

BOUNDING VOLUME BY SYSTOLES OF 3-MANIFOLDS

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ABSTRACT. We prove a new systolic volume lower bound for non-orientable n -manifolds, involving the stable 1-systole and the codimension 1 systole with coefficients in \mathbb{Z}_2 . As an application, we prove that Lusternik-Schnirelmann category and systolic category agree for non-orientable closed manifolds of dimension 3, extending our earlier result in the orientable case. Finally, we prove the homotopy invariance of systolic category.

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1. INTRODUCTION

The systolic project in its modern form was initiated by M. Gromov [Gr83], when he proved a volume lower bound for a closed essential Riemannian manifold M , which is curvature-free and depends

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only on the least length of a noncontractible loop in M , *i.e.* the 1-systole $\text{sys}\pi_1(M)$:

$$(\text{sys}\pi_1(M))^n \leq C_n \text{vol}_n(M). \quad (1.1)$$

See [BeCG03] for a recent application. Alternative approaches to the proof of (1.1) may be found in [AK00, We05, Gu06].

More generally, one considers higher dimensional systoles, and seeks analogous volume lower bounds. The defining text for this material is the monograph [Gr99] (which is an extended English version of [Gr81]), with additional details in the earlier texts [Gr83, Gr96]. Recently there has been a considerable amount of activity related to systolic inequalities. The Loewner inequality and its generalisations are studied in [Am04, IK04, KL05, BaCIK05, BaCIK07, KS06, KRS06]. Near-optimal asymptotic bounds are studied in [Ba04, Bal04, BaB05, KS05, KSV06]. See [Ka07] for an overview of systolic problems.

2. MOTIVATION

The notion of systolic category cat_{sys} was introduced by the authors in [KR06], *cf.* (3.5) below. It can be regarded as a differential-geometric analogue of the Lusternik-Schnirelmann category (LS category) cat_{LS} , *cf.* [LS34]. While the definitions of the two categories use completely different language, the values of the two invariants turn out to be very close in a number of interesting cases. Thus, the insight gained from the study of one of them can be used to glean a fresh perspective on the other.

The significance of systolic category as compared to LS category can be illustrated by the recent progress on a long-standing conjecture concerning the latter [DKR07]. Such progress was motivated in part by systolic geometry. Namely, in collaboration with A. Dranishnikov, we proved the 1992 conjecture of Gomez-Larrañaga and Gonzalez-Acuña, to the effect that n -manifolds ($n \geq 3$) of LS category 2 necessarily have free fundamental group.

Conversely, the knowledge of the value of LS category for 4-manifolds M with non-free fundamental groups lends plausibility to the possible existence of systolic inequalities on M corresponding to the partition $4 = 1 + 1 + 2$, *cf.* (3.2). Such an inequality is currently known to exist only in a limited number of cases, due essentially to Gromov [Gr83, Theorem 7.5.B].

We will use the modern definition of the Lusternik-Schnirelmann category, *cf.* [CLOT03], which differs by a unit from the original definition. Thus, cat_{LS} of a contractible space is equal to zero. The two invariants

share a number of characteristics, including lower bound by cup-length and sensitivity to Massey products, *cf.* [KR06, Theorem 11.1], [Ka06].

Remark 2.1. The lowest known dimension of a manifold whose systolic category is strictly smaller than its LS category, is dimension 16, see [KR06, Example 9.5] or [Ka07, p. 104-105].

We prove that systolic category and LS category agree for non-orientable closed manifolds of dimension 3, extending our earlier result in the orientable case [KR06, Corollary 6.2]. The required lower bound for the systolic category of a non-orientable manifold follows from a new inequality (4.2) involving systoles of dimension and codimension one. A different but similar inequality in the orientable case was studied in [BaK03, BaK04]. The proof exploits harmonic forms, the Cauchy-Schwartz inequality, and the coarea formula, see Section 5.

The paper is organized as follows. In Section 3, we recall the definition of the systolic invariants. In Section 4, we recall the definition of the Bergé-Martinet constant, and present an optimal systolic inequality (4.2) combining systoles of dimension and codimension 1, valid for non-orientable manifolds. Its proof appears in Section 5. The positivity of systoles is proved in Section 6. In Section 7, we prove the homotopy invariance of systolic category, which parallels that of the LS category. Some open questions are posed in Section 8.

