

PFC/JA-82-29

**Lagrangian Formulation of Transport Theory:
Like-Particle Collisional Transport and Variational Principle**

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Abstract

A general formulation of transport theory, specifically treating like-particle collisions, is developed for non-symmetric geometries, using the Lagrangian picture. This theory provides a particularly simple method for calculating transport coefficients which is useful for devices such as stellarators and tandem mirrors. We prove that the explicit piece of the transport coefficient, processes such as test particle diffusion $\langle \Delta\psi \Delta\psi / \Delta t \rangle$, provides a rigorous upper bound to the full diagonal transport coefficient. The more complex compound, implicit processes (involving perturbed distribution functions and not accessible from a simple Monte Carlo procedure), are always inward (for normal gradients) and can only reduce the full flux from its test particle value. The implicit fluxes have an associated variational principle, maximizing this inward flux. This variational principle differs from the usual minimum principle for entropy production, although the two principles are equivalent, as is shown. An application to systems with weak asymmetry, such as stellarators, EBT, and tandem mirrors, is considered.

This work was supported by the Office of Fusion Energy, U.S. Department of Energy Contract #DE-AC02-78ET-51013.

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I. Introduction

A general feature of scattering and transport processes in plasma confinement configurations is that in each scattering event the *relative* change in quantities such as velocity or mean radial location is small. This is true even for asymmetric magnetic geometries with stochastic magnetic regions and for turbulent scattering resulting from low frequency drift wave type fluctuations. Thus a Fokker-Planck description of the scattering would be appropriate for all the processes that are likely to be present. Although some of the Fokker-Planck coefficients, such as the radial test particle diffusion coefficient, $\langle \frac{\Delta\psi\Delta r}{\Delta t} \rangle$, are fairly easily related to parts of the transport, the full transport coefficients contain contributions from several types of processes. In many cases these more complex processes can be comparable to test particle diffusion. The theory which determines the transport coefficients from the Fokker-Planck coefficients has been termed the Lagrangian formulation of transport theory ^[1] in reference to the intrinsically Lagrangian nature of Fokker-Planck coefficients. The Lagrangian theory was worked out for the Lorentz gas case in Ref. [1]. The present paper continues this development, treating like-particle collisions and formulating a variational principle, as well as considering an application.

In Section II the Lagrangian representation of the like-particle collision operator is obtained and used to develop the formal transport theory. The actual calculations are carried out for the lowest collisionality, banana, regime. The leading order operator, which neglects radial scattering, satisfies a Boltzmann type H-theorem and provides a basis for expansion about a local thermal equilibrium. In Section III, the variational principle appropriate to the Lagrangian formulation is given. This variational principle is different from that employed in the Eulerian ^[2] form of transport theory in that only part of the transport coefficient is obtained from a variational form; the rest is given explicitly. The Eulerian formulation gives the entire transport coefficient as a variational minimum form corresponding to local entropy production. A practical consequence of the Lagrangian version is that the explicit expressions provide absolute upper bounds to the full diagonal transport coefficients.

That like-particle collisions do not produce a particle flux in symmetric systems (to second order in the gyroradius) is well known. In the macroscopic, Eulerian, picture, this result is related to momentum conservation in the collision process. In Section IV, we derive an analogous condition in the Lagrangian framework for the vanishing of the particle flux. It can be stated as follows. If the average magnetic surface of a particle, J_3 , is given by a function that is *both* a constant of the orbital motion and conserved by collisions, the like-particle collisions will not produce a particle flux. Some examples of the microscopic processes which lead to this cancellation are described.

In Section V, an application is considered of neoclassical transport due to a weak asymmetry. Such transport occurs in systems such as stellarators, EBT and tandem mirrors. A previous result ^[3] for tandem mirrors which required extreme calculations can be obtained immediately.

II. Like-Particle Collision Operator and Formal Transport Theory

This section gives the derivation of the generalized or action space collision operator appropriate for the low collisionality regimes of fusion interest. The objective is to eliminate the fast collisionless processes, reducing the kinetic equation to a simple form,

$$\frac{\partial f}{\partial t} = C_J(f, f), \quad (2.1)$$

containing only the collisional time scale. The operator C_J describes collisional scattering of the collisionless constants of motion, or actions, such that two, J_1 and J_2 , are essentially velocity-like while the third, J_3 , is basically radial. In particular, we will utilize

$$\begin{aligned} J_1 &\equiv \frac{\pi}{\Omega} |\mathbf{b} \times \mathbf{v}|^2 = \text{MAGNETIC MOMENT} \\ J_2 &\equiv m \oint dsu = \text{PARALLEL INVARIANT} \\ J_3 &\equiv \frac{2\pi q}{c} \langle \psi \rangle = \text{AVERAGE MAGNETIC SURFACE}, \end{aligned} \quad (2.2)$$

where the average used to define J_3 is taken over a particle orbit and includes the drift motion. The variable ψ is the usual poloidal flux function in a tokamak, but may more generally be taken to be the Clebsch magnetic coordinate that labels magnetic surfaces. The radial variable J_3 is distinguished from the other two by its relatively slower scattering rate. A simple perturbation theory based on the smallness of radial scattering relative to velocity scattering is used to solve Eq. (2.1) and generate the transport equations for particles and energy.

For the formal development of this section, we will assume the existence of these three integral invariants and their conjugate angle variables $\theta_1, \theta_2, \theta_3$, an assumption that runs contrary to our stated motivation of developing a formulation for treating asymmetric systems and turbulence, where the invariants undoubtedly do not exist. Actually, one can anticipate that any practically useful confinement configuration will have asymptotic or quasi-invariants analogous to the asymptotic magnetic geometries in configurations where true magnetic surfaces do not exist. The perturbations of these invariants which can cause transport directly (without collisions) can be treated by a Fokker-Planck theory as a natural extension of the formulation here to be developed. The present model gives a precise framework on which to base such extensions. In some cases, as shown in the examples of Sec. V, these are quite straightforward.

We begin with the velocity space representation of the collision operator, utilizing the form

$$C(f, f) = \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3x' \int d^3v' m^3 \delta(\mathbf{x} - \mathbf{x}') \left(\mathbf{r}(\mathbf{v}, \mathbf{v}') \cdot \frac{\partial}{\partial \mathbf{v}} - Q(\mathbf{v}, \mathbf{v}') \right) f(\mathbf{x}, \mathbf{v}) f(\mathbf{x}', \mathbf{v}') \quad (2.3)$$

where the tensor $\tau(\mathbf{v}, \mathbf{v}')$, is given by

$$\tau(\mathbf{v}, \mathbf{v}') = \frac{2\pi q^4}{m^2} \Lambda \frac{I |\mathbf{v} - \mathbf{v}'|^2 - (\mathbf{v} - \mathbf{v}')(\mathbf{v} - \mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|^3}, \quad (2.4)$$

with

$$\mathbf{Q}(\mathbf{v}, \mathbf{v}') = -\frac{\partial}{\partial \mathbf{v}'} \cdot \boldsymbol{\tau}^T = \frac{\partial}{\partial \mathbf{v}} \cdot \boldsymbol{\tau}^T, \quad (2.5)$$

and Λ , the Coulomb logarithm. Equation (2.3) is essentially in Fokker-Planck form, with $\boldsymbol{\tau}$ a differential diffusion tensor, and \mathbf{Q} a differential friction vector, per unit field particle density, $m^3 d^3 v' f(\mathbf{x}', \mathbf{v}')$. The spatial position of the field particle \mathbf{x}' , has been denoted explicitly, and the delta function $\delta(\mathbf{x} - \mathbf{x}')$, inserted to represent a zero range interaction. On this scale the collision occurs when test and field particles are coincident in space.

In action-angle variables, the distribution function becomes $f(\mathbf{x}, \mathbf{v}) \rightarrow f(J, \theta)$, and the volume element is preserved, $m^3 d^3 x d^3 v = d^3 J d^3 \theta$. The angle dependence of f is rapidly mixed on the collisionless time scale (and then irreversibly destroyed by collisions), making f a function of the actions alone. In the low collisionality regime, this collisionless relaxation is the dominant process. On the slower time scales, $f = f(J)$ evolves according to Eq. (2.1) with C_J determined from Eq. (2.3) averaged over angles,

$$C_J(f, f) \equiv \int d^3 \theta C(f, f) = \frac{\partial}{\partial J} \cdot \int d^3 \theta d^3 J' \delta(\mathbf{x} - \mathbf{x}') \left[\bar{D} \cdot \frac{\partial}{\partial J} - \tilde{f} \right] f(J) f(J'), \quad (2.6)$$

where the generalized (differential) diffusion tensor is

$$\bar{D} \equiv (\nabla_{\mathbf{v} J})^T \cdot \boldsymbol{\tau} \cdot \nabla_{\mathbf{v} J}, \quad (2.7)$$

and the generalized (differential) friction vector is

$$\tilde{f} \equiv (\nabla_{\mathbf{v} J})^T \cdot \mathbf{Q}. \quad (2.8)$$

The angle average $d^3 \theta$ corresponds to a sequential average over the fast time scales determined by gyration, bounce motion and drift motion, respectively. It is convenient to write the delta function in flux coordinates, s, β, ψ , such that $\mathbf{B} = \nabla \beta \times \nabla \psi$, and s is the arc length along \mathbf{B} ,

$$\delta(\mathbf{x} - \mathbf{x}') = B \delta(s - s') \delta(\beta - \beta') \delta(\psi - \psi'). \quad (2.9)$$

The bounce phase θ_2 can be parameterized by s , such that $d\theta_2 = \omega_2 ds/u$. There is a similar relation between θ_3 and β . On the other hand, ψ is very nearly the radial action modulo $2\pi q/c$. We denote this by defining the radial orbit width ΔJ_3 , according to,

$$J_3 = \frac{2\pi q}{c} \psi + \Delta J_3. \quad (2.10)$$

By using this definition and Eq. (2.9), the J_3 integral in Eq. (2.6) can be done to give,

$$C_J(f, f) = \frac{\partial}{\partial J} \cdot \int d^2 J'_\perp d^3 \theta d^3 \theta' \frac{2\pi q B}{c} \delta(s - s') \delta(\beta - \beta') f(J'_\perp, J_3 + \Delta) \left(\tilde{D} \cdot \frac{\partial}{\partial J} - \tilde{f} \right) f(J), \quad (2.11)$$

with

$$\Delta = \Delta J'_3 - \Delta J_3. \quad (2.12)$$

Finally, defining the angle averaging operator,

$$\langle A \rangle = \int d^3 \theta d^3 \theta' \frac{2\pi q B}{c} \delta(s - s') \delta(\beta - \beta') A, \quad (2.13)$$

Eq. (2.11) becomes

$$C_J(f, f) = \frac{\partial}{\partial J} \cdot \int d^2 J'_\perp \left[\langle f(J'_\perp, J_3 + \Delta) \tilde{D} \rangle \cdot \frac{\partial}{\partial J} - \langle f(J'_\perp, J_3 + \Delta) \tilde{f} \rangle \right] f(J). \quad (2.14)$$

This is a generalized Fokker-Planck form and lends itself to a relatively simple physical interpretation. The velocity like actions J_1, J_2 (denoted by the subscript \perp) are straightforward generalizations of the perpendicular and parallel velocity components. If one neglects the J_3 variations, and accordingly deletes the coefficients $\tilde{D}_{i3}, \tilde{D}_{3j}, \tilde{f}_3$, Eq. (2.14) is similar to the familiar gyrophase averaged Fokker-Planck equation and has the usual associated H-theorem and related properties. The coefficient $\int d^2 J'_\perp \langle f' \tilde{D}_{33} \rangle$ describes the diffusion of test particle average surfaces. For example, for trapped particles in tokamaks, this is explicitly the banana center diffusion coefficient. Notice that the field particle distribution is evaluated at $J'_3 = J_3 + \Delta$, since when the collision occurs the field and test particle would not have the same average surface in general. The shift, Δ , gives the displacement of the mean surfaces, J_3 and J'_3 , when the particles collide at $\psi = \psi'$. There are also collisional cross processes of the form $\int d^2 J'_\perp \langle f' \tilde{D}_{3\perp} \rangle$ and $\int d^2 J'_\perp \langle f' \tilde{D}_{\perp 3} \rangle$ reflecting a correlation in the scattering process between velocity and radial steps. These processes are directly responsible for the neoclassical pinch and bootstrap effects in a tokamak^[4]. Finally, there is a radial flow coefficient, $\int d^2 J'_\perp \langle f' \tilde{f}_3 \rangle$, closely related to the collisional cross process, where appropriate distortions of the field particle distribution can drive radial test particle flows.

