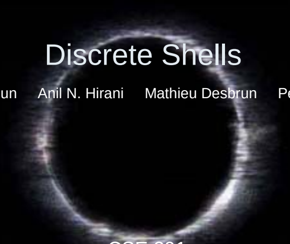

Discrete Shells

Eitan Grinspun Anil N. Hirani Mathieu Desbrun Peter Schroder

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Henrick Shyu

Contents

Overview (1 of 2)

- Thin shells
  - High ratio of width to thickness (> 100)
  - Curved undeformed configuration
  - Examples
    - Leaves, fingernails, hats, cans, cardboard
- Remarkably difficult to simulate
  - Degeneracy in one dimension (“thinness”)
    - Cannot model as a three-dimensional solid
  - Structural / flexural rigidity
    - V-shaped cardboard vs flat cardboard

Overview (2 of 2)

- Cloth
  - Somewhat similar to shells…
  - Successful numerical treatment as mass-spring network
    - Calculate forces for shearing, stretching, bending
    - Unfortunately, insensitive to sign of dihedral angle
- So what?
  - A minor change to the bending energy yields thin shell simulation

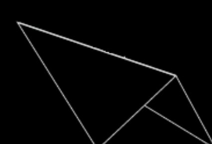
Previous Work

- Continuum-based approaches
  - Kirchoff-Love consitutive equations
  - Cirak et al. 2000, *Subdivision surfaces*
  - Seth Green et al. 2002, *Subdivision-based multilevel methods for large scale engineering simulation of thin shells*
  - Grinspun et al. 2002, *CHARMS*

Complex, challenging, costly to simulate

Discrete Shell Model

- 2-manifold triangle mesh
- Governed by
  - Membrane energies (intrinsic)
    - Stretching – length preserving
    - Shearing – area preserving
  - Flexural energies (extrinsic)
    - Bending – angle preserving
- Deformation defined by piecewise-affine deformation map
  - Mapping of every face of the undeformed to the deformed surface



Dynamics (1 of 2)

Shell model as sum of membrane and flexural energies

$$W = W_M + k_B W_B$$

W_M is the membrane energy
W_B is the flexural energy
k_B is the bending stiffness

The membrane energy can be expressed as

$$W_M = k_L W_L + k_A W_A$$

W_L is the stretching energy
k_L is the stretching stiffness

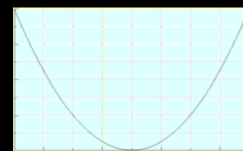
W_A is the shearing energy
k_A is the shearing stiffness

Dynamics (2 of 2)

$$W_L = \sum_e (1 - \|e\|/\|\bar{e}\|)^2 \|\bar{e}\|$$
$$W_A = \sum_A (1 - \|A\|/\|\bar{A}\|)^2 \|\bar{A}\|$$

||e|| is the *deformed* edge length
||e|| bar is the *undeformed* edge length

||A|| is the *deformed* area
||A|| bar is the *undeformed* area



$y = (1-x/x_0)^2 x_0$

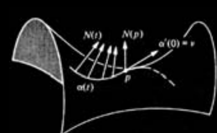
Resists stretching and shearing deformation

Bending Energy (1 of 5)

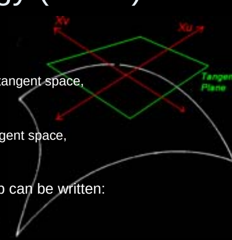
Bending Energy Intuition
Measure of the difference in curvature.

Curvature
Differential of the Gauss map. Shape Operator does this.

Gauss Map
Maps points in R^3 to unit sphere by associated with points with their oriented unit normal vector.



Bending Energy (2 of 5)



Shape Operator continued…

Since the differential of a curve C(t) lies in it's tangent space, we can write:

$$C'(t) = \mathbf{X}_u A + \mathbf{X}_v B$$

where X_u and X_v are basis of the tangent space, A and B are coefficients.

Similarly, the differential of the gauss map can be written:

$$N_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v$$
$$N_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v$$

or

$$\begin{pmatrix} N_u \\ N_v \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \mathbf{x}_u \\ \mathbf{x}_v \end{pmatrix}$$

The matrix A is the shape operator.