All manifolds are assumed to be closed, connected, and smooth. All polyhedra are assumed to be compact and connected, unless explicitly mentioned otherwise. To the extent that our paper aims to address both a topological and a geometric audience, we attempt to give some indication of proof of pertinent results that may be more familiar to one audience than the other.

3. SYSTOLES AND SYSTOLIC CATEGORY

Let X be a (finite) polyhedron equipped with a piecewise Riemannian metric \mathcal{G} . We will now define the systolic invariants of (X, \mathcal{G}) . Note that we adopt the usual convention that the infimum calculated over an empty set is infinite. Thus by definition, a k -systole is infinite in the case when $H_k(X) = 0$.

Definition 3.1. The homotopy 1-systole, denoted $\text{sys}\pi_1(X, \mathcal{G})$, is the least length of a non-contractible loop in X . The homology 1-systole, denoted $\text{sysh}_1(X, \mathcal{G})$, is defined in a similar way, in terms of loops which are not zero-homologous.

Clearly, we have $\text{sys}\pi_1 \leq \text{sysh}_1$.

Definition 3.2. Let $k \in \mathbb{N}$. Higher homology k -systoles $\text{sysh}_k = \text{sysh}_k(X, \mathcal{G}, A)$, with coefficients over a ring $A = \mathbb{Z}$ or \mathbb{Z}_2 , are defined similarly to sysh_1 , as the infimum of k -areas of k -cycles, with coefficients in A , which are not zero-homologous.

More generally, let B be the group of deck transformations of a regular covering space of X . Then the homology groups of the covering space of X can be identified with the homology groups of X with coefficients in the group ring $A[B]$. Allowing more general coefficients in such a group ring, we can therefore define the corresponding systole

$$\text{sysh}_k(X, \mathcal{G}; A[B]). \quad (3.1)$$

Note that we adopt the usual convention, convenient for our purposes, that the infimum over an empty set is infinity.

More detailed definitions appear in the survey [CK03] by C. Croke and the first author, and in [Ka07]. We do not consider higher ‘‘homotopy’’ systoles.

Definition 3.3 (*cf.* [Fe69, BaK03]). Given a class $\alpha \in H_k(X; \mathbb{Z})$ of infinite order, we define the stable norm $\|\alpha_{\mathbb{R}}\|$ by setting

$$\|\alpha_{\mathbb{R}}\| = \lim_{m \rightarrow \infty} m^{-1} \inf_{\alpha(m)} \text{vol}_k(\alpha(m)),$$

where $\alpha_{\mathbb{R}}$ denotes the image of α in real homology, while $\alpha(m)$ runs over all Lipschitz cycles with integral coefficients representing $m\alpha$. The stable homology k -systole, denoted $\text{stsys}_k(\mathcal{G})$, is defined by minimizing the stable norm $\|\alpha_{\mathbb{R}}\|$ over all integral k -homology classes α of infinite order.

We have $\text{stsys}_k \leq \text{sysh}_k$ in the absence of torsion. The stable systoles can be significantly different from the ordinary ones. The first examples of such a phenomenon, already for the 1-dimensional homology systoles, were discovered by Gromov and described by M. Berger in [Be93]. On the other hand, for $\mathbb{R}\mathbb{P}^n$ we have $\text{sysh}_1 < \infty$ while $\text{stsys}_1 = \infty$.

Recall that, in our convention, the systolic invariants are infinite when defined over an empty set of loops or cycles.

Remark 3.4. M. Berger [Be72] defined invariants which eventually came to be known as the k -systoles, in the framework of Riemannian manifolds. All systolic notions can be defined similarly for polyhedra, *cf.* [Gr96] and [Ba02, Ba06]. Note that every smooth manifold is triangulable and therefore can be viewed as a polyhedron. When $k = n$ is the dimension, $\text{sysh}_n(M, \mathcal{G})$ is equal to the volume $\text{vol}_n(M, \mathcal{G})$ of a compact Riemannian n -manifold (M, \mathcal{G}) . For an n -polyhedron X , however, the volume may not agree with the n -systole $\text{sys}_n(X)$, as the former is

always finite, while the latter may be infinite, when X does not possess a fundamental class. Moreover, it can happen that $\text{sys}_n X \neq \text{vol}_n X$ even if $\text{sys}_n X$ is finite: for example, if X is a wedge of two n -spheres.