The expansion procedure for solving Eqs. (2.1) and (2.14) is based on the smallness of the radial step size to the radius, $\Delta J_3/J_3$, or equivalently, the ratio of radial-to-velocity scattering rates. This means that

the operator C_J can be ordered in $\Delta J_3/J_3$ by simply counting the number of J_3 derivatives (after expanding $f'(J_\perp', J_3 + \Delta) = f'(J_\perp, J_3) + \Delta \frac{\partial f'}{\partial J_3} + \frac{1}{2} \Delta^2 \frac{\partial^2 f'}{\partial J_3^2} + \dots$). The zero order (velocity) collision operator is then, (dropping the subscript J)

$$C_0(f, f) = \frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp \left(D \cdot \frac{\partial}{\partial J_\perp} - f \right) f f' \Big|_{J_3=J_3} \quad (2.15)$$

where

$$D(J, J) = \langle \tilde{D} \rangle, \quad (2.16)$$

$$f(J, J) = \langle \tilde{f} \rangle. \quad (2.17)$$

For the first order operator we have

$$\begin{aligned} C_1(f, f) &= \frac{\partial}{\partial J_3} \int d^2 J'_\perp \left(e_3 \cdot D \cdot \frac{\partial}{\partial J_\perp} - f_3 \right) f f' \Big|_{J_3=J_3} \\ &\quad + \frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp D \cdot e_3 f' \frac{\partial f}{\partial J_3} \Big|_{J_3=J_3} \\ &\quad + \frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp \frac{\partial f'}{\partial J_3} \langle \Delta \left(\tilde{D} \cdot \frac{\partial}{\partial J_\perp} - \tilde{f} \right) \rangle f \Big|_{J_3=J_3} \end{aligned} \quad (2.18)$$

And, finally, the second order operator is,

$$\begin{aligned} C_2(f, f) &= \frac{\partial}{\partial J_3} \int d^2 J'_\perp D_{33} f' \frac{\partial f}{\partial J_3} \Big|_{J_3=J_3} \\ &\quad + \frac{\partial}{\partial J_3} \int d^2 J'_\perp \frac{\partial f'}{\partial J_3} \langle \Delta \left(e_3 \cdot \tilde{D} \cdot \frac{\partial}{\partial J_\perp} - \tilde{f}_3 \right) \rangle f \Big|_{J_3=J_3} \\ &\quad + \frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp \frac{\partial f'}{\partial J_3} \langle \Delta \tilde{D} \cdot e_3 \rangle \frac{\partial f}{\partial J_3} \Big|_{J_3=J_3} \\ &\quad + \frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp \frac{\partial^2 f'}{\partial J_3^2} \frac{1}{2} \langle \Delta^2 \left(\tilde{D} \cdot \frac{\partial}{\partial J_\perp} - \tilde{f} \right) \rangle f \Big|_{J_3=J_3}. \end{aligned} \quad (2.19)$$

The H-theorem that is satisfied by C_0 is proved using the Landau form rather than the Fokker-Planck form of Eq. (2.15). This alternate form can be obtained directly from Eq. (2.15) by writing out the second term

$$-\frac{\partial}{\partial J_\perp} \cdot \int d^3 \theta d^3 \theta' d^3 J' \delta(J'_3 - J_3) \frac{2\pi qB}{c} \delta(s - s') \delta(\beta - \beta') f'(J_\perp', J_3)$$

$$(\nabla_{\mathbf{v}} \mathbf{J})^T \cdot (-\nabla_{\mathbf{v}} \cdot \boldsymbol{\tau}^T) f(\mathbf{J}_{\perp}, J_3).$$

Utilizing $d^3\theta'd^3J' = m^3 d^3x'd^3v'$, the derivative $\nabla_{\mathbf{v}}$ can be integrated by parts, giving

$$-\frac{\partial}{\partial J_{\perp}} \cdot \int d^3\theta d^3\theta' d^3J' \frac{2\pi qB}{c} \delta(s-s') \delta(\beta-\beta') f(\mathbf{J}_{\perp}, J_3) \left[\delta(J'_3 - J_3) (\nabla_{\mathbf{v}} \mathbf{J})^T \cdot \boldsymbol{\tau} \cdot \nabla_{\mathbf{v}} \cdot \mathbf{J}_{\perp}' \right. \\ \left. + \frac{\partial}{\partial J_{\perp}} f'(\mathbf{J}_{\perp}', J_3) + (\nabla_{\mathbf{v}} \mathbf{J})^T \cdot \boldsymbol{\tau} \cdot \nabla_{\mathbf{v}} J_3 f(\mathbf{J}_{\perp}', J_3) \frac{\partial}{\partial J_3} \delta(J'_3 - J_3) \right].$$

The last term is discarded upon integration by parts in J'_3 because the coefficients, upon averaging, are weakly J_3 dependent. Defining the frictional tensors,

$$\tilde{\mathbf{F}} = (\nabla_{\mathbf{v}} \mathbf{J})^T \cdot \boldsymbol{\tau} \cdot \nabla_{\mathbf{v}} \mathbf{J}, \quad (2.20)$$

$$\mathbf{F} = \langle \tilde{\mathbf{F}} \rangle \quad (2.21)$$

the remaining term is simply $-\partial/\partial J_{\perp} \cdot \int d^2J'_{\perp} \mathbf{F} \cdot \partial/\partial J_{\perp}' \cdot ff'$, and the zero order operator becomes,

$$C_0(f, f) = \frac{\partial}{\partial J_{\perp}} \cdot \int d^2J'_{\perp} \left(\mathbf{D} \cdot \frac{\partial}{\partial J_{\perp}} - \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{J}'_{\perp}} \right) ff' \Big|_{J'_3=J_3} \quad (2.22)$$

We now proceed by expanding f in powers of $\Delta J_3/J_3$,

$$f = f_0 + f_1 + f_2 + \dots,$$

and regarding the time derivative as second order, describing the transport time scale.

The zero order equation is

$$0 = C_0(f_0, f_0) \quad (2.23)$$

The C_0 operator satisfies a local H-theorem ^[1] which implies that the leading order distribution function, f_0 , must be a local Maxwellian

$$f_0 = n(J_3) [2\pi m T(J_3)]^{-3/2} \exp[-H_0(\mathbf{J}_{\perp})/T(J_3)] \quad (2.24)$$

where we took into account the smallness of J_3 derivative of H_0 .

The first order equation is now

$$C_o(f_1, f_o) + C_o(f_o, f_1) = -C_1(f_o, f_o). \quad (2.25)$$

It is convenient to write

$$f_1 = f_o \hat{f}_1,$$

so that the linearized operator on the left side of Eq. (2.25) becomes

$$C_l(\hat{f}_1) \equiv C_o(\hat{f}_1 f_o, f_o) + C_o(f_o, \hat{f}_1 f_o) \\ \frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp f_o f'_o \left(D \cdot \frac{\partial}{\partial J_\perp} \hat{f}_1 - F \cdot \frac{\partial}{\partial J_\perp} \hat{f}_1 \right) \quad (2.26)$$

which is in manifestly self-adjoint form. In obtaining Eq. (2.26) we have used the property

$$D \cdot \omega_\perp = F \cdot \omega_\perp, \quad (2.27)$$

which, in turn, is a consequence of $\tau \cdot (v - v') = (v - v') \cdot \tau = 0$ as follows from Eq. (2.4). Note also that

$$\omega_\perp = \frac{\partial}{\partial J_\perp} H_o \quad (2.28)$$

and

$$\frac{\partial}{\partial J_\perp} f_o = -\omega_\perp \frac{f_o}{T} \quad (2.29)$$

For manipulating the first order equation and formalizing the transport theory, it is convenient to define

$$\frac{\partial f_o}{\partial J_3} = f_o \left[A_1 + A_2 \left(\frac{H}{T} - \frac{3}{2} \right) \right] \quad (2.30)$$

where A_1 and A_2 are the thermodynamic forces or the density and temperature gradients $\frac{d \ln n}{d J_3}$, $\frac{d \ln T}{d J_3}$, respectively.

It is also convenient to express the first order operator in Landau form. Consider then, transforming the quantity

$$\int d^2 J'_\perp \langle A \cdot \left(\tilde{D} \cdot \frac{\partial}{\partial J_\perp} - \tilde{f} \right) f \rangle_{J_3=J_3}$$

where A is $e_3 f'$ and $e_\perp \frac{\partial f'}{\partial J_3} \Delta$ respectively for the first and the third item of Eq. (2.18). Utilizing $d^3 \theta' d^3 J' = m^3 d^3 x' d^3 v'$ and Eqs. (2.5), (2.8) and (2.13), we have:

$$\int d^2 J'_\perp \langle A \cdot \left(\tilde{\mathbf{D}} \cdot \frac{\partial}{\partial \mathbf{J}_\perp} - \tilde{f} \right) \rangle f |_{J'_3=J_3} = \int d^3 \theta \int d^3 \mathbf{x}' \int d^3 v' m^3 \frac{2\pi q B}{c} A \delta(J_3 - J'_3) \delta(s - s') \delta(\beta - \beta') \cdot \left[\tilde{\mathbf{D}} \cdot \frac{\partial f}{\partial \mathbf{J}_\perp} + f (\nabla_{\mathbf{v}} \mathbf{J})^T \nabla_{\mathbf{v}} \cdot \boldsymbol{\tau}^T \right] \quad (2.31)$$

Partial integration of the derivative $\nabla_{\mathbf{v}}$ gives:

$$\int d^2 J'_\perp \langle A \cdot \left(\tilde{\mathbf{D}} \cdot \frac{\partial}{\partial \mathbf{J}_\perp} - \tilde{f} \right) \rangle f |_{J'_3=J_3} = \int d^3 \theta \int d^3 \mathbf{x}' \int d^3 v' m^3 \frac{2\pi q B}{c} \delta(\beta - \beta') \delta(s - s') \left\{ -f A(J'_3 = J_3) \cdot \tilde{\mathbf{F}} \cdot \mathbf{e}_3 \frac{\partial}{\partial J'_3} \delta(J_3 - J'_3) + \delta(J_3 - J'_3) \left[A \cdot \tilde{\mathbf{D}} \cdot \frac{\partial f}{\partial \mathbf{J}_\perp} - f \tilde{\mathbf{F}} \cdot \frac{\partial}{\partial \mathbf{J}_\perp} A \right] \right\}. \quad (2.32)$$

The first term inside the curly brackets is discarded as previously [Eq. (2.22)]. Applying Eq. (2.32) for $A = \mathbf{e}_3 f'$ yields:

$$\int d^2 J'_\perp f' \left(\mathbf{e}_3 \cdot \mathbf{D} \cdot \frac{\partial}{\partial \mathbf{J}_\perp} - f_3 \right) f |_{J'_3=J_3} = \int d^2 J'_\perp \mathbf{e}_3 \cdot \left(f' \mathbf{D} \cdot \frac{\partial f}{\partial \mathbf{J}_\perp} - f \mathbf{F} \cdot \frac{\partial f'}{\partial \mathbf{J}_\perp} \right) |_{J'_3=J_3} \quad (2.33)$$

and for $A = \mathbf{e}_\perp \frac{\partial f'}{\partial J'_3} \Delta$

$$\int d^2 J'_\perp \frac{\partial f'}{\partial J'_3} \langle \Delta \mathbf{e}_\perp \cdot \left(\tilde{\mathbf{D}} \cdot \frac{\partial}{\partial \mathbf{J}_\perp} - \tilde{f} \right) \rangle f |_{J'_3=J_3} = \int d^2 J'_\perp \left(\frac{\partial f'}{\partial J'_3} \langle \Delta \mathbf{e}_\perp \cdot \tilde{\mathbf{D}} \rangle \cdot \frac{\partial f}{\partial \mathbf{J}_\perp} - f \langle \Delta \mathbf{e}_\perp \cdot \tilde{\mathbf{F}} \rangle \cdot \frac{\partial}{\partial \mathbf{J}_\perp} \frac{\partial f'}{\partial J'_3} \right) |_{J'_3=J_3} - \int d^3 \theta \int d^3 \theta' \int d^3 J' \frac{2\pi q B}{c} \delta(\beta - \beta') \delta(s - s') \delta(J_3 - J'_3) f (\nabla_{\mathbf{v}} \mathbf{J}_\perp)^T \cdot \boldsymbol{\tau} \cdot (\nabla_{\mathbf{v}} \Delta) \frac{\partial f'}{\partial J'_3}$$

