
* & * & * & * & -d_2^{-1} Z_i
\end{bmatrix} < 0, \tag{41}
\]
\[
\begin{bmatrix}
\dot{X}_i \\
* \quad W_i^T + W_i - Z_i
\end{bmatrix}
\geq 0,
\begin{bmatrix}
\dot{X}_i \\
* \quad W_i^T + W_i - Z_i
\end{bmatrix}
\geq 0,
\]
(42)

where
\[
\Gamma_{11} = -\beta P_i - (d_2 - d_1 + 1)Q_i + R_i + \beta d^2 (\hat{N}_{11} + \hat{N}_{11}) + d_2 \beta d^2 \hat{X}_{11},
\]
\[
\Gamma_{12} = \beta d^2 (-\hat{N}_{12} + \hat{N}_{12} + \hat{N}_{22}) + d_2 \beta d^2 \hat{X}_{12},
\]
\[
\Gamma_{22} = -\beta d^2 Q_i - \beta d^2 (\hat{N}_{12} + \hat{N}_{12} + \hat{N}_{22}) + d_2 \beta d^2 \hat{X}_{22}.
\]

then there exists an observer in the form of (3) such that error system (4) is exponentially stable for any switching signals with the average dwell time scheme (16), where \( \mu \geq 1 \) satisfies

\[
\begin{bmatrix}
\Xi_{11} \\
* \quad \Xi_{12} \\
* \quad \Xi_{22} \quad H_i P_i \\
* \quad * \quad -P_i
\end{bmatrix}
< 0, \quad i, j \in \mathbb{N}, \quad i \neq j,
\]
(43)

\[
\beta Q_i \leq \mu Q_j, \quad \beta R_i \leq \mu R_j, \quad \beta Z_i \leq \mu Z_j, \quad i, j \in \mathbb{N}, \quad i \neq j.
\]
(44)

where
\[
\Xi_{11} = E_i P_i E_i + \mu P_i + (d_2 - d_1 + 1)Q_i + R_i + d_2 Z_i,
\]
\[
\Xi_{12} = -E_i P_i H_i + \mu P_i - (d_2 - d_1 + 1)Q_i - R_i - d_2 Z_i,
\]
\[
\Xi_{22} = -E_i P_i + (d_2 - d_1 + 1)Q_i + R_i + d_2 Z_i,
\]
\[
P_i = (W_i^T)^{-1} P_i (W_i)^{-1}, \quad R_i = (W_i^T)^{-1} R_i (W_i)^{-1}, \quad Q_i = (W_i^T)^{-1} Q_i (W_i)^{-1}, \quad Z_i = Z_i^{-1}
\]

Moreover, if the above conditions (41), (42) have a feasible solution, the observer gain matrices \( L_i \) can be obtained by
\[
L_i = Y_i (C_i W_i)^{+},
\]
(45)

where \((C_i W_i)^{+}\) denotes the pseudo-inverse matrix of \( C_i W_i \).

**Proof.** Applying Theorem 1 to system (4) and replacing \( A_i \) in Theorem 1 with \( A_i - L_i C_i \), one can obtain

\[
\begin{bmatrix}
\Gamma_{11} \\
\Gamma_{12} \\
\Gamma_{22}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\sigma_i I \\
A^T_i G_i \\
-A_i^T B_i
\end{bmatrix} \\
A^T_i B_i \\
A^T_i F_i
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix}
\sigma_i I \\
A^T_i G_i \\
-A_i^T B_i
\end{bmatrix} \\
A^T_i B_i \\
A^T_i F_i
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix}
\sigma_i I \\
A^T_i G_i \\
-A_i^T B_i
\end{bmatrix} \\
A^T_i B_i \\
A^T_i F_i
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix}
\sigma_i I \\
A^T_i G_i \\
-A_i^T B_i
\end{bmatrix} \\
A^T_i B_i \\
A^T_i F_i
\end{bmatrix} < 0
\]
(46)

Using \( \text{diag} \{1, 1, 1, I_i \} \) and \( \text{diag} \{1, 1, 1, I_i \} \) to pre- and post-multiply both sides of (46), respectively, and applying Lemma 2, we obtain that (46) holds if the following inequality is satisfied:

\[
\begin{bmatrix}
\Gamma_{11} \\
\Gamma_{12} \\
\Gamma_{22}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
\sigma_i I \\
A^T_i G_i \\
-A_i^T B_i
\end{bmatrix} \\
A^T_i B_i \\
A^T_i F_i
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix}
\sigma_i I \\
A^T_i G_i \\
-A_i^T B_i
\end{bmatrix} \\
A^T_i B_i \\
A^T_i F_i
\end{bmatrix} < 0
\]
(47)