The idea is to bound the total volume from below in terms of lower-dimensional systolic invariants. Here we wish to incorporate all possible curvature-free systolic inequalities, stable or unstable. More specifically, we proceed as follows.

Definition 3.5. Given $k \in \mathbb{N}, k > 1$ we set

$$\text{sys}_k(X, \mathcal{G}) = \inf\{\text{sysh}_k(X, \mathcal{G}; \mathbb{Z}[B]), \text{sysh}_k(X, \mathcal{G}; \mathbb{Z}_2[B]), \text{stsys}_k(X, \mathcal{G})\},$$

where the infimum is over all groups B of regular covering spaces of X . Furthermore, we define

$$\text{sys}_1(X, \mathcal{G}) = \min\{\text{sys}\pi_1(X, \mathcal{G}), \text{stsys}_1(X, \mathcal{G})\}.$$

Note that the systolic invariants thus defined are positive (or infinite), see Section 6.

Let X be an n -dimensional polyhedron, and let $d \geq 2$ be an integer. Consider a partition

$$n = k_1 + \dots + k_d, \tag{3.2}$$

where $k_i \geq 1$ for all $i = 1, \dots, d$. We will consider scale-invariant inequalities “of length d ” of the following type:

$$\text{sys}_{k_1}(\mathcal{G}) \text{sys}_{k_2}(\mathcal{G}) \dots \text{sys}_{k_d}(\mathcal{G}) \leq C(X) \text{vol}_n(\mathcal{G}), \tag{3.3}$$

satisfied by all metrics \mathcal{G} on X , where the constant $C(X)$ is expected to depend only on the topological type of X , but not on the metric \mathcal{G} . Here the quantity sys_k denotes the infimum of all non-vanishing systolic invariants in dimension k , as defined above.

Definition 3.6. Systolic category of X , denoted $\text{cat}_{\text{sys}}(X)$, is the largest integer d such that there exists a partition (3.2) with

$$\prod_{i=1}^d \text{sys}_{k_i}(X, \mathcal{G}) \leq C(X) \text{vol}_n(X, \mathcal{G})$$

for all metrics \mathcal{G} on X . If no such partition and inequality exist, we define systolic category to be zero.

In particular, $\text{cat}_{\text{sys}} X \leq \dim X$.

Remark 3.7. Clearly, systolic category equals 1 if and only if the polyhedron possesses an n -dimensional homology class, but the volume cannot be bounded from below by products of systoles of positive codimension. Systolic category vanishes if X is contractible. Another

example of a 2-polyhedron X with $\text{cat}_{\text{sys}} X = 0$ is a wedge of the disk and the circle, *cf.* Corollary 7.4.

4. INEQUALITY COMBINING DIMENSION AND CODIMENSION 1

Given a maximal rank lattice L in a normed space $(\mathbb{R}^b, \|\cdot\|)$, let $\lambda_1(L)$ denote the least length of a non-zero vector of L .

Definition 4.1. The Bergé-Martinet constant [BeM89], denoted γ'_b , is defined as follows:

$$\gamma'_b = \sup \{ \lambda_1(L) \lambda_1(L^*) \mid L \subseteq \mathbb{R}^b \}, \quad (4.1)$$

where the supremum is extended over all lattices L in \mathbb{R}^b with its Euclidean norm.

Here the dual lattice $L^* \subset \mathbb{R}^b$ by definition consists of elements $y \in \mathbb{R}^b$ satisfying $\langle x, y \rangle \in \mathbb{Z}$ for all $x \in L$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. A lattice attaining the supremum in (4.1) is called *dual-critical*.