Noting that $\nabla_{\mathbf{v}} \Delta \frac{\partial f'}{\partial J'_3} = \nabla_{\mathbf{v}} J'_3 \frac{\partial f'}{\partial J'_3} = \nabla_{\mathbf{v}} \mathbf{J} \cdot \mathbf{e}_3 \frac{\partial f'}{\partial J'_3}$ we can rewrite the last expression as follows

$$\int d^2 J'_\perp \frac{\partial f'}{\partial J'_3} \langle \Delta \mathbf{e}_\perp \cdot \left(\tilde{\mathbf{D}} \cdot \frac{\partial}{\partial \mathbf{J}_\perp} - \tilde{f} \right) \rangle f |_{J'_3=J_3} = \int d^2 J'_\perp \left(\frac{\partial f'}{\partial J'_3} \langle \Delta \mathbf{e}_\perp \cdot \tilde{\mathbf{D}} \rangle \cdot \frac{\partial}{\partial \mathbf{J}_\perp} - f \langle \Delta \mathbf{e}_\perp \cdot \tilde{\mathbf{F}} \rangle \cdot \frac{\partial}{\partial \mathbf{J}_\perp} \frac{\partial f'}{\partial J'_3} \right) |_{J'_3=J_3} - \int d^2 J'_\perp f \mathbf{F} \cdot \mathbf{e}_3 \frac{\partial f'}{\partial J'_3} |_{J'_3=J_3} \quad (2.34)$$

Utilizing the results of Eqs. (2.33) and (2.34), we can rewrite the first order equation as follows:

$$\begin{aligned}
C_\ell(\hat{f}_1) = & -\frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp \left(\mathbf{D} \cdot \mathbf{e}_3 \frac{\partial}{\partial J_3} - \mathbf{F} \cdot \mathbf{e}_3 \frac{\partial}{\partial J'_3} \right) f_o f'_o \Big|_{J'_3=J_3} \\
& -\frac{\partial}{\partial J_3} \int d^2 J'_\perp \left(\mathbf{e}_3 \cdot \mathbf{D} \cdot \frac{\partial}{\partial J_\perp} - \mathbf{e}_3 \cdot \mathbf{F} \cdot \frac{\partial}{\partial J'_\perp} \right) f_o f'_o \Big|_{J'_3=J_3} \\
& -\frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp \left(\frac{\partial f'_o}{\partial J'_3} \langle \Delta \bar{\mathbf{D}} \rangle \cdot \frac{\partial f_o}{\partial J_\perp} - f_o \langle \Delta \bar{\mathbf{F}} \rangle \cdot \frac{\partial}{\partial J_\perp} \frac{\partial f'_o}{\partial J'_3} \right) \Big|_{J'_3=J_3}
\end{aligned}$$

or, since the second term vanishes by virtue of Eqs. (2.27) - (2.29), we have finally utilized Eq. (2.30):

$$\begin{aligned}
C_\ell(\hat{f}_1) = & -A_1 \frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp f_o f'_o (\mathbf{D} \cdot \mathbf{e}_3 - \mathbf{F} \cdot \mathbf{e}_3) \\
& -A_2 \frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) \mathbf{D} \cdot \mathbf{e}_3 - \left(\frac{H'}{T} - \frac{3}{2} \right) \mathbf{F} \cdot \mathbf{e}_3 \right] \\
& +A_2 \frac{\partial}{\partial J_\perp} \cdot \int d^2 J'_\perp \frac{f_o f'_o}{T} \langle \Delta \bar{\mathbf{F}} \cdot \boldsymbol{\omega}_\perp \rangle
\end{aligned} \tag{2.35}$$

Here and in the following we drop the indication, $J'_3 = J_3$, with the understanding that both primed and unprimed distributions are to be evaluated at J_3 .

Inversion of the collision operator C_ℓ determines \hat{f}_1 subject to certain integrability conditions determined by the annihilators of C_ℓ , namely the particle and energy moments. It is evident that the particle moment, $\int d^2 J_\perp$, also annihilates the source terms in Eq. (2.35). We then compute the energy moment $\int d^2 J_\perp H(\mathbf{J})$, of the source term; integrating by parts

$$\begin{aligned}
\int d^2 J_\perp H C_\ell(\hat{f}_1) = 0 = & A_1 \int d^2 J_\perp \int d^2 J'_\perp f_o f'_o (\boldsymbol{\omega}_\perp \cdot \mathbf{D} \cdot \mathbf{e}_3 - \boldsymbol{\omega}_\perp \cdot \mathbf{F} \cdot \mathbf{e}_3) \\
& +A_2 \int d^2 J_\perp \int d^2 J'_\perp f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) \boldsymbol{\omega}_\perp \cdot \mathbf{D} \cdot \mathbf{e}_3 - \left(\frac{H'}{T} - \frac{3}{2} \right) \boldsymbol{\omega}_\perp \cdot \mathbf{F} \cdot \mathbf{e}_3 \right] \\
& -A_2 \int d^2 J_\perp \int d^2 J'_\perp \frac{f_o f'_o}{T} \langle \Delta \boldsymbol{\omega}_\perp \cdot \bar{\mathbf{F}} \cdot \boldsymbol{\omega}_\perp \rangle
\end{aligned}$$

The variables $\mathbf{J}_\perp, \mathbf{J}'_\perp$, are interchanged in parts of the last expression yielding

$$\begin{aligned}
\int d^2 J_\perp H C_\ell(\hat{f}_1) = 0 = & A_1 \int d^2 J_\perp \int d^2 J'_\perp f_o f'_o (\boldsymbol{\omega}_\perp \cdot \mathbf{D} \cdot \mathbf{e}_3 - \boldsymbol{\omega}'_\perp \cdot \mathbf{F}^T \cdot \mathbf{e}_3) \\
& +A_2 \int d^2 J_\perp \int d^2 J'_\perp f_o f'_o \left(\frac{H}{T} - \frac{3}{2} \right) (\boldsymbol{\omega}_\perp \cdot \mathbf{D} \cdot \mathbf{e}_3 - \boldsymbol{\omega}'_\perp \cdot \mathbf{F}^T \cdot \mathbf{e}_3) \\
& -A_2 \int d^2 J_\perp \int d^2 J'_\perp f_o f'_o \langle (\boldsymbol{\omega}_\perp \cdot \bar{\mathbf{F}} \cdot \boldsymbol{\omega}'_\perp - \boldsymbol{\omega}'_\perp \cdot \bar{\mathbf{F}}^T \cdot \boldsymbol{\omega}_\perp) \Delta J'_3 \rangle
\end{aligned} \tag{2.36}$$

where we have used

$$H'(J_{\perp} \rightarrow J_{\perp}, J'_3 = J_3) = H \quad (2.37)$$

$$\tilde{F}(J_{\perp} \rightarrow J_{\perp}, J'_3 = J_3) = \tilde{F}^T \quad (2.38)$$

$$\Delta J_3(J_{\perp} \rightarrow J_{\perp}, J_3 = J'_3) = \Delta J'_3 \quad (2.39)$$

Furthermore, using the transposition of the identity given by Eq. (2.28)

$$\omega_{\perp} \cdot \tilde{D}^T = \omega_{\perp} \cdot \tilde{D} = \omega_{\perp} \cdot \tilde{F}^T \quad (2.40)$$

it becomes evident that the right side of Eq. (2.36) is identically zero; therefore, the second integrability condition is automatically satisfied.

Continuing with the formal transport theory, we write the first order equation as:

$$C_i(\tilde{f}_1) = \alpha_1 A_1 + \alpha_2 A_2 \quad (2.41)$$

where the driving functions, α_i are given by:

$$\begin{aligned} \alpha_1 &= -\frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} f_0 f'_0 (D \cdot e_3 - F \cdot e_3) |_{J'_3=J_3} \\ &= -\frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} f'_0 \left\{ \langle \tilde{D} \cdot (e_3 - \Delta \omega_{\perp}/T) \rangle - \langle \tilde{f} \Delta \rangle \right\} f_0 |_{J'_3=J_3} \end{aligned} \quad (2.42)$$

and

$$\begin{aligned} \alpha_2 &= -\frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} f_0 f'_0 \left[\left(\frac{H}{T} - \frac{3}{2} \right) D \cdot e_3 - \left(\frac{H'}{T} - \frac{3}{2} \right) F \cdot e_3 \right] |_{J'_3=J_3} \\ &\quad - \frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} \frac{f_0 f'_0}{T} \langle \Delta \tilde{F} \cdot \omega_{\perp} \rangle |_{J'_3=J_3} \\ &= -\frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} f'_0 \left\{ \langle \tilde{D} \cdot \left[\left(\frac{H}{T} - \frac{3}{2} \right) e_3 - \left(\frac{H'}{T} - \frac{3}{2} \right) \Delta \omega_{\perp}/T \right] \rangle \right. \\ &\quad \left. - \langle \tilde{f} \left(\frac{H'}{T} - \frac{3}{2} \right) \Delta \rangle \right\} f_0 |_{J'_3=J_3} \end{aligned} \quad (2.43)$$

Finally, defining the individual responses g_i according to:

$$C_\ell(g_i) = a_i, \quad (2.44)$$

f_1 can now be expressed as a sum over the thermodynamic forces,

$$f_1 = f_o \sum_i A_i g_i \quad (2.45)$$

In principle the function g_i can be determined by inverting the operator C_ℓ to give the first order distribution. Since C_ℓ is self-adjoint, the inversion can be accomplished approximately through a variational principle which gives the implicit transport coefficient directly.

The second order equation is:

$$\begin{aligned} \frac{\partial f_o}{\partial t} - C_2(f_o, f_o) - C_1(f_o, f_1) - C_1(f_1, f_o) - C_o(f_1, f_1) \\ = C_o(f_o, f_2) - C_o(f_2, f_o) \\ \equiv C_\ell(\hat{f}_2) \end{aligned} \quad (2.46)$$

where $\hat{f}_2 = f_2/f_o$ and the sum $C_o(f_o, f_2) + C_o(f_2, f_o) \equiv C_\ell(\hat{f}_2)$ is the self adjoint operator defined in Eq. (2.26).

The integrability conditions for the solution of \hat{f}_2 now provide the transport equations. Note that the transport equations require knowledge of f only up to the first order since $C_\ell(\hat{f}_2)$ is annihilated by the particle and the energy moments.