Bending Energy (3 of 5)

Bending Energy Calculation
The squared difference of mean curvature.

Mean Curvature Intuition
The mean curvature is half the principle curvatures.

Principle Direction

Calculated as the eigenvectors of the differential of the gauss map

Bending Energy (4 of 5)

Principle Curvature
The differential of the gauss map, in the basis of the principle directions. Solvable using the Weingarten equations.

The diagonal elements of the shape operator are the principle curvatures.

Mean Curvature Calculation
The mean curvature calculated at point p is

$$H(\mathbf{p}) = \frac{1}{2}\,\mathrm{Tr}\,(S(\mathbf{p})),$$

Tr(S) denotes the matrix trace (sum of diagonal elements) of the shape operator evaluated at p.

Note: The equation actually still holds even if the shape operator is not evaluated in the basis of the principle directions.

Bending Energy (5 of 5)

Bending Energy

Squared difference of mean curvature

$$[\mathrm{Tr}(\varphi^*\mathrm{S}) - \mathrm{Tr}(\bar{\mathrm{S}})]^2 = 4(H \circ \varphi - \bar{H})^2$$

S and **S bar** are the shape operators evaluated over the undeformed and deformed surfaces, respectively

H and **H bar** are the mean curvatures

φ represents a diffeomorphism which is a map between topological space that is differentiable and has a differentiable inverse.
-used here to have consistent basis of coordinates

Tr(A) is the matrix trace

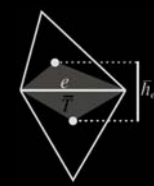
Flexural Energy (1 of 6)

- Continuous Flexural Energy
  - Integrate over reference domain

$$\int_{\bar{\Omega}} 4(H \circ \varphi - \bar{H})^2 d\bar{A}$$

- Discrete Flexural Energy

  Partition surface into disjoint union of diamond-shaped tiles for each edge **e**



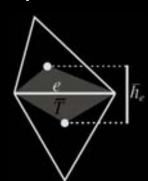
T is the surface of the diamond
h is 1/6 of the sum of the heights of the two triangles sharing edge **e**

The area of T is $\frac{1}{2}\bar{h}_e\|\bar{e}\|$

Flexural Energy (2 of 6)

Cohen-Steiner and Morvan proved that over such a diamond, the mean curvature integral is

$$\int_{\bar{T}} \bar{H} d\bar{A} = \bar{\theta}\|\bar{e}\|$$



Similarly, the difference of the mean curvature integral is

$$\int_{\bar{T}} (H \circ \varphi - \bar{H}) d\bar{A} = (\theta - \bar{\theta})\|\bar{e}\|$$

where θis the complement of the dihedral angle of edge **e**

But we want to solve for

$$\int_{\bar{T}} (H \circ \varphi - \bar{H})^2 d\bar{A}$$

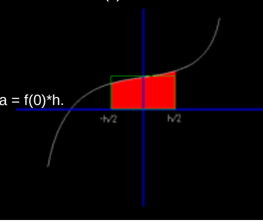
Flexural Energy (3 of 6)

Concept of Area Average Value

Say we want to calculate $\int_{-h/2}^{h/2} f(x)dx$

If h is very small, we can assume f(x) is linear.

Approximate as
average value*area = f(0)*h.



Flexural Energy (4 of 6)

Extending this concept, we can write for g(f(x)).

$$\int_{-h/2}^{h/2} g(f(x))dx = g(f(0)) * h$$

(Note that we're interested in the case where $g(x) = x^2$.)
And since

$$f(0) = f(0)*h/h$$
$$= \int_{-h/2}^{h/2} f(x)dx \ /h$$

we can now write

$$\int_{-h/2}^{h/2} g(f(x))dx = g\left(\int_{-h/2}^{h/2} f(x)dx \ /h \right) * h$$

= g(area average integral)*area

Flexural Energy (5 of 6)

Area-average Integral

Divide mean curvature integral by area of T.

$$\frac{\int_{\bar{T}} (H \circ \varphi - \bar{H}) d\bar{A}}{\mathit{Area(T)}} = \frac{(\theta - \bar{\theta})\|\bar{e}\|}{\frac{1}{2}\bar{h}_e\|\bar{e}\|} = \frac{\theta - \bar{\theta}}{\frac{1}{2}\bar{h}_e}$$