Both the Hermite constant γ_b and the Bergé-Martinet constant γ'_b are asymptotically linear in b . Their values are known in dimensions up to 4 as well as certain higher dimensions.

Example 4.2. In dimension 3, the value of the Bergé-Martinet constant, $\gamma'_3 = \sqrt{\frac{3}{2}} = 1.2247\dots$, is slightly below the Hermite constant $\gamma_3 = 2^{\frac{1}{3}} = 1.2599\dots$. It is attained by the face-centered cubic lattice, which is not isodual [MH73, p. 31], [BeM89, Proposition 2.13(iii)], [CoS94].

We generalize an inequality proved in the orientable case in [BaK03, BaK04].

Theorem 4.3. *Let M be an n -dimensional manifold with first Betti number $b \geq 1$. Then every metric \mathcal{G} on M satisfies the systolic inequality*

$$\text{stsys}_1(\mathcal{G}) \text{sys}_{n-1}(\mathcal{G}; \mathbb{Z}_2) \leq \gamma'_b \text{vol}_n(\mathcal{G}), \quad (4.2)$$

where γ'_b is the Bergé-Martinet constant of (4.1). Furthermore, inequality (4.2) is optimal.

This inequality is proved in Section 5.

Corollary 4.4. *We have $\text{cat}_{\text{sys}}(M) \geq 2$ for all manifolds M with positive first Betti number. \square*

Corollary 4.5. *If M is a closed 3-dimensional manifold with nontrivial free fundamental group, then $\text{cat}_{\text{sys}} M = 2$.*

Proof. The classifying space of the free group is a wedge of circles. Thus a manifold M with free fundamental group is not essential. Therefore its systolic category is at most 2.

The systolic inequality provided by Corollary 4.4, corresponding to the partition $3 = 1 + 2$ as in (3.2), shows that $\text{cat}_{\text{sys}} M = 2$. \square

Corollary 4.6. *Systolic category and LS category coincide for all closed connected 3-manifolds, orientable or not.* \square

Proof. The fact that the LS category of a 3-manifold is determined by its fundamental group was proved by J. Gomez-Larrañaga and F. Gonzalez-Acuña [GG92] (see also [OR01]). In particular, $\text{cat}_{\text{LS}} M^3 = 3$ if and only if $\pi_1(M)$ is not free.

By Gromov’s inequality for essential manifolds [Gr83], combined with I. Babenko’s converse to it [Ba93], an n -manifold has systolic category n if and only if it is essential. Here the systolic inequality corresponds to the partition $n = 1 + 1 + \dots + 1$ as in (3.2). In [KR06, Corollary 7.3], we proved that every 3-manifold with non-free fundamental group is essential. Hence, if $\text{cat}_{\text{LS}}(M^3) = 3$ then M^3 is essential, and hence $\text{cat}_{\text{sys}} M = 3$. Furthermore, if $\text{cat}_{\text{LS}} M = 2$ then $\pi_1(M)$ is nontrivial and free, and hence $\text{cat}_{\text{sys}} M = 2$ by Corollary 4.5. Finally, if $\text{cat}_{\text{LS}} M^3 = 1$ then M is a homotopy sphere, and thus $\text{cat}_{\text{sys}} M = 1$. \square

Remark 4.7. The class of 3-dimensional Poincaré complexes is essentially larger than the class of 3-manifolds. For example, the Sphere Theorem does not hold for 3-dimensional Poincaré complexes by J. Hillman’s work [Hi04]. Hillman’s example Y is irreducible, essential, and virtually free. Thus Y does not easily fit into the algebraic dichotomy in the context of 3-manifolds discussed in [KR06, Proposition 7.2], *cf.* the “Tits alternative” of [Hil03]. It remains to be seen how the existence of such an example affects the calculation of the two categories.

Question 4.8. Does the conclusion of Corollary 4.6 hold more generally for 3-dimensional Poincaré complexes?