We use the Landau form for $C_1(f_o, f_1) + C_1(f_1, f_o)$, utilizing Eq. (2.19) in taking the particle moment of the second order equation

$$\frac{\partial n_3}{\partial t} + \frac{\partial \Gamma}{\partial J_3} = 0 \quad (2.47)$$

where,

$$\Gamma = \Gamma^e + \Gamma^i \quad (2.48)$$

with the implicit flux Γ^i given by,

$$\Gamma^i = - \int d^2 J_\perp \int d^2 J'_\perp \left(e_3 \cdot D \cdot \frac{\partial}{\partial J_\perp} - e_3 \cdot F \cdot \frac{\partial}{\partial J_\perp} \right) (f_o f'_1 + f_1 f'_o) \quad (2.49)$$

and the explicit flux Γ^e by,

$$\begin{aligned}\Gamma^e &= - \int d^2 J_{\perp} \int d^2 J'_{\perp} D_{33} f'_o \frac{\partial f_o}{\partial J_3} \\ &\quad - \int d^2 J_{\perp} \int d^2 J'_{\perp} \frac{\partial f'_o}{\partial J'_3} \langle \Delta \left(\mathbf{e}_3 \cdot \tilde{\mathbf{D}} \cdot \frac{\partial}{\partial J_{\perp}} - \tilde{f}_3 \right) \rangle f_o |_{J'_3=J_3}\end{aligned}\quad (2.50)$$

Alternately employing Eq. (2.34) once more to reexpress the second term of Γ^e in a Landau form

$$\begin{aligned}\Gamma^e &= - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o \frac{\partial f'_o}{\partial J'_3} (D_{33} - F_{33}) \\ &\quad - \int d^2 J_{\perp} \int d^2 J'_{\perp} \left(\frac{\partial f'_o}{\partial J'_3} \langle \Delta \mathbf{e}_3 \cdot \tilde{\mathbf{D}} \rangle \cdot \frac{\partial f_o}{\partial J_{\perp}} - f_o \langle \Delta \mathbf{e}_3 \cdot \tilde{\mathbf{F}} \rangle \cdot \frac{\partial}{\partial J_{\perp}} \frac{\partial f'_o}{\partial J'_3} \right) |_{J'_3=J_3}\end{aligned}\quad (2.51)$$

Employing Eq. (2.30), Eq. (2.50) can be written in Fokker-Planck form as

$$\begin{aligned}\Gamma^e &= - A_1 \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o (D_{33} - \langle \Delta \tilde{f}_3 \rangle - \langle \Delta \mathbf{e}_3 \cdot \tilde{\mathbf{F}} \cdot \boldsymbol{\omega}_{\perp} \rangle / T) \\ &\quad - A_2 \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) D_{33} \right. \\ &\quad \left. - \left(\frac{H'}{T} - \frac{3}{2} \right) \langle \Delta \tilde{f}_3 \rangle - \left(\frac{H'}{T} - \frac{3}{2} \right) \langle \Delta \mathbf{e}_3 \cdot \tilde{\mathbf{F}} \cdot \boldsymbol{\omega}_{\perp} \rangle / T \right]\end{aligned}\quad (2.52)$$

or in Landau form

$$\begin{aligned}\Gamma^e &= - A_1 \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o (D_{33} - F_{33}) \\ &\quad - A_2 \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) D_{33} - \left(\frac{H'}{T} - \frac{3}{2} \right) F_{33} \right] \\ &\quad + A_2 \int d^2 J_{\perp} \int d^2 J'_{\perp} \frac{f_o f'_o}{T} \langle \Delta \mathbf{e}_3 \cdot \tilde{\mathbf{F}} \cdot \boldsymbol{\omega}_{\perp} \rangle\end{aligned}\quad (2.53)$$

The explicit flux Γ^e can also be expressed in terms of the explicit transport coefficients T_{11}^e and T_{12}^e as follows

$$\Gamma^e = T_{11}^e A_1 + T_{12}^e A_2 \quad (2.54)$$

with,

$$\begin{aligned}
T_{12}^e &\equiv - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left(D_{33} - \langle \Delta(e_3 \cdot \tilde{F} \cdot \omega_{\perp} / T + \tilde{f}_3) \rangle \right) \\
&= - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o (D_{33} - F_{33})
\end{aligned} \tag{2.55}$$

and

$$\begin{aligned}
T_{12}^e &\equiv - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) D_{33} - \left(\frac{H'}{T} - \frac{3}{2} \right) \langle \Delta(e_3 \cdot \tilde{F} \cdot \omega_{\perp} / T + \tilde{f}_3) \rangle \right] \\
&= - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) D_{33} - \left(\frac{H'}{T} - \frac{3}{2} \right) F_{33} - \langle \Delta e_3 \cdot \tilde{F} \cdot \omega_{\perp} / T \rangle \right]
\end{aligned} \tag{2.56}$$

The implicit flux Γ^i as in Eq. (2.49) can be written, interchanging J_{\perp} and J'_{\perp} in the second part,

$$\Gamma^i = - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left(e_3 \cdot D \cdot \frac{\partial \hat{f}_1}{\partial J_{\perp}} - e_3 \cdot F^T \cdot \frac{\partial \hat{f}_1}{\partial J'_{\perp}} \right),$$

partially integrating,

$$\Gamma^i = \int d^2 J_{\perp} \hat{f}_1 \frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} f_o f'_o (D \cdot e_3 - F \cdot e_3),$$

and using Eq. (2.42)

$$\Gamma^i = - \int d^2 J_{\perp} \hat{f}_1 \alpha_1 \tag{2.57}$$

Formally, this becomes

$$\Gamma^i = T_{11}^i A_1 + T_{12}^i A_2 \tag{2.58}$$

with, implicit transport coefficients,

$$T_{11}^i = -(\alpha_1, g_2), \tag{2.59}$$

and

$$T_{12}^i = -(\alpha_1, g_2), \tag{2.60}$$

where we used the inner product notation, $(s, h) \equiv \int d^2 J_{\perp} s h$ to write the coefficients.

We proceed now, taking the energy moment of the second order equation

$$\begin{aligned} \int d^2 J_{\perp} H \frac{\partial f_o}{\partial t} - \int d^2 J_{\perp} H C_2(f_o, f_o) - \int d^2 J_{\perp} H [C_1(f_o, f_1) + C_1(f_1, f_o)] \\ - \int d^2 J_{\perp} H C_o(f_1, f_1) = 0 \end{aligned} \quad (2.61)$$

The second contributing term can now be explicitly written in Landau form as,

$$\begin{aligned} - \int d^2 J_{\perp} H C_2(f_o, f_o) = & - \frac{\partial}{\partial J_3} \int d^2 J_{\perp} \int d^2 J'_{\perp} H \left(D_{33} \frac{\partial}{\partial J_3} - \frac{\partial}{\partial J'_3} \right) f_o f'_o \\ & - \frac{\partial}{\partial J_3} \int d^2 J_{\perp} \int d^2 J'_{\perp} H \left[\frac{\partial f'_o}{\partial J'_3} \langle \Delta e_3 \cdot \tilde{D} \rangle \cdot \frac{\partial f_o}{\partial J'_3} - f_o \langle e_3 \cdot \tilde{F} \rangle \cdot \frac{\partial}{\partial J_{\perp}'} \frac{\partial f'_o}{\partial J_3} \right] \\ & - \int d^2 J_{\perp} H \frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} \left[\frac{\partial f'_o}{\partial J'_3} \langle \Delta \tilde{D} \cdot e_3 \rangle \frac{\partial f_o}{\partial J_3} - f_o \langle \Delta \tilde{F} \cdot e_3 \rangle \frac{\partial^2 f'_o}{\partial J_3^2} \right] \\ & - \int d^2 J_{\perp} H \frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} \left[\frac{\partial^2 f'_o}{\partial J_3^2} \langle \frac{\Delta^2}{2} \tilde{D} \rangle \cdot \frac{\partial f_o}{\partial J_{\perp}} - f_o \langle \frac{\Delta^2}{2} \tilde{F} \rangle \cdot \frac{\partial}{\partial J_{\perp}'} \frac{\partial^2 f'_o}{\partial J_3^2} \right] \end{aligned} \quad (2.62)$$

The last two terms are partially integrated, the variables J_{\perp} and J'_{\perp} are interchanged in their second parts and also the weak dependence of the quantity Δ , and so of the tensors D and F , is assumed. After employing the relations given by Eqs. (2.29) and (2.30), this leads to,

$$\begin{aligned} - \int d^2 J_{\perp} H C_2(f_o, f_o) = & - \frac{\partial}{\partial J_3} T A_1 \left\{ \int d^2 J_{\perp} \int d^2 J'_{\perp} \frac{H}{T} f_o f'_o (D_{33} - F_{33}) \right. \\ & \left. \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \langle \Delta \frac{\omega_{\perp}}{T} \cdot \tilde{D} \cdot e_3 \rangle \right\} \\ & - \frac{\partial}{\partial J_3} T A_2 \left\{ \int d^2 J_{\perp} \int d^2 J'_{\perp} \frac{H}{T} f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) D_{33} - \left(\frac{H'}{T} - \frac{3}{2} \right) F_{33} \right] \right. \\ & - \int d^2 J_{\perp} \int d^2 J'_{\perp} \frac{H}{T} f_o f'_o \langle \Delta e_3 \cdot \tilde{F} \cdot \omega_{\perp} / T \rangle \\ & - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left(\frac{H}{T} - \frac{3}{2} \right) \langle \Delta \frac{\omega_{\perp}}{T} \cdot \tilde{D} \cdot e_3 \rangle \\ & \left. + \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \langle \frac{\Delta^2}{2} \frac{\omega_{\perp}}{T} \cdot \tilde{D} \cdot \frac{\omega_{\perp}}{T} \rangle \right\} \end{aligned} \quad (2.63)$$

The third term of the energy moment equation contributes to the implicit fluxes and it can be written in Landau form as,

$$\begin{aligned}
-\int d^2J_{\perp}H[C_1(f_o, f_1) + C_1(f_1, f_o)] &= -\frac{\partial}{\partial J_3} \int d^2J_{\perp} \int d^2J'_{\perp} H e \cdot \left(D \cdot \frac{\partial}{\partial J_{\perp}} - F \cdot \frac{\partial}{\partial J'_{\perp}} \right) (f_o f'_1 + f_1 f'_o) \\
&\quad - \int d^2J_{\perp} H \frac{\partial}{\partial J_{\perp}} \cdot \int d^2J'_{\perp} \left(D \cdot e_3 \frac{\partial}{\partial J_3} - F \cdot e_3 \frac{\partial}{\partial J'_3} \right) (f_o f'_1 + f_1 f'_o) \\
&\quad - \int d^2J_{\perp} H \frac{\partial}{\partial J_{\perp}} \cdot \int d^2J'_{\perp} \langle \Delta \left(\tilde{D} \cdot \frac{\partial}{\partial J_{\perp}} - \tilde{F} \cdot \frac{\partial}{\partial J'_{\perp}} \right) \rangle \\
&\quad \quad \left(f_o \frac{\partial f'_1}{\partial J'_3} + f_1 \frac{\partial f'_o}{\partial J'_3} \right) \tag{2.64}
\end{aligned}$$

Following the same procedure as for the previous contributing term yields,

$$\begin{aligned}
-\int d^2J_{\perp}H[C_1(f_o, f_1) + C_1(f_1, f_o)] &= \frac{\partial}{\partial J_3} T \int d^2J_{\perp} \hat{f}_1 \frac{\partial}{\partial J_1} \cdot \int d^2J'_{\perp} f_o f'_o \left(\frac{H}{T} D \cdot e_3 - \frac{H'}{T} F \cdot e_3 \right) \\
&\quad - \frac{\partial}{\partial J_3} T \int d^2J_{\perp} \hat{f}_1 \frac{\partial}{\partial J_{\perp}} \cdot \int d^2J'_{\perp} f_o f'_o \langle \Delta \tilde{F} \cdot \omega_{\perp} / T \rangle \tag{2.65}
\end{aligned}$$

The same procedure applied on the third term leads to a zero contribution since $\omega_{\perp} \cdot D = \omega_{\perp} \cdot F^T$; therefore,

$$-\int d^2J_{\perp} H C_o(f_1, f_1) = 0 \tag{2.66}$$

Then multiplying the continuity equation by $\frac{3}{2}T$ and subtracting it from the energy equation yields,

$$\frac{\partial}{\partial t} \frac{3}{2} n_3 T + \frac{\partial}{\partial J_3} \left(\frac{3}{2} \Gamma T + q \right) = 0 \tag{2.67}$$

where q is the heat flux given in terms of the thermodynamic forces and the transport coefficients as follows,

$$\frac{q}{T} = T_{21} A_1 + T_{22} A_2 \tag{2.68}$$

The transport coefficients are decomposed into their implicit and explicit parts, as

$$T_{21} = T_{21}^e + T_{21}^i \tag{2.69}$$

$$T_{22} = T_{22}^e + T_{22}^i \tag{2.70}$$

The explicit parts are given by,

$$\begin{aligned}
T_{21}^e = & - \int d^3 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) D_{33} - \left(\frac{H'}{T} - \frac{3}{2} \right) F_{33} \right]_{|J'_3=J_3} \\
& + \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \langle \Delta \frac{\omega_{\perp}}{T} \tilde{\mathbf{D}} \cdot \mathbf{e}_3 \rangle_{|J'_3=J_3}
\end{aligned} \tag{2.71}$$

which is exactly the same as the particle transport coefficient T_{12}^e , and

$$\begin{aligned}
T_{22}^e = & - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left(\frac{H}{T} - \frac{3}{2} \right) \left[\left(\frac{H}{T} - \frac{3}{2} \right) D_{33} - \left(\frac{H'}{T} - \frac{3}{2} \right) F_{33} \right]_{|J'_3=J_3} \\
& + 2 \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left(\frac{H}{T} - \frac{3}{2} \right) \langle \Delta \mathbf{e}_3 \cdot \tilde{\mathbf{F}} \cdot \boldsymbol{\omega}_{\perp} / T \rangle_{|J'_3=J_3} \\
& - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \langle \frac{\Delta^2 \omega_{\perp}}{2} \frac{\omega_{\perp}}{T} \cdot \tilde{\mathbf{D}} \cdot \frac{\omega_{\perp}}{T} \rangle_{|J'_3=J_3}
\end{aligned} \tag{2.72}$$

The explicit heat conduction coefficient T_{22}^e can also be expressed in Fokker-Planck form, as

$$\begin{aligned}
T_{22}^e = & - \int d^2 J_{\perp} \int d^2 J'_{\perp} \left(\frac{H}{T} - \frac{3}{2} \right) \left[\left(\frac{H}{T} - \frac{3}{2} \right) D_{33} - \langle \Delta \left(\mathbf{e}_3 \cdot \tilde{\mathbf{F}} \cdot \boldsymbol{\omega}_{\perp} / T + \tilde{f}_3 \right) \rangle \right]_{|J'_3=J_3} \\
& - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \langle \frac{\Delta^2 \omega_{\perp}}{2} \frac{\omega_{\perp}}{T} \cdot \tilde{\mathbf{D}} \cdot \frac{\omega_{\perp}}{T} \rangle_{|J'_3=J_3}
\end{aligned} \tag{2.73}$$

The implicit transport coefficients are given by,

$$T_{21}^i = -(\alpha_2, g_1) = -(\alpha_1, g_2) = T_{12}^i \tag{2.74}$$

and,

$$T_{22}^i = -(\alpha_2, g_2) \tag{2.75}$$

where the inner product notation has been used.