Integral Squared

Area-average integral multiplied by area (dropping all constants).

$$\int_T (H \circ \varphi - \bar{H})^2 d\bar{A} \approx \frac{(\theta - \bar{\theta})^2}{\bar{h}_e^{\,2}} \bar{h}_e\|e\| = (\theta - \bar{\theta})^2\|\bar{e}\|/\bar{h}_e$$

Discrete flexural energy over the surface

Sum over all edges

$$W_B(\mathbf{x}) = \sum_e (\theta_e - \bar{\theta}_e)^2 \|\bar{e}\| / \bar{h}_e$$

Flexural Energy (6 of 6)

- Simplification

$$W_B(\mathbf{x}) = \sum_e (\theta_e - \bar{\theta}_e)^2 \|\bar{e}\|$$

- Appealing because it doesn't depend on underlying mesh triangulization
- Produces satisfactory results
- Not as accurate
  - Does not converge to its continuous equivalent underrefinement

Implementation

- Implementation
  - Take working code for a cloth simulator (eg., Baraff)
  - Replace the bending energy

$$W_B(\mathbf{x}) = \sum_e {\theta_e}^2 \rightarrow W_B(\mathbf{x}) = \sum_e (\theta_e - \bar{\theta}_e)^2 \|\bar{e}\| / \bar{h}_e$$

- Hurdles
  - Cloth simulators generally work with flat planes
    - Doesn't work for any surface which cannot be unfolded into a flat sheet
    - Solution: Simply express the undeformed configuration in 3D coordinates

Newmark Time Stepping

- Newmark scheme

$$\begin{aligned} \mathbf{x}_{i+1} &= \mathbf{x}_i + \Delta t_i \dot{\mathbf{x}}_i + \Delta t_i^2 ((1/2 - \beta)\ddot{\mathbf{x}}_i + \beta \ddot{\mathbf{x}}_{i+1}), \\ \dot{\mathbf{x}}_{i+1} &= \dot{\mathbf{x}}_i + \Delta t_i ((1-\gamma)\ddot{\mathbf{x}}_i + \gamma \ddot{\mathbf{x}}_{i+1}), \end{aligned}$$

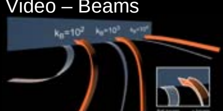
- Either an explicit (β=0) or implicit (β>1) integrator.
  - This implementation uses β=1/4
- Numerical damping parameter γ
  - This implementation minimizes damping with γ=1/2
- β and γ adjustable for increased stability/accuracy

Automatic Differentiation

- Implicit integrators require evaluating force gradients
  - Derive second derivatives of energy
  - Cumbersome, error-prone
- Automatic Differentiation
  - Convenient, automatically evaluates derivative
  - Based on observation that every computation algorithm can be written as composition of simple, easily differential steps, to which the chain rule is applied
  - Slightly slower
  - Available at: http://multires.caltech.edu/software

Results

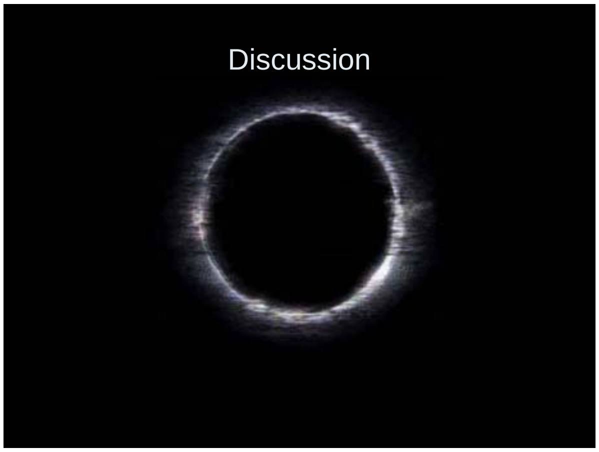
- Computation time
  - Few minutes to few hours on 2Ghz Pentium 4
- Video – Beams



- Video – Hat

Conclusion

- First work to geometrically derive a discrete model for thin shells aimed at computer animation
- Simple implementation
- Separation of membrane and bending energies
- Captures characteristic behaviors of shells
  - Flexural rigidity
  - crumpling


Discussion