5. PROOF OF OPTIMAL INEQUALITY

With the proof of Theorem 4.3 in mind, let $H_1(M; \mathbb{Z})_{\mathbb{R}}$ be the integer lattice in $H_1(M; \mathbb{R})$, and similarly for cohomology. Given a metric \mathcal{G} , one defines the stable norm $\|\cdot\|$ in homology and the comass norm $\|\cdot\|^*$ in cohomology. The normed lattices

$$(H_1(M; \mathbb{Z})_{\mathbb{R}}, \|\cdot\|) \quad \text{and} \quad (H^1(M; \mathbb{Z})_{\mathbb{R}}, \|\cdot\|^*)$$

are dual, whether or not M is orientable [Fe74, item 5.8]. An intuitive explanation for such duality may be found in [Gr99, Proposition 4.35, p. 261].

Let $\|\cdot\|_2^*$ be the L^2 -norm in $H^1(M; \mathbb{R})$, *i.e.*

$$\|\omega\|_2^* = \inf_{\xi \in \omega} |\xi|_2$$

where the infimum is over all closed forms $\xi \in \omega$, and $|\cdot|_2$ is the L^2 -norm for forms. We have $\|\omega\|_2^* = |\eta|_2$ for the harmonic form $\eta \in \omega$ [LM89], and in particular the norm $\|\cdot\|_2^*$ is Euclidean. We will consider the invariant $\lambda_1(L)\lambda_1(L^*)$ for the lattice $L = H_1(M; \mathbb{Z})_{\mathbb{R}} \subset H_1(M; \mathbb{R})$.

Lemma 5.1. *Let $\omega \in H^1(M; \mathbb{Z})_{\mathbb{R}}$ be a cohomology class whose modulo 2 reduction $\bar{\omega} \in H^1(M; \mathbb{Z}_2)$ is nonzero. Then*

$$\text{sys}_{n-1}(\mathcal{G}; \mathbb{Z}_2) \leq \|\omega\|_2^* (\text{vol}_n(\mathcal{G}))^{1/2}.$$

Proof. Let $\eta \in \omega$ be the harmonic 1-form for the metric \mathcal{G} on M . Then η can be represented as df for some map

$$f : M \rightarrow S^1 = \mathbb{R}/\mathbb{Z}.$$

Using the Cauchy-Schwartz inequality, we obtain

$$\|\omega\|_2^* (\text{vol}_n(\mathcal{G}))^{1/2} = |\eta|_2 (\text{vol}_n(\mathcal{G}))^{1/2} \geq \int_M |df| d \text{vol}_n, \quad (5.1)$$

where $|\cdot|$ is the pointwise norm defined by the Riemannian metric. We now use the coarea formula, *cf.* [Fe69, 3.2.11], [Ch93, p. 267]:

$$\int_M |df| d \text{vol}_n = \int_{S^1} \text{vol}_{n-1}(f^{-1}(t)) dt. \quad (5.2)$$

Note that, for every regular value t of f , the \mathbb{Z}_2 -homology class of the hypersurface $f^{-1}(t) \subset M$ is Poincaré \mathbb{Z}_2 -dual to $\bar{\omega}$. Hence,

$$\text{vol}_{n-1}(f^{-1}(t)) \geq \text{sys}_{n-1}(\mathcal{G}; \mathbb{Z}_2)$$

for all regular values t of f . By Sard's Theorem, the set of regular values of f has measure 1 in S^1 . Thus,

$$\int_M |df| d \text{vol}_n \geq \text{sys}_{n-1}(\mathcal{G}; \mathbb{Z}_2). \quad (5.3)$$

The lemma results by combining inequalities (5.1) and (5.3). \square

Proof of Theorem 4.3. Let $\|\cdot\|_2$ be the norm in homology dual to the L^2 -norm $\|\cdot\|_2^*$ in cohomology. Let $\alpha \in H_1(M; \mathbb{Z})_{\mathbb{R}}$ be an element of least norm, so that

$$\|\alpha\|_2 = \lambda_1(H_1(M; \mathbb{Z})_{\mathbb{R}}, \|\cdot\|_2).$$

Clearly, $\|\cdot\|_2^* \leq \|\cdot\|^* \text{vol}_n(\mathcal{G})^{1/2}$, and so, dually, $\|\alpha\| \leq \|\alpha\|_2 \text{vol}_n(\mathcal{G})^{1/2}$. Choose ω so that