III. Variational Principle, Entropy Production and Explicit Bounds for Transport Coefficients

A large part of the transport problem, that involving the explicit fluxes, has been reduced to quadrature in Section II. The remaining, or implicit, contributions to transport are fundamentally more difficult to compute since they require the inversion of an integrodifferential collision operator, eq. (2.26). In practice, since the linearized collision operator is self-adjoint, there exists a variational procedure that can be used for approximate evaluations of the implicit flux. This variational principle is different from that employed in the Eulerian formulation of transport theory and there is a potentially useful practical consequence that stems from this difference. The Eulerian variational procedure gives the entire transport coefficient as a variational minimum form corresponding to local entropy production.^[2] In the Lagrangian formulation, given here, the variational procedure applies to the implicit piece alone. It is actually maximum principle for a flux which is manifestly *parallel* to the radial density and temperature gradients (inward for normal profiles), in contrast to the explicit and total fluxes which, of course, are always antiparallel to the gradient. Thus, maximizing the implicit inflow corresponds to minimizing the total flux and the local entropy production. The explicit expressions provide absolute upper bounds to the full diagonal fluxes (diffusion coefficient and thermal conductivity). The present section is concerned with developing these results.

We consider first the implicit problem which gives the fluxes as inner products

$$T_{ij}^i = -(\alpha_i, g_j), \quad (3.1)$$

with the first order distribution function, g_j , given by equation (2.44). Note that the standard sign convention for the transport coefficients, eqs. (2.54) and (2.68), are employed so that a positive coefficient corresponds to a flux parallel to the gradient. A variational form for the functions, \hat{g}_i, \hat{g}_j , which reduces to T_{ij}^i when $\hat{g}_i = g_i$ and $\hat{g}_j = g_j$, is

$$G_{ij} = -(\alpha_i, \hat{g}_j) - (\alpha_j, \hat{g}_i) + (\hat{g}_i, C_l(\hat{g}_j)). \quad (3.2)$$

For the diagonal component, G_{ii} , in eq. (3.2) let $\hat{g}_i = g_i + \delta g_i$. There results,

$$\begin{aligned} G_{ii} &= -2(\alpha_i, g_i) - (g_i, C_l(g_i)) - 2(\alpha_i, \delta g_i) \\ &\quad + (\delta g_i, C_l(g_i)) + (g_i, C_l(\delta g_i)) + (\delta g_i, C_l(\delta g_i)) \\ &= -(\alpha_i, g_i) + 2(-\alpha_i + C_l(g_i), \delta g_i) + (\delta g_i, C_l(\delta g_i)), \end{aligned} \quad (3.3)$$

where we have used the self-adjointness of C_l on the two dimensional action space $\underline{J}_\perp = (J_1, J_2)$ in obtaining this last expression. Requiring that the first variation of G_{ii} be zero,

$$\delta G_{ii} = 0 = 2(-\alpha_i + C_i(g_i), \delta g_i), \quad (3.4)$$

implies eq. (2.44), since eq. (3.4) must be satisfied for arbitrary δg_i . Furthermore, since the operator C_i is negative definite (this was shown for the specific Lagrangian operator in Ref. [1], but is essentially the same demonstration that is made in proving the conventional H-theorem), it follows that the second variation of G_{ii} is negative,

$$\delta^2 G_{ii} = (\delta g_i, C_i(\delta g_i)) < 0, \quad (3.5)$$

so that the extremum of G_{ii} is a maximum. Finally, the negativity of C_i implies that T_{ii}^i is positive, the diagonal implicit fluxes run up the applied gradient. The overall diagonal coefficients must be negative, opposing the applied gradients (to assure positivity of the local entropy production, as shown below). This establishes the results stated above.

The off-diagonal coefficients cannot be computed from a single variation, since two trial functions, \hat{g}_i and \hat{g}_j , are required to evaluate eq. (3.2). However, when variational expressions for \hat{g}_i and \hat{g}_j are obtained from the diagonal problems, eq. (3.2) provides a variationally correct expression for T_{ij}^i . It is not, however, a minimum or a maximum principle. The explicit expression is not necessarily a bound for the overall coefficient.

This variational procedure can be interpreted in terms of entropy production for a system held away from thermal equilibrium by the thermodynamic forces A_i . The entropy for a system where the distribution function depends on the actions alone is given by,

$$S \equiv - \int d^3 J f \ln f, \quad (3.6)$$

so that with f evolving according to eq. (2.1) the rate of change of entropy is

$$\frac{dS}{dt} = - \int d^3 J \frac{\partial f}{\partial t} (1 + \ln f) = - \int d^3 J (1 + \ln f) C_J(f, f). \quad (3.7)$$

We first show that this quantity is positive definite. Using eq. (2.6) for the collision operator, eq. (3.7) can be written,

$$\frac{dS}{dt} = - \int d^3 J d^3 \theta d^3 J' d^3 \theta' \delta(\mathbf{x} - \mathbf{x}') (1 + \ln f) \frac{\partial}{\partial \mathbf{v}} \cdot \boldsymbol{\tau} \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'} \right) f f', \quad (3.8)$$

or, since $\boldsymbol{\tau}(\mathbf{v}, \mathbf{v}')$ is an invariant under interchange of \mathbf{v} and \mathbf{v}' ,

$$\begin{aligned} \frac{dS}{dt} = & -\frac{1}{2} \int d^3J d^3\theta d^3J' d^3\theta' \delta(\mathbf{x} - \mathbf{x}') (2 + \ln f + \ln f') \\ & \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'} \right) \cdot \boldsymbol{\tau} \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'} \right) f f'. \end{aligned} \quad (3.9)$$

This can now be integrated by parts, since $d^3J d^3\theta = m^3 d^3x d^3v$, to give

$$\begin{aligned} \frac{dS}{dt} = & \frac{1}{2} \int d^3J d^3\theta d^3J' d^3\theta' \delta(\mathbf{x} - \mathbf{x}') \frac{1}{f f'} \left\{ \left[\left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'} \right) f f' \right] \cdot \boldsymbol{\tau} \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'} \right) f f' \right\}, \\ & \geq 0, \end{aligned} \quad (3.10)$$

since $\boldsymbol{\tau}$ is a positive definite tensor.

We are presently concerned with systems that can be treated by transport theory and eq. (2.1) is solved by expansion in powers of $\Delta J_3/J_3$. The expression for dS/dt can then be expanded as well, with non zero entropy production appearing at second order. To leading order, eq. (3.7) becomes

$$\begin{aligned} \frac{dS}{dt} = & - \int d^3J (1 + \ln f_0) [C_i(f_0, f_1) + C_1(f_1, f_0) + C_2(f_0, f_0) + C_i(f_2)], \\ = & - \int d^3J (1 - \frac{3}{2} \ln 2\pi m + \ln(n/T^3) - H/T) [C_i(f_0, f_1) + C_1(f_1, f_0) + C_2(f_0, f_0)], \end{aligned} \quad (3.11)$$

since the particle and energy moments annihilate C_i .

From the transport theory of Section II one has the following results,

$$\int d^2J_{\perp} (C_i(f_0, f_1) + C_1(f_1, f_0)) = -\frac{\partial}{\partial J_3} \Gamma^i = -\frac{\partial}{\partial J_3} \sum_j T_{ij}^i A_j, \quad (3.12)$$

$$\int d^2J_{\perp} C_2(f_0, f_0) = -\frac{\partial}{\partial J_3} \Gamma^e = -\frac{\partial}{\partial J_3} \sum_j T_{ij}^e A_j, \quad (3.13)$$

$$\int d^2J_{\perp} H (C_i(f_0, f_1) + C_1(f_1, f_0)) = -\frac{\partial}{\partial J_3} \left(\frac{3}{2} T \Gamma^i + T q^i \right), \quad (3.14)$$

$$\int d^2J_{\perp} H C_2(f_0, f_0) = -\frac{\partial}{\partial J_3} \left(\frac{3}{2} T \Gamma^e + T q^e \right), \quad (3.15)$$

so that eq. (3.11) can be written

$$\begin{aligned} \frac{dS}{dt} = & \int dJ_3 \left(1 - \frac{3}{2} \ln(2\pi m) + \ln(n/T^{3/2}) \right) \frac{\partial}{\partial J_3} \Gamma \\ & - \int dJ_3 \frac{1}{T} \frac{\partial}{\partial J_3} \left(\frac{3}{2} T \Gamma + T q \right). \end{aligned} \quad (3.16)$$

Integrating the remaining radial, J_3 , integrals by parts gives

$$\frac{dS}{dt} = \left(\ln(n/T^{3/2}) - \frac{1}{2} - \frac{3}{2} \ln(2\pi m) \right) \Gamma \Big|_{J_3^{MIN}}^{J_3^{MAX}} - q \Big|_{J_3^{MIN}}^{J_3^{MAX}} - \int dJ_3 \left(\frac{d \ln n}{dJ_3} \Gamma + \frac{d \ln T}{dJ_3} q \right). \quad (3.17)$$

The first term corresponds to a flux of entropy into the system from the external world. The second term represents external heat supplied across the boundary. The last term is the internal entropy production, per unit J_3 , written as a sum of products of fluxes times thermodynamic forces. This term can also be written as a quadratic form,

$$\frac{dS_{INT}}{dt} = - \int dJ_3 \sum_{i,j} A_i T_{ij} A_j \quad (3.18)$$

Since the rate of entropy production must be positive for any choice of density and temperature profiles, it follows that $dS_{INT}/dt \geq 0$ holds separately, and furthermore, that the integrand of eq. (3.18) must be negative for all choices of the forces A_i . Therefore, one must have $T_{ii} < 0$. The full diagonal flux must oppose the applied gradient.

Notice that since the implicit transport coefficients, T_{ii}^i , are positive, they contribute negatively to the entropy production. The maximum principle for the implicit coefficients is equivalent to a minimum principle for the overall entropy production. The latter form of the variational principle, for the full coefficient, is familiar from the Eulerian formulation of transport theory. The present formulation is complementary and potentially useful in that the modules of the explicit coefficient, $|T_{ii}^e|$, provide an upper bound to the diagonal flux.

IV. Sufficient Conditions for Vanishing Particle Flux

In the previous section, the formal transport theory was developed. In this section, we are going to examine sufficient conditions under which the flux, due to the like particle collisions, vanishes.