Denote \( W_i = (G_i)^{-1} \) and assign the following new matrices:

\[
W_i^T N_{11} W_i = \hat{N}_{11}, \quad W_i^T N_{12} W_i = \hat{N}_{12}, \quad W_i^T N_{22} W_i = \hat{N}_{22}, \quad W_i^T X_{11} W_i = \hat{X}_{11}, \quad W_i^T X_{12} W_i = \hat{X}_{12}, \quad W_i^T X_{22} W_i = \hat{X}_{22},
\]
\[
W_i^T P_i W_i = \hat{P}_i, \quad W_i^T R_i W_i = \hat{R}_i, \quad W_i^T Q_i W_i = \hat{Q}_i, \quad Y_i = L_i C_i W_i, \quad Z_i = Z_i^{-1}
\]

Using \( \text{diag} \{W_i^T, W_i^T, W_i^T, I, W_i^T, I, I, W_i^T, I\} \) to pre- and post-multiply both sides of (47), respectively, we get (41).

In addition, using \( \text{diag} \{W_i^T, W_i^T, W_i^T\} \) and \( \text{diag} \{W_i, W_i, W_i\} \) to pre- and post-multiply both sides of the two inequalities in (15), respectively, and applying Lemma 2, we obtain (42). From (4b), one has that
\[ V_{ii}(\tilde{x}(k_0)) - \mu V_{ii}(\tilde{x}(k_0 - 1)) = (E_p x(k_0 - 1) - H_p \tilde{x}(k_0 - 1))^T P_i (E_p x(k_0 - 1) - H_p \tilde{x}(k_0 - 1)) \\
- \mu (x(k_0 - 1) - \tilde{x}(k_0 - 1))^T P_j (x(k_0 - 1) - \tilde{x}(k_0 - 1)) \] (48)

Then following the proof line of Theorem 1, one obtains that
\[ V_{ii}(\tilde{x}(k_0)) \leq \mu V_{ii}(\tilde{x}(k_0 - 1)) \]
if the following inequalities hold
\[
\begin{bmatrix}
X_{i1} & X_{i2} \\
* & X_{i2}
\end{bmatrix} < 0, \quad i, j \in \mathbb{N}, \quad i \neq j.
\] (49)

where
\[
X_{i1} = E_p^T P_i E_q - \mu P_i + (d_2 - d_1 + 1) Q_i + R_i + d_2 Z_i,
\]
\[
X_{i2} = -E_p^T P_i H_q + \mu P_j - (d_2 - d_1 + 1) Q_i - R_i - d_2 Z_i,
\]
\[
X_{i2} = H_q^T P_i H_q - \mu P_j + (d_2 - d_1 + 1) Q_i + R_i + d_2 Z_i.
\]

By Lemma 1, it is easy to get that (43) is equivalent to (49).

This completes the proof. \(\square\)

We are now in a position to give a procedure for determining the desired matrices \(L_i\) and \(H_q(j, i \in \mathbb{N})\).

**Algorithm 1**

Step 1: Given a constant \(0 < \beta < 1\), we can obtain the matrices \(\bar{P}_i, \bar{Q}_i, \bar{R}_i, \bar{Z}_i, W_i, Y_i, N_i, N_{i2}\) and \(X_i(i \in \mathbb{N})\) by solving (41), (42).

Step 2: With the matrices obtained in Step 1, we can get \(L_i\) by (45).

Step 3: Solving (43), (44), the matrices \(H_q(j, i \in \mathbb{N})\) and \(\mu\) can be obtained. Then compute the dwell time in (16).

When the proposed observer (3) has the same jumps at switching instants as those in system (1), i.e., \(H_q = E_q(j, i \in \mathbb{N})\), the condition (43) can be written as (17). In the following numerical simulations, \(H_q = E_q\) will be chosen and (17) will be used.

4. Numerical examples

In this section, we present two examples to illustrate the effectiveness of the proposed approach.

**Example 1.** Consider system (8) with the following parameters:
\[
A_1 = \begin{bmatrix} 0.56 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad D_{f1} = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.12 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.1 \end{bmatrix},
\]
\[
A_2 = \begin{bmatrix} 0.65 & 0.1 \\ -0.1 & 0.3 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.11 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad D_{f2} = \begin{bmatrix} -0.13 & 0 \\ 0.1 & 0.13 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 1.25 & 0 \\ 0 & 1.3 \end{bmatrix}.
\]

Let \(\beta = 0.8\), \(d_1 = 1\), \(d_2 = 3\), \(a_1 = a_2 = 0.6\). Solving the matrix inequalities in Theorem 1, we get \(\mu = 3.7873\). From (16), it can be obtained that \(\tau_{s*} = 6.9677\).