$$\|\omega\|_2^* = \lambda_1(H^1(M; \mathbb{Z})_{\mathbb{R}}, \|\cdot\|_2^*).$$

By Lemma 5.1, we obtain

$$\begin{aligned} \text{stsys}_1(\mathcal{G}) \text{sys}_{n-1}(\mathcal{G}; \mathbb{Z}_2) &= \|\alpha\| \text{sys}_{n-1}(\mathcal{G}; \mathbb{Z}_2) \\ &\leq \|\alpha\| \|\omega\|_2^* \text{vol}_n(\mathcal{G})^{1/2} \\ &\leq \|\alpha\|_2 \|\omega\|_2^* \text{vol}_n(\mathcal{G}). \end{aligned}$$

The theorem now follows from the inequality

$$\begin{aligned} \|\alpha\|_2 \|\omega\|_2^* &= \lambda_1(H_1(M; \mathbb{Z})_{\mathbb{R}}, \|\cdot\|_2) \lambda_1(H^1(M; \mathbb{Z})_{\mathbb{R}}, \|\cdot\|_2^*) \\ &\leq \gamma'_b \end{aligned}$$

by Definition 4.1 of the Bergé-Martinet constant. The optimality of the inequality results by considering a suitable product metric on the product $\mathbb{T}^b \times \mathbb{RP}^2$, where \mathbb{T}^b is dual-critical. \square

6. POSITIVITY OF SYSTOLES

Proposition 6.1. *The homotopy 1-systole and the stable systoles defined in Section 3 are nonzero for all polyhedra X .*

Proof. We cover X by a finite number of open, contractible sets U_i . By the Lebesgue Lemma, there exists $\delta > 0$ such that every subset of X of diameter at most δ is contained in some U_i . Therefore the diameter of any non-contractible loop L must be more than δ , and thus the length of L must exceed 2δ , proving the positivity of the 1-systole.

For the stable k -systoles, the positivity follows by a ‘‘calibration’’ argument. Namely, suppose classes $\alpha \in H_k(X; \mathbb{Z})_{\mathbb{R}}$ and $\omega \in H^k(X; \mathbb{Z})_{\mathbb{R}}$ pair non-trivially and positively. (For a theory of differential forms on polyhedra see *e.g.* [Ba02].) Then

$$1 \leq \int_{\alpha} \omega \leq C \|\omega\|^* \text{vol}_k(\alpha),$$

and, moreover,

$$1 \leq \int_{\alpha} \omega \leq m^{-1} C \|\omega\|^* \text{vol}_k(\alpha(m))$$

for all $m \in \mathbb{N}$. Minimizing over all singular Lipschitz cycles $\alpha(m) \in m[\alpha_{\mathbb{R}}]$, we obtain the necessary bound $1 \leq C \|\omega\|^* \|\alpha_{\mathbb{R}}\|$. Now the result follows because the abelian group $H^k(X; \mathbb{Z})_{\mathbb{R}}$ is of finite rank. \square

For the ordinary k -systoles, one cannot use differential forms, as in the proof of Proposition 6.1, due to possible torsion classes in homology. Nevertheless, the positivity of sysh_k holds as well. In fact, H. Federer [Fe69, item 4.2.2(1)] proved that cycles with small mass are homologically trivial. However, we need a slightly stronger conclusion, to obtain a uniform lower bound for systoles of covering spaces. The following lemma is a consequence of the construction used in the proof of the deformation theorem of Federer and Fleming [FF60, Fe69], cf. [Wh99]. The proof was summarized in [Gr83, Prop. 3.1.A].