The explicit and implicit transport coefficients related to the particle fluxes are

$$T_{11}^e = - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o (D_{33} - F_{33}) \quad (4.1)$$

$$T_{11}^i = \int d^2 J_{\perp} g_1 \frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} f_o f'_o (D \cdot e_3 - F \cdot e_3) \quad (4.2)$$

$$T_{12}^e = - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) D_{33} - \left(\frac{H'}{T} - \frac{3}{2} \right) F_{33} - \langle \Delta e_3 \cdot \tilde{F} \cdot \omega_{\perp} / T \rangle \right] \quad (4.3)$$

$$T_{12}^i = \int d^2 J_{\perp} g_1 \frac{\partial}{\partial J_{\perp}} \cdot \int d^2 J'_{\perp} f_o f'_o \left[\left(\frac{H}{T} - \frac{3}{2} \right) D \cdot e_3 - \left(\frac{H'}{T} - \frac{3}{2} \right) F \cdot e_3 - \langle \Delta \tilde{F} \cdot \omega_{\perp} / T \rangle \right] \quad (4.4)$$

where the explicit formulas for the driving functions α_1 and α_2 are given by Eqs. (2.42) and (2.43). The coefficients T_{11}^e , T_{11}^i and T_{12}^e , T_{12}^i can be combined and after partially integrating the implicit pieces, one has,

$$T_{11} = - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left\langle \left[\frac{\partial g_1}{\partial J_{\perp}} \cdot (\nabla_{\mathbf{v}} J_{\perp})^T + \nabla_{\mathbf{v}} \Delta J_3 \right] \cdot \tau \cdot (\nabla_{\mathbf{v}} \Delta J_3 - \nabla_{\mathbf{v}} \Delta J'_3) \right\rangle \quad (4.5)$$

$$T_{12} = - \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left\langle \left[\frac{\partial g_1}{\partial J_{\perp}} \cdot (\nabla_{\mathbf{v}} J_{\perp})^T + \nabla_{\mathbf{v}} \Delta J_3 \right] \cdot \tau \cdot \left[\left(\frac{H}{T} - \frac{3}{2} \right) \nabla_{\mathbf{v}} \Delta J_3 - \left(\frac{H'}{T} - \frac{3}{2} \right) \nabla_{\mathbf{v}} \Delta J'_3 - (\Delta J'_3 - \Delta J_3) v m / T \right] \right\rangle \quad (4.6)$$

Bringing the terms $(\frac{H}{T} - \frac{3}{2})$ inside the velocity derivatives $\nabla_{\mathbf{v}}$, yields for T_{12} ,

$$T_{12} = \int d^2 J_{\perp} \int d^2 J'_{\perp} f_o f'_o \left\langle \left[\frac{\partial g_1}{\partial J_{\perp}} \cdot (\nabla_{\mathbf{v}} J_{\perp})^T + \nabla_{\mathbf{v}} \Delta J_3 \right] \cdot \tau \cdot \left[\nabla_{\mathbf{v}} \Delta J_3 \left(\frac{H}{T} - \frac{3}{2} \right) - \nabla_{\mathbf{v}} \Delta J'_3 \left(\frac{H'}{T} - \frac{3}{2} \right) \right] \right\rangle \quad (4.7)$$

We now observe in Eqs. (4.5) and (4.7) that if the vector quantity in the first brackets stays unaffected by interchanging J_{\perp} and J'_{\perp} , then since the second brackets are antisymmetric in J_{\perp} and J'_{\perp} , the net result would be zero. Since all the quantities involved in the first bracket are functions of J only, this bracket should be a vector independent of J_{\perp} but in general, J_3 dependence would be allowed. If, instead, the first bracket has the

form of a scalar function of J_3 multiplied by the velocity vector \mathbf{v} , that makes again the net result zero since $\mathbf{v} \cdot \boldsymbol{\tau} = \mathbf{v}' \cdot \boldsymbol{\tau}$. Therefore,

$$\frac{\partial g_1}{\partial J_\perp} \cdot (\nabla_{\mathbf{v}} J_\perp)^T + \nabla_{\mathbf{v}} \Delta J_3 = c(J_3) + 2d(J_3)\mathbf{v} \quad (4.8)$$

gives a sufficient condition for vanishing particle fluxes. This can also be written in a different form,

$$\left(1 - \frac{\partial g_1}{\partial J_3}\right) \nabla_{\mathbf{v}} \Delta J_3 = -\nabla_{\mathbf{v}} g_1 + c(J_3) + 2d(J_3)\mathbf{v} \quad (4.9)$$

or, realizing that $\frac{\partial g_1}{\partial J_3}$ is of order $(\frac{e_p}{a})^2$,

$$\nabla_{\mathbf{v}} \Delta J_3 = -\nabla_{\mathbf{v}} g_1 + c(J_3) + 2d(J_3)\mathbf{v} \quad (4.10)$$

which, is correct up to the order $0(J_3 \frac{e_p}{a})$ for calculating ΔJ_3 . Integrating Eq. (4.10) gives,

$$\Delta J_3 = -g_1(\mathbf{J}_\perp, J_3) + \mathbf{v} \cdot c(J_3) + v^2 d(J_3) + h(J_3) \quad (4.11)$$

This expression suggests a sufficient condition on the form for ΔJ_3 which leads to vanishing particle fluxes. It must be expressible as a sum of two types of terms, one being a constant of the orbital motion, or a function of the actions alone, the other being conserved under collisions, i.e., momentum and energy. If we then write ΔJ_3 in the form

$$\Delta J_3 = \mathbf{v} \cdot c + v^2 d - \xi(\mathbf{J}_\perp), \quad (4.12)$$

where $\xi(\mathbf{J}_\perp)$ is any function of the actions, it then follows upon evaluating α_1 in Eq. (2.42) that $\alpha_1 = C_\ell(\xi)$. Therefore, up to a constant, $g_1 = \xi$, and when ΔJ_3 has the form (4.12), one can actually solve for g_1 in closed form. Equation (4.12), then for arbitrary ξ , is a sufficient condition for the vanishing of the particle flux due to like particle collisions.

A somewhat different form of the condition is obtained by writing

$$J_3 = \frac{2\pi q}{c} \psi + \mathbf{v} \cdot c + v^2 d - \xi(\mathbf{J}_\perp), \quad (4.13)$$

from which it follows that $J_\psi = 2\pi q/c \psi + \mathbf{v} \cdot c + v^2 d$ is a constant of the orbital motion. In fact, given J_ψ , one can easily find J_3 by observing that $\xi = \langle \mathbf{v} \cdot c + v^2 d \rangle$, where the brackets here denote a simple angle average. Thus, the condition can be stated as follows: if the average surface, J_3 , can be determined from a function, J_ψ , that is *both* a constant of the orbital motion and conserved by collisions, then there is no net particle flux.

Now a question arises. There is always an outward diffusion flux of test particles as would be computed in a Monte Carlo code, represented by coefficient D_{33} above. In the cases where the net flow is zero, what physical process produces an inward flow to cancel this outward diffusion flux? To illustrate the theorem, we answer this question for two concrete examples.

First, consider classical diffusion across a straight magnetic field. We take the z -direction for the magnetic field, and the x -direction for the density gradients, so that y is ignorable. The guiding center radius, X , is given in terms of the particle location, x , by

$$X = x + \frac{1}{\Omega} v_y = x + \frac{v_{\perp}}{\Omega} \cos \phi, \quad (4.14)$$

where ϕ is the gyrophase. Apart from a normalization constant, J_3 is equal to the guiding center position, X . Note that this has the form of Eq. (4.13) with $\xi = 0$, so that the net diffusion flux is zero as is well known. The relevant velocity derivatives are

$$\begin{aligned} \nabla_{\mathbf{v}} J_3 &\rightarrow \nabla_{\mathbf{v}} X = \frac{1}{\Omega} \mathbf{e}_y, \\ \nabla_{\mathbf{v}} J'_3 &= \nabla_{\mathbf{v}} X' = \frac{1}{\Omega} \mathbf{e}_y. \end{aligned} \quad (4.15)$$

Using Eqs. (2.7) and (2.20), one has $\tilde{\mathbf{D}} \cdot \mathbf{e}_3 - \tilde{\mathbf{F}} \cdot \mathbf{c}_3 = 0$, and thus from Eq. (2.42), $\alpha_1 = 0$. Therefore, the implicit flux Γ^i , is identically zero. The complicated compound processes that create Γ^i are not present. The flux here is all explicit which means it arises from simple integrations over the Fokker-Planck coefficients describing the elementary scattering processes.

Consider, T_{11}^e given by Eq. (2.55), with $T_{11}^i = 0$. This is the negative of the particle diffusion coefficient. Since $D_{33} = F_{33}$, we have $T_{11}^e = 0$, as expected, and also the formation that the test particle diffusion flux is canceled by the process embodied in the coefficient, F_{33} . But F_{33} is the coefficient $\langle \Delta J_3 \Delta J'_3 / \Delta t \rangle$ of correlation between test particle jumps, ΔJ_3 and field particle jumps, $\Delta J'_3$. The condition $D_{33} = F_{33}$ simply says that in *each scattering event* the radial steps of the guiding centers are equal and opposite for the two particles undergoing a collision. That the microscopic coefficients, D_{33} , F_{33} , cancel directly without integration is a reflection of the event-by-event nature of this cancellation of flux. These physical arguments are well known. This discussion is intended to show how naturally they appear in the Lagrangian framework and for comparison to more complex cases.

Now consider the tokamak. The third action can now be constructed from the canonical toroidal momentum invariant

$$J_\phi = \frac{2\pi q}{c}\psi + 2\pi m R b \cdot v \equiv \frac{2\pi q}{c}\psi + \frac{2\pi m I u}{B}$$

where $I \equiv RB_T$ and u is the parallel velocity, as

$$J_3 = \frac{2\pi q}{c}\psi + \frac{2\pi m I u}{B} - \left\langle \frac{2\pi m I u}{B} \right\rangle. \quad (4.16)$$

The angular brackets here imply bounce averages, $\langle \rangle \rightarrow \omega_2 \oint \frac{ds}{u}$, so that $\langle \frac{2\pi m I u}{B} \rangle$ measures the mean toroidal momentum. It is zero for trapped particles. Here J_3 has the form of Eq. (4.13) but with $\xi = \langle \frac{2\pi m I u}{B} \rangle$, non-zero for circulating particles. Therefore $\alpha_1 = C_\ell(\langle \frac{2\pi m I u}{B} \rangle)$ does not vanish and there are implicit contributions. In fact, the perturbed distribution, $g_1 = \langle \frac{2\pi m I u}{B} \rangle$, is quite large for circulating particles and represents a sizable toroidal flow.

We have a situation here where a minority species, the trapped particles, have large radial excursions while most of the particles are circulating near a magnetic flux surface with ΔJ_3 very small. Thus, in a single scattering event between a trapped test particle and a circulating field particle, the large trapped particle jump will not be balanced by any immediate back reaction of the field particles as was in the case for a straight magnetic field. Looking at the explicit coefficient, T_{11}^e of Eq. (2.55) it is then clear that D_{33} term will contribute when the test particles are trapped and be of order $\sqrt{2\epsilon}$. The F_{33} term, however, requires that both field and test particles be trapped and is then of order 2ϵ . To leading order then

$$T_{11}^e \simeq - \int_{\text{TR}} d^2 J_\perp \int d^2 J'_\perp f_0 f'_0 D_{33}. \quad (4.17)$$

It is now clear that the test particle diffusion flow, Eq. (4.17) must, in the tokamak, be canceled by some implicit inward flow. This is noteworthy since the argument leading to Eq. (4.17) has been invoked in the stellarator problem^[5] to justify use of (4.17) as the entire transport coefficient. The argument is incomplete without some rationale for ignoring implicit flows.