Take
\[
f_1(x(k)) = \begin{bmatrix} x_1(a_1 - a_1 |\sin(0.5\pi k)|) \\ x_2(a_1 - a_1 |\sin(0.5\pi k)|) \end{bmatrix}, \quad f_2(x(k)) = \begin{bmatrix} x_1(a_2 - a_2 |\cos(0.5\pi k)|) \\ x_2(a_2 - a_2 |\cos(0.5\pi k)|) \end{bmatrix}.
\]

Choosing \(\tau_s = 7\), simulation results are shown in Figs. 1 and 2, where the initial conditions are \(x(0) = [5 \ 8], \tilde{x}(k) = \phi(k) = 0, k \in [-3, 0]\). It can be seen that the system is exponentially stable under the switching signal with the average dwell time \(\tau_s = 7\).

In order to illustrate the delay effect, we give the admissible average dwell time \(\tau_{s*}\) for different \(d_2\) in Table 1.

It can be found that if the value of \(d_2\) gets larger, the value of \(\tau_{s*}\) tends to get larger. When \(d_2 \geq 6\), the admissible average dwell time \(\tau_{s*}\) cannot be found by Theorem 1.

**Example 2.** Consider system (1) with the following parameters:
\[
A_1 = \begin{bmatrix} 0.56 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.5 & 0 \\ 0.6 & 0.8 \end{bmatrix}, \quad D_{f1} = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.12 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.65 & 0.1 \\ -0.1 & 0.3 \end{bmatrix},
\]
\[
A_{d2} = \begin{bmatrix} 0.11 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.5 & 0.7 \\ 0 & -0.6 \end{bmatrix},
\]
\[
D_{f2} = \begin{bmatrix} -0.13 & 0 \\ 0.1 & 0.13 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.1 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 1.25 & 0 \\ 0 & 1.3 \end{bmatrix}.
\]
Let $\beta = 0.8$, $d_1 = 1$, $d_2 = 3$, $a_1 = a_2 = 0.6$. Solving (41), (42) in Theorem 2, we get

\[
W_1 = \begin{bmatrix} 1.8233 & 0.1105 \\ 0.1182 & 2.2640 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 2.0135 & -0.0169 \\ -0.0024 & 2.0000 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 0.2131 & 0.0283 \\ 0.0939 & -0.5301 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.3509 & 0.2016 \\ -0.1397 & -0.4630 \end{bmatrix}
\]

Then by (45), $L_1$ and $L_2$ can be obtained:

\[
L_1 = \begin{bmatrix} 0.2227 & -0.0085 \\ 0.4899 & -0.2968 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.3488 & 0.2365 \\ -0.1393 & 0.2243 \end{bmatrix}
\]

From (16), (43), and (44), we get $\mu = 3.3261$ and $\tau^*_a = 6.3858$. 

---

**Table 1**

<table>
<thead>
<tr>
<th>$d_2$</th>
<th>$d_3 = 3$</th>
<th>$d_3 = 4$</th>
<th>$d_3 = 5$</th>
<th>$d_3 \geq 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau^*_a$</td>
<td>6.9677</td>
<td>9.4204</td>
<td>14.4190</td>
<td>Infeasible</td>
</tr>
</tbody>
</table>

---

**Fig. 1.** Switching signal with the average dwell time $\tau^*_a = 7$.

**Fig. 2.** State $x(k)$ of the system.
Fig. 3. Switching signal with the average dwell time $\tau_a = 6.5$.

Fig. 4. State $x(k)$ of the system and its estimation $\hat{x}(k)$.

Fig. 5. State of the error dynamic system.
Take

\[
\begin{align*}
 f_1(x(k)) &= \begin{bmatrix} x_1(a_1 - a_1 |\sin(0.5\pi k)|) \\ x_2(a_1 - a_2 |\cos(0.5\pi k)|) \end{bmatrix}, \\
 f_2(x(k)) &= \begin{bmatrix} x_1(a_2 - a_2 |\cos(0.5\pi k)|) \\ x_2(a_2 - a_2 |\cos(0.5\pi k)|) \end{bmatrix}, \\
 f_3(x(k)) &= \begin{bmatrix} \dot{x}_1(a_1 - a_1 |\sin(0.5\pi k)|) \\ \dot{x}_2(a_1 - a_1 |\sin(0.5\pi k)|) \end{bmatrix}.
\end{align*}
\]

Choosing \( \tau_0 = 6.5 \), simulation results are shown in Figs. 3–5, where \( x(0) = [5.8]^T, x(\theta) = [0 0]^T, \theta \in [-d_2, 0]; x(\theta) = [0 0]^T, \theta \in [-d_2, 0], \) and \( u(k) = 3 \sin(2k) \).

We can see that the proposed observer can estimate the state of the system. This demonstrates the effectiveness of the proposed method.

5. Conclusion

In this paper, we have studied the problem of observer design for discrete-time nonlinear impulsive switched systems with time-varying delays. A delay-dependent exponential stability criterion for the considered system is established by using the average dwell time approach. Then, an observer is proposed. The designed observer can estimate the state of the system. Finally, two numerical examples are given to show the effectiveness of the proposed approach.

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References