Lemma 6.2. *Let V be a k -dimensional polyhedron in \mathbb{R}^N . Then there exists a continuous map f of V into a $(k-1)$ -dimensional polyhedron K^{k-1} in \mathbb{R}^n such that*

$$\text{dist}(v, f(v)) \leq C_N(\text{vol } V)^{1/k}$$

for all $v \in V$ and for some constant C_N depending only on the ambient dimension, where dist denotes the Euclidean distance in \mathbb{R}^N . \square

Proposition 6.3. *The homology k -systoles $\text{sysh}_k(X, \mathcal{G}; A)$ defined in Section 3, Definition 3.2, are nonzero for all polyhedra X . In fact, a uniform lower bound for sysh_k is valid for all covering spaces of X .*

Proof. We may view X as a polyhedron in \mathbb{R}^N , since the metric \mathcal{G} is bilipschitz equivalent to the restricted metric. Note that X has a regular neighborhood $U \subset \mathbb{R}^N$, i.e. X is a deformation retract of U . Since X is compact, there exists $\varepsilon > 0$ such that $\text{dist}(X, \mathbb{R}^N \setminus U) > \varepsilon$. Let $j : X \rightarrow U$ denote the inclusion.

Note that we can regard any singular chain in X , with coefficients in A with $A = \mathbb{Z}$ or \mathbb{Z}_p , as a singular, not necessarily connected polyhedron. Consider a such a polyhedron $\varphi : F^k \rightarrow X$ representing a non-zero k -dimensional homology with coefficients in A . We now view φ as a singular polyhedron in \mathbb{R}^N .

Choose $\sigma > 0$ such that $C_N \sigma^{1/k} < \varepsilon$ where C_N is the constant from Lemma 6.2. Assume that $\text{vol}(\varphi(F)) < \sigma$. Then by Lemma 6.2, there is a map $f : \varphi(F) \rightarrow K^{k-1} \subset \mathbb{R}^N$ such that $\text{dist}(a, f(a)) < \varepsilon$ for all $a \in \varphi(F)$. Thus $f(\varphi(F)) \subset U$. Moreover, for all $a \in \varphi(F)$, the segment joining a and $f(a)$ is contained in U . Therefore the maps

$$F \xrightarrow{\varphi} X \xrightarrow{j} U \quad \text{and} \quad F \xrightarrow{\varphi} \varphi(F) \xrightarrow{f} U$$

are homotopic. But $f(\varphi(F)) \subset K^{k-1}$, and so $f \circ \varphi : F \rightarrow U$ represents the zero element in $H_k(U; A)$. Hence $j \circ \varphi$ represents the zero element in $H_k(U; A)$. But then φ represents the zero element in $H_k(X; A)$. This is a contradiction.

The deformation of F into a polyhedron of positive codimension is contained in U . Since X is a retract of U , and because of the Cellular Approximation Theorem, we can map the deformation into X in a way that the last-moment map sends F into the $(k - 1)$ -skeleton $X^{(k-1)}$ of X .

It follows that the same δ works for the systoles of arbitrary covering spaces of X , by the covering homotopy property, *cf.* [BaCIK07, Section 2]. \square

7. HOMOTOPY INVARIANCE OF SYSTOLIC CATEGORY

The homotopy invariance of systolic category follows from the techniques developed by I. Babenko in [Ba93], *cf.* the compression theorem of [KR06, Theorem 5.1]. More precisely, we have the following theorem, proved in [Ba93] for the 1-systole by essentially the same argument.

Theorem 7.1. *The optimal systolic ratio associated with a partition of $n = \dim(M)$ is a homotopy invariant of a closed manifold M .*

Proof. Let $f : M^n \rightarrow N^n$ be a homotopy equivalence of closed PL manifolds. By A. Wright [Wr74, Theorem 7.3], f is homotopic to a PL monotone map. Recall that a continuous map is called monotone if the inverse image of every point is connected and compact. Thus, we can assume the every n -dimensional simplex of N has exactly one inverse image simplex. Then we can pull back systolic inequalities from one manifold to the other, in the following sense. The pullback metric has *the same volume* as the target metric. Meanwhile, the projection map is distance decreasing, and therefore the target manifold has smaller systoles than the source manifold.