To identify the inward flow, we write the terms responsible for the implicit flow from the first term of $C_1(f_1, f_0) + C_1(f_0, f_1)$, in the Fokker-Planck form of Eq. (2.18). Using $f_1 = f_0 \hat{f}_1$ this is,

$$\begin{aligned} T_{ii}^i = & \int d^2 J_\perp d^2 J'_\perp \left[\hat{f}'_1 f_0 (e_3 \cdot D \cdot \frac{\partial}{\partial J_\perp} - e_3 \cdot f) f_0 \right. \\ & \left. + f_0 f'_0 e_3 \cdot D \cdot \frac{\partial}{\partial J_\perp} \hat{f}_1 + \hat{f}_1 f'_0 (e_3 \cdot D \cdot \frac{\partial}{\partial J_\perp} - e_3 \cdot f) f_0 \right]. \end{aligned} \quad (4.18)$$

The first term in Eq. (4.18) is zero, as follows from integrating the friction term by parts and using Eq. (2.27). The second term contains the flux resulting from the cross-process described in detail in Ref. [1]. This involves

only circulating particles and will be ignored in the present discussion. The third term contains the inward flow we seek. We consider in detail the contribution

$$- \int d^2 J_{\perp} d^2 J'_{\perp} f'_1 e_3 \cdot f f_0 \quad (4.19)$$

The remaining piece of the third term in Eq. (4.18) has a similar interpretation as will be clear shortly. Now Eq. (4.19) can be written out in more detail as

$$- \int d^2 J_{\perp} d^2 J'_{\perp} f'_1 \langle \nabla \cdot \mathbf{v} J_3 \cdot \mathbf{Q} \rangle f_0, \quad (4.20)$$

where \mathbf{Q} is the velocity space friction vector defined in Eq. (2.5). Basically, the toroidal flow reflected in the field particle distribution, f'_1 will produce, through collisions, a force of friction on the test particles. This toroidal force drives a radial flow of the trapped test particles in a manner exactly analogous to the Ware effect.^[4] In fact, if one identifies $\int d^2 J'_{\perp} f'_1 \mathbf{Q}$ with the force, replacing this force by qE in Eq. (4.20) will give precisely the Ware effect flow for the trapped particle portion of the phase space. By making this argument in more detail, we can determine the sign of the flow and show that it must be inward.

To this end, since the flow is independent of charge, we consider the ions. They are subject, to first order in ρ_p/a , only to like particle collisions. Thus, g_1 is known, as above, and the density gradient driven ion perturbation is $f_1 = \langle \frac{2\pi m I u}{B} \rangle f_0 d \ln n / d J_3$. This represents a sizable toroidal flow which is of order unity in the inverse aspect ratio expansion. One part of the bootstrap current, in fact, originates from this flow. The electrons are dragged along by the ions, tending to short out the current, but leave a residual electric current of order $\sqrt{2}\epsilon$ in the same direction as the original ion flow. The bootstrap current flows in the same direction as the Ohmic current which is, of course, parallel to the electric force. Since the friction force produced by the ion perturbation f'_1 on the test ions is also in this same direction, it follows that the resulting radial trapped particle flow has the same direction as the Ware pinch effect and must be inward.

To summarize, to obtain a net particle flux of zero in a tokamak, the outward diffusion flux of trapped particles is canceled by an inward pinch driven by a toroidal friction force resulting from the ion contribution to the bootstrap current. Related considerations would apply to the circulating particles. The point to note is that the canceling inflow is a complicated two-step process, and not a simple back reaction of field particles in the individual scattering events. This indicates that a minor breakdown of the axisymmetry condition could strongly influence the result.

V. Systems with Weak Axial Symmetry

We consider systems which are weakly asymmetric, such that J_2 is of the form:

$$J_2 = J_{2s}(H, \mu, \psi) + J_{2a}(H, \mu, \psi, \beta) \quad (5.1)$$

with $J_{2a} \ll J_{2s}$. Here, β describes azimuthal position on a flux surface and $\mathbf{B} = \nabla\psi \times \nabla\beta$. For simplicity, we consider J_{2a} to have a single Fourier component in β :

$$J_{2a} = \hat{J}_2(H, J_1, \psi) \cos m\beta \quad (5.2)$$

Systems which can be described by Eqs. (5.1) and (5.2) include stellarators, EBT and tandem mirrors. In particular, for tandem mirrors, plug electrons are so described with $m = 2$, and solenoid and passing electrons as well as neoclassical solenoid ions (ions in a machine where the azimuthal drift frequency is small compared to the bounce frequency) are so described with $m = 4$; in these examples J_2 is the longitudinal invariant. Resonant solenoid ions [in which the bounce and drift frequencies ω_b and ω_d are nearly harmonically related, $\omega_d \approx (2n + 1)\omega_b$] are described by Eqs. (5.1) and (5.2) with $m = 2$, if one defines J_2 and β relative to a closed, exactly resonant orbit rather than with respect to a field line.

We label a species as non-resonant if the β component of the drift velocity never vanishes, $\partial J_{2s}/\partial\psi \neq 0$. Electrons in a standard (positive-plug) tandem mirror are an example. For such a species, we can Taylor expand J_2 in the neighborhood of a particular flux surface $\psi = \bar{\psi}$:

$$J_2 \simeq J_{2s}(H, J_1, \bar{\psi}) + \frac{\partial J_{2s}}{\partial\psi_0}(\psi - \bar{\psi}) + \hat{J}_2(H, J_1, \bar{\psi}) \cos m\beta + 0[J_{2s}(\frac{\psi - \bar{\psi}}{\psi})^2] \quad (5.3)$$

Using the equations of motion ${}^{(6)}\dot{\psi} = \gamma \partial J_2 / \partial \beta$, $\dot{\beta} = -\gamma \partial J_2 / \partial \psi$ where $\gamma \equiv \omega_2 c / 2\pi q$, we easily find

$$\psi = -\frac{\hat{J}_2}{\partial J_{2s} / \partial \psi_0} (\cos m\beta) [1 + 0(\delta)] + \psi_0 \quad (5.4)$$

$$\beta = \beta_0 - \gamma \frac{\partial J_{1s}}{\partial \psi_0}(\psi) t [1 + 0(\delta)] \quad (5.5)$$

and thus $J_3 = \langle \psi \rangle_{\theta_3} = \langle \psi \rangle_t$ satisfies

$$J_3 = \psi + \alpha \cos m\beta + 0(\alpha\delta) \quad (5.6)$$

where

$$\alpha \equiv \frac{\hat{J}_2(H, J_1, \psi)}{\partial J_{2s} / \partial \psi}$$

and $\delta \equiv (\psi - \bar{\psi})/\psi$ can self-consistently be taken to be of order α . Here brackets with subscripts denote an average over the subscripted variable. Also, for the purpose of actual evaluation, the radial variable J_3 is taken to have the dimensions of magnetic flux rather than action as was convenient for the formal developments in previous sections. Thus we find

$$\nabla_v J_3 = (\nabla_v \alpha) \cos m\beta + O(\alpha\delta). \quad (5.7)$$

Eq. (5.5) implies that, to $O(\delta)$, θ_3 is proportional to β ; one can define θ_3 so that $\theta_3 \simeq m\beta$. Then from eq. (5.7) we see that $\langle \nabla_v J_3 \rangle_{\theta_3} = O + O(\alpha^2)$; furthermore, $\nabla_v J_1$ and $\nabla_v J_2$ are, to leading order in δ , independent of θ_3 . Thus, from inspection of the definitions of D and F , D_{3i} and F_{3i} are zero (to order α^2) for $i = 1, 2$, while $f_1/f_0 = O(\alpha)$, so that only the explicit pieces of eq. (2.48) survive to leading order in α , or order α^2 . That is, the particle flux can be written in the form [cf eqs. (2.51) and (2.53)]

$$\begin{aligned} \Gamma = & - \int d^2 J_{\perp} \int d^2 J'_{\perp} \left[\left(D_{33} \frac{\partial}{\partial J_3} - F_{33} \frac{\partial}{\partial J_3} \right) f_0 f'_0 \right. \\ & \left. + \frac{A_L M}{T} \langle (\delta - \delta') \nabla_v \delta \cdot \tau \cdot v' \rangle f_0 f'_0 \right] \end{aligned} \quad (5.8)$$

The same arguments establish that the energy flux is given to leading order in α by the explicit contributions, $T_{2j} = T_{2j}^e [1 + O(\alpha)]$ with T_{2j}^e given by eqs. (2.71)-(2.73).

The fluxes just derived are of course those due to like-particle (electron-electron) scattering only. The contribution due to electron-ion scattering is simpler. Electron-ion scattering, to leading order in $\sqrt{m_e/m_i}$ is described by the Lorentz (pitch-angle scattering) collision operator. The fluxes for this operator were derived in the Lagrangian formulation by Molvig and Bernstein ^[1] Actually, the fluxes can be written down by inspection of the like-particle fluxes just derived, by treating ions as infinitely massive and at rest, and taking $f'_0 = n\delta(\vartheta)$ and $f'_1 = 0$. Then (following the above arguments) only the explicit test-particle fluxes survive; in particular, the particle flux eq. (5.7) reduces to

$$\Gamma_{ei} = -n_3 \int d^2 J_{\perp} D_{33} \frac{\partial f_0}{\partial J_3}, \quad (5.9)$$

an intuitively almost obvious result. But now D_{33} [as defined by eqs. (2.7) and (2.4)] takes the simple form

$$D_{33} = \frac{2\pi q^4 \Lambda}{m^2} \left\langle \frac{\delta^2 \mu v_{\parallel}^2}{4Bv^3} \right\rangle$$

where $\delta_\mu = \partial\delta/\partial\mu$. Thus, we have

$$\Gamma_{ei} = -\frac{2\pi q^4 n n_3}{2m} \left(\frac{m^{\frac{3}{2}}}{2\pi\tau} \int d\mu dE e^{-E/T} \mu \delta_\mu^2 \left(A_1 - \frac{3}{2} A_2 + A_2 \frac{E}{T} \right) \right) \times \int \frac{ds}{B} \frac{(E - \mu B - q\phi)^{\frac{3}{2}}}{(E - q\phi)^{\frac{3}{2}}} \quad (5.10)$$

Equation (5.10) is identified with the expression derived using conventional (Eulerian) techniques in Ref. [3]. It is to be added to eq. (5.7) to obtain the total flux.

For a resonant species, $\partial J_{2s}/\partial\psi$ vanishes on one or more resonant surfaces $\psi_r(H, J_1)$, and excursions in ψ are largest near such surfaces. Taylor-expanding about a resonant surface, we have

$$J_2 \simeq J_{2r} + \frac{\partial J_{2r}^2}{\partial\psi_r^2} \frac{(\psi - \psi_r)^2}{2} + \hat{J} \cos m\beta \quad (5.11)$$

where $J_2 \equiv J_{2s}(H, J_1, \psi_r)$. The equations of motion for ψ and β are those of a pendulum; orbits are divided into trapped and passing according to whether $|J_2 - J_{2r}|$ is less or greater than, respectively, the separatrix value $\delta J_2^{(s)} = \hat{J}_2$. We find, for trapped particles,

$$\psi = \psi_r + \delta cn \left[mK_l(\theta_3 - \theta_0) | a \right] \quad (5.12)$$

so that

$$J_3^{(t)} \equiv \langle \psi \rangle_{\theta_3} = \psi_r = \psi - \delta cn [mK_l(\theta_3 - \theta_0) | a] \quad (5.13)$$

and

$$\cos \frac{m\beta}{2} = (a)^{\frac{1}{2}} sn \left[mK_l(\theta_3 - \theta_0) | a \right] \quad (5.14)$$

where now

$$\delta \equiv \left[\frac{2(J_2 - J_{2r} + \hat{J}_2)}{\partial^2 J_{2r} / \partial\psi_r^2} \right]^{\frac{1}{2}} \quad (5.15)$$

$$a \equiv \frac{J_2 - J_{2r} + \hat{J}_2}{2\hat{J}} \quad (5.16)$$

$$K_l \equiv \frac{2}{\pi} K(a), \quad (5.17)$$

cn, sn, dn are the Jacobi elliptic functions, $K(a)$ is the elliptic integral of the first kind [7]; and where θ_0 is a value of θ at which $\cos m\beta/2 = 0$. Proceeding similarly for the untrapped particles,

$$\psi = \psi_r \pm \delta dn[mK(\theta_3 - \theta_0)|a^{-1}] \quad (5.18)$$

and

$$\cos \frac{m\beta}{2} = \pm sn[mK(\theta_3 - \theta_0)|a^{-1}] \quad (5.19)$$

where

$$K_u \equiv \frac{\pi J_{2r}}{\psi} \frac{2}{\pi} K(A^{-1}). \quad (5.20)$$

There are two passing-particle orbits corresponding to the same J_1, J_2 on opposite sides of the resonance surface; these are denoted by the + and - signs. Since the average of dn is K_u^{-1} , we find from Eq. (5.18),

$$J_3^{(u)} = \psi_r \pm \delta/K_u. \quad (5.21)$$

We evaluate $\nabla_v J_3$ by using eq. (5.13) or (5.21) (for trapped or passing particles, respectively) and (5.14) or (5.19) to eliminate ψ_r and θ_3 from eq. (5.12) or (5.18) and then differentiate the resultant expression. For trapped particles we obtain

$$\nabla_v J_3^{(\epsilon)} = (J_3 - \psi)^{-1} [\nabla_v(\delta^2/2) - 2\nabla_v(\hat{J}/J_{\psi\psi}) \cos^2 m\beta/2]. \quad (5.22)$$

where $J_{\psi\psi} \equiv d^2 J_{2r}/\delta\psi_r^2$. Upon using eqs. (5.13)-(5.15), this becomes

$$\nabla_v J_3^{(\epsilon)} = (C_i \delta)^{-1} \nabla_v(\delta^2/2) + 2a\delta^{-1} (C_i^{-1} - C_i) \nabla_v(\hat{J}/J_{\psi\psi}) \quad (5.23a)$$

$$= (C_i \delta)^{-1} \nabla_v(\delta^2/2) [1 + O(\hat{J}/J_{2r})]. \quad (5.23b)$$

where $C_i \equiv cn[mK_i(\theta_3 - \theta_0)|a]$. In writing (5.23b) we have made use of the fact that $\nabla_v(J_2 - J_{2r})$ is of the same order as $\nabla_v J_{2r}$. From eq. (5.23a) it is easily verified that $\langle \nabla_v J_3^{(\epsilon)} \rangle = 0$.

Proceeding similarly for the passing particles, we find

$$\nabla_v J_3^{(u)} = \pm \left\{ \delta^{-1} \nabla_v(\delta^2/2) (K - u^{-1} - d_u) + \delta \nabla_v a [K_u^{-2} \frac{dK_u}{da} + \frac{1}{2a} (d_u - d_u^{-1})] \right\} \quad (5.24)$$

where $d_u = dn[mK_t(\theta_3 - \theta_0) | a^{-1}]$. This expression does not vanish upon averaging over θ_3 . Noting that $\langle d_u^{-1} \rangle_{\theta_3} = K_u^{-1}(1 - a^{-1})^{-\frac{1}{2}}$, we obtain

$$\langle \nabla_v J_3^{(u)} \rangle_{\theta_3} = \pm K_u^{-1} \nabla_v a \left\{ K_u^{-1} \frac{dK_u}{da} + \frac{1}{2a} [1 - (1 - a^{-1})^{-\frac{1}{2}}] \right\} \quad (5.25)$$

However, using the small-parameter expansions of the elliptic function dn and the small-argument expansion of the elliptic integral $K(a^{-1})$ which appears in (5.21), eqs. (5.24) and (5.25) become

$$\nabla_v J_3^{(u)} = \mp \frac{1}{4} \nabla_v \left(\frac{\delta}{a} \right) \cos[2m(\theta_3 - \theta_0)] [1 + O(a^{-1})] \quad (5.26a)$$

$$= \mp \frac{\delta \nabla_v a}{\delta a^2} \cos[2m(\theta_3 - \theta_0)] [1 + O(a^{-1}) + O\left(\frac{J_2 - J_{2r}}{J_2}\right)] \quad (5.26b)$$

and

$$\langle \nabla_v J_3^{(u)} \rangle_{\theta_3} = \pm \frac{\delta \nabla_v a}{32a^3} [1 + O(a^{-1})]. \quad (5.27)$$

We use these expressions to estimate the relative sizes of the explicit particle-flux terms in eq. (2.51). We first observe that the ratio of the F_{33} term Γ_F to the test-particle (D_{33}) term Γ_t is at most of order

$$\frac{\Gamma_F}{\Gamma_t} \simeq \frac{[\int d^2 J_{\perp} \langle (\nabla_v J_3)^2 \rangle_{\theta_3}^{\frac{1}{2}} f]^2}{[\int d^2 J_{\perp} \langle (\nabla_v J_3)^2 \rangle_{\theta_3} f] [\int d^2 J_{\perp} f]} \quad (5.28a)$$

$$\simeq \hat{J} / \bar{J}_2 \quad (5.28b)$$

where \bar{J}_2 denotes the typical difference in J_2 between one resonance surface and the next or, if there is only one resonance acting on the bulk of the distribution, a typical (averaged over f) value of J_2 . The estimate (5.28b) is obtained simply by noting from eqs. (5.23b) and (5.25) that $\langle (\nabla_v J_3)^2 \rangle_{\theta_3}$ is sizable only in a layer of width of order \hat{J}_2 about a resonance surface. From eq. (5.19) and the large-argument expansion of dn it can be verified that \hat{J}_2 / \bar{J}_2 is the order of the fractional variation of ψ for a highly entrapped ($J_2 - J_{2r} \simeq \bar{J}_2$) particle.

The ratio of the " Δ " term Γ_{Δ} [last term in eq. (4.1)] to the temperature-gradient portion $\Gamma_{\nabla T}$ of the test-particle flux is

$$\begin{aligned} \frac{\Gamma_{\Delta}}{\Gamma_{\Delta t}} &\simeq \frac{\int d^2 J_{\perp} d^2 J'_{\perp} \langle \Delta \nabla_v J_3 \rangle_{\theta_3} v' f f'}{\int d^2 J_{\perp} d^2 J'_{\perp} \langle (\nabla_v J_3)^2 \rangle_{\theta_3} \frac{v^2}{2} f f'} \\ &\simeq \frac{\int d^2 J_2 \langle (J_3 - \psi) \nabla_v J_3 \rangle_{\theta_3}}{v \int d^2 J_{\perp} \langle (\nabla_v J_3)^2 \rangle_{\theta_3}} \end{aligned} \quad (5.29)$$

where $\nu = (2T/m)^{1/2}$. Noting from eqs. (5.13), (5.14), (5.21), (5.22), (5.24) and (5.25) that $J_3 - \psi$ and $\nabla_\nu J_3$ are both appreciable in a layer with width of order \tilde{J} about a resonant surface and fall off rapidly outside the layer, we see that

$$\frac{\Gamma_\Delta}{\Gamma_{\Delta T}} \simeq \frac{[(\langle J_3 - \psi \rangle \nabla_\nu J_3)_{\theta_3}]_s}{\nu [(\langle \nabla_\nu J_3 \rangle^2)_{\theta_3}]_s}$$

where the subscripts denote maximum values, i.e., values for either trapped or passing particles near the separatrix. From the expressions given above we estimate $[(\langle J_3 - \psi \rangle \nabla_\nu J_3)_{\theta_3}]_s \simeq \nabla_\nu \delta^2/2 \simeq J_{2r}/(\nu J_{\psi\psi})$ and similarly $[(\langle \nabla_\nu J_3 \rangle^2)_{\theta_3}]_s \simeq J_{2r}^2/(\nu^2 \tilde{J} J_{\psi\psi})$ so that

$$\frac{\Gamma_\Delta}{\Gamma_{\Delta T}} \simeq \frac{\tilde{J}}{J_{2r}} \quad (5.30)$$

Thus, to leading order in $\tilde{J}/\bar{J}_2 (\simeq \tilde{J}_2/J_{2r})$, the explicit fluxes given by the test-particle contribution:

$$\Gamma^e = - \int d^2 J_\perp d^2 J'_\perp f'_0 D_{33} \frac{\partial f_0}{\partial J_3} [1 + O(\tilde{J}/\bar{J}_2)] \quad (5.31)$$

By completely analogous procedures one can readily show that the explicit heat flux q^e given by eq. (2.62) is dominated by the test-particle term,

$$q^e = - \frac{\partial}{\partial J_3} \int d^2 J_\perp d^2 J'_\perp H D_{33} f'_0 \frac{\partial f_0}{\partial J_3} [1 + O(\tilde{J}/\bar{J}_2)] \quad (5.32)$$

We consider now the implicit flux. We note from eqs. (5.23a), (5.27) and the definitions (5.16) and (3.18) that $\langle \nabla_\nu J_3 \rangle_{\theta_3}$ is zero for trapped particles and is nonzero for untrapped particles only near the separatrix, falling off as the $J/2$ power of the distance (in J_2) from resonance. Hence, from eq. (2.56) and the definitions of D and F , we see that the implicit flux comes only from a thin boundary layer outside the separatrix, whereas the explicit flux involves all of the trapped particles as well. We might therefore expect the implicit and explicit fluxes to be in the ratio of the boundary-layer thickness to the separatrix width.

To be more quantitative, we use the first-order equation (2.25) [with eqs. (2.26) and (2.22)] to estimate

$$|\nabla_\nu J_\perp \cdot \frac{\partial \hat{f}_1}{\partial J_\perp} - \nabla_\nu J'_\perp \cdot \frac{\partial \hat{f}_1}{\partial J'_\perp}| \simeq \langle \nabla_\nu J_3 \rangle_{\theta_3} \frac{\partial f_0}{\partial J_3} - \langle \nabla_\nu J'_3 \rangle_{\theta_3} \frac{\partial f'_0}{\partial J_3} \quad (5.33)$$

Then, from eqs. (2.56), (5.30) and the definitions of D and F we have

$$\frac{\Gamma^i}{\Gamma^e} \simeq \frac{\int d^2 J_\perp \langle \nabla_\nu J_3 \rangle_{\theta_3}^2}{\int d^2 J_\perp \langle (\nabla_\nu J_3)^2 \rangle_{\theta_3}} \quad (5.34)$$

We evaluate this expression taking $\langle (\nabla_v J_3^{(u)})^2 \rangle_{\theta_3}$ and $+\langle \nabla_v J_3^{(u)} \rangle_{\theta_3}$ from eqs. (5.25) and (139), recalling that $\langle \nabla_v J_3^{(l)} \rangle_{\theta_3} = 0$, and estimating, from eq. (5.23b), that $\langle (\nabla_v J_3^{(l)} J^2) \rangle_{\theta_3} \gtrsim (\hat{J}/2J_{\psi\psi})(\nabla_v a)^2$, we find that the expression (5.33) becomes a pure number,

$$\frac{\Gamma^i}{\Gamma^e} \simeq \frac{1}{272} \quad (5.35)$$

Thus, the implicit flux is not asymptotically small in some parameter, but is estimated to be numerically insignificant. The same conclusion is obtained for the heat flux.

In summary, the particle flux for a resonant species is given approximately by eq. (5.30). If the resonance under consideration is between harmonics of the bounce and drift frequencies, as in tandem-mirror ion transport, eq. (5.30) is to be summed over resonances; each term in the sum is evaluated with a different definition of J_2 (integration along a closed orbit for the resonance in question). This result is closely related to the form for the resonant-ion flux in the banana regime obtained in Ref. [8]. In our notation, the result Γ_{rb} of Ref. 8 can be written in the form

$$\Gamma_{rb} = - \int d^2 J_{\perp} d^2 J'_{\perp} f'_0 \langle \tilde{D}_{33} \rangle \partial f_0 / \partial \psi$$

where now $f_0 = f_0(J_{\perp}, \psi)$ and

$$\tilde{D}_{33} = \hat{D}_{33}(\mathbf{x}_0, \mathbf{v}_0) \equiv \frac{1}{t} \int_0^t dt' [\nabla_v \hat{\psi}(\mathbf{x}, \mathbf{v}, t - t')]^T \cdot \boldsymbol{\tau} \cdot \nabla_v \hat{\psi}(\mathbf{x}, \mathbf{v}, t - t'). \quad (5.36)$$

Here $\hat{\psi}(\mathbf{x}, \mathbf{v}, t - t')$ denotes the value of the flux coordinate after evolving for a time $t - t'$ from an initial phase-space position \mathbf{x}, \mathbf{v} under collision-free dynamics, and \mathbf{x}, \mathbf{v} are to be evaluated at the phase-space point into which the initial $\mathbf{x}_0, \mathbf{v}_0$ evolves in time t' under collision-free dynamics. The difference between eqs. (5.30) and (5.35) is that the object $\nabla_v J_3$ which appears in D_{33} in eq. (5.30) describes the response of a particle-oscillation center to a collisional change in velocity, whereas the quantity $\nabla_v \hat{\psi}$ in eq. (5.36) describes the response of the instantaneous particle position to previous collisional changes in velocity. To the extent that $\nabla_v J_3$ and $\nabla_v \hat{\psi}$ can be equated, \tilde{D}_{33} in eq. (5.36) becomes identical with \tilde{D}_{33} in eq. (2.7); since $\partial/\partial\psi = \partial/\partial J_3$ to leading order in δ , eq. (5.34) then becomes equivalent to eq. (5.30).

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