Now suppose that M satisfies a systolic inequality

$$\prod_i \text{sys}_{k_i}(M) \leq C \text{vol}_n(M), \tag{7.1}$$

relative to a suitable partition $n = k_1 + \dots + k_d$, for *all* metrics, with a constant C . In particular, it satisfies it for all pullback metrics. Then the target manifold will satisfy the same inequality (with the same partition of n) with exactly the same constant C . Thus the associated optimal systolic ratio, which is the least such constant C , is a homotopy invariant. \square

Since Wright's result is not available for polyhedra, we have to make do with a fixed simplicial map. The disadvantage here is that the pullback metric may have greater volume than the target, but anyway it is controlled by the number of simplices in the inverse image of a

top dimensional simplex. Thus the optimal systolic ratio is no longer a homotopy invariant, but nevertheless the homotopy invariance of systolic category persists, in the following sense.

Theorem 7.2. *Given two n -dimensional polyhedra X and Y , assume that there exists a simplicial map $f : X \rightarrow Y$ that induces an isomorphism in π_1 and a monomorphism in homology with coefficients in $\mathbb{Z}[B]$ and $\mathbb{Z}_2[B]$, where B runs over all groups of regular covering maps of X (and therefore Y). Then $\text{cat}_{\text{sys}} X \leq \text{cat}_{\text{sys}} Y$.*

Proof. The case $\text{cat}_{\text{sys}} X = 0$ is trivial, so assume that $\text{cat}_{\text{sys}} X > 0$. Consider a partition $n = k_1 + \dots + k_d$. Suppose that X satisfies a systolic inequality

$$\prod_i \text{sys}_{k_i}(X, \mathcal{G}) \leq C(X) \text{vol}_n(X, \mathcal{G})$$

for all piecewise Riemannian metrics. Choose a piecewise Riemannian metric on Y and consider the pull back (degenerate) metric \mathcal{H} induced by f on X . By the monomorphism hypothesis, we have $\text{sys}_k(X, \mathcal{H}) \geq \text{sys}_k(Y)$. Meanwhile, $\text{vol}_n(X, \mathcal{H}) \leq k \text{vol}_n(Y)$ where k is number of n -simplices in X . A small perturbation of \mathcal{H} will yield a non-degenerate metric \mathcal{G} , satisfying the inequality $\text{vol}_n(X, \mathcal{G}) \leq 2k \text{vol}_n(Y)$. Moreover, we can also assume that $\text{sys}_k(X, \mathcal{G}) \geq \text{sys}_k(Y) - \delta$ where $\delta \leq \text{sys}_k(X)$ for all k . Hence,

$$\prod \text{sys}_{k_i}(Y) \leq \prod (\text{sys}_{k_i}(X, \mathcal{G}) + \delta) \leq 2C \text{vol}_n(X, \mathcal{G}) \leq 4kC \text{vol}_n(Y),$$

and thus $\text{cat}_{\text{sys}} X \leq \text{cat}_{\text{sys}} Y$. \square

We have the following two immediate corollaries.

Corollary 7.3. *The systolic category of n -dimensional polyhedra is a homotopy invariant.* \square

Corollary 7.4. *Let X be an n -dimensional polyhedron which is homotopy equivalent to a polyhedron Y of dimension at most $n - 1$. Then $\text{cat}_{\text{sys}} X = 0$.*

8. NEW DIRECTIONS

Question 8.1. In the context of higher Massey products, is there a generalization of the result [KR06, Theorem 11.1], to a case where triple products vanish but there is a higher nontrivial product?

Question 8.2. Considering finite covers which are intermediate between the free abelian cover and the manifold. Is it true that if the

fiber class is non-zero in the free abelian cover, then it is already non-trivial in a finite cover? In such case we would immediately get a lower bound of $b_1(X) + 1$ for cat_{LS} , which parallels the bound for systolic category resulting from [IK04].

Question 8.3. Consider an (absolute) degree 1 map $f : X \rightarrow T^2$ from a non-orientable surface to the torus. Consider also the product map

$$f : X \times S^2 \rightarrow T^2 \times S^2.$$

The range has systolic category 3, by real cup length argument. However, the domain has real cup length only 2. Therefore we do not have an immediate lower bound of 3 for systolic category, unlike LS category. Is there such a bound? What is the systolic category of $\mathbb{RP}^2 \times S^2$?

Question 8.4. Given a function f on a manifold, how does $\text{cat}_{\text{sys}}(f^c)$ of the sublevel set f^c change as a function of c ?

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