

# Determinants of Matrices from Pascal's Triangle

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## Abstract

When devising problems for a linear algebra course it is desirable to have an extensive source of integral matrices of determinant one. In this short paper we give a straightforward method of generating some such matrices from binomial coefficients.

## 1 Introduction

We present a simple result on the determinant of certain matrices devised from Pascal's triangle. The inspiration behind this result was the quest for a ready supply of suitable matrices for linear algebra problem sets. The goal being integral matrices that are invertible over  $\mathbb{Z}$  so that one can test knowledge of row and column operations whilst minimising the risk of computational errors.

Experimentation led to the conjecture that the following procedure worked: write Pascal's triangle in left-justified form and choose a square block which abuts the left-hand edge.

Using the theory of numerical polynomials we are able to prove an expanded version of this initial conjecture. The procedure now works as follows:

1. Write Pascal's triangle in left-justified form. We label the columns starting with zero so that the  $i$ th column corresponds to  $\binom{n}{i}$ :

1	0	0	...
1	1	0	...
1	2	1	...
1	3	3	...
1	4	6	...

2. Extend the table in the vertical direction by defining  $\binom{n}{i} := n(n-1)\cdots(n-i+1)/i!$  for negative  $n$ . We have the symmetry formula:  $\binom{n}{i} = (-1)^i \binom{i-n-1}{i}$ .

1	-2	3	...
1	-1	1	...
1	0	0	...
1	1	0	...
1	2	1	...
1	3	3	...
1	4	6	...

- Choose a size of matrix, say  $k$ .
- Shift each of the columns  $1, \dots, k-1$  vertically by some arbitrary amount (technically one can do this for the zeroth column as well but of course it makes no difference).

$$\begin{array}{cccc} 1 & 3 & 0 & \cdots \\ 1 & 4 & 0 & \cdots \\ 1 & 5 & 1 & \cdots \\ 1 & 6 & 3 & \cdots \\ 1 & 7 & 6 & \cdots \end{array}$$

- From the resulting matrix, choose a  $k \times k$  block which abuts the left-hand edge:

$$\begin{pmatrix} 1 & 5 & 1 \\ 1 & 6 & 3 \\ 1 & 7 & 6 \end{pmatrix}$$

- This matrix will have determinant 1:

$$\begin{aligned} & 1 \cdot 6 \cdot 6 + 5 \cdot 3 \cdot 1 + 1 \cdot 1 \cdot 7 - 1 \cdot 3 \cdot 7 - 5 \cdot 1 \cdot 6 - 1 \cdot 6 \cdot 1 \\ & = 36 + 15 + 7 - 21 - 30 - 6 \\ & = 1. \end{aligned}$$

We actually go a little further than this and examine what happens if after shifting the columns one selects elements from each column periodically, with the period varying with the column. For example, one might take every 3rd element from the first column, every 5th from the second, and so on. The resulting determinant depends only on these selection rules and the order in which they are applied. Specifically, if in the  $i$ th column one takes every  $l_i$ th term the resulting matrix has determinant  $\prod l_i^{l_i}$ .

## 2 Numerical Polynomials

The key theorem is more general than the above discussion implies, although it is not complicated. We have concentrated in the introduction on the application to binomial coefficients due to the original motivation and due to their simplicity. The proof depends on the structure of the ring  $\text{Int}(\mathbb{Z})$  of rational polynomials which map  $\mathbb{Z}$  to itself. This ring has been extensively studied and much is known about it. Perhaps most surprising is its applications in the field of algebraic topology. The work in this paper depends on the following result:

**Proposition 2.1** For  $i \geq 0$ , define the polynomial  $b_i(w) \in \mathbb{Q}[w]$  by:

$$b_i(w) := \frac{w(w-1)\cdots(w-i+1)}{i!}$$

then  $b_i(w) \in \text{Int}(\mathbb{Z})$  and the sequence  $(b_i(w))$  is a basis for  $\text{Int}(\mathbb{Z})$  as a  $\mathbb{Z}$ -module.

*Proof.* That  $b_i(w)$  lies in  $\text{Int}(\mathbb{Z})$  comes from the fact that its value at  $k \in \mathbb{N}$  is the integer  $\binom{k}{i}$ , whilst at negative  $k$ , its value is  $(-1)^i \binom{i-k-1}{i}$ , also an integer.

As  $b_i(w)$  has degree  $i$ , the sequence  $(b_i(w))$  is a  $\mathbb{Q}$ -basis for  $\mathbb{Q}[w]$ . Therefore there cannot any relations between the  $b_i(w)$  in  $\mathbb{Q}[w]$  and hence neither can there be any relations between them in  $\text{Int}(\mathbb{Z})$ . Hence the  $\mathbb{Z}$ -submodule of  $\text{Int}(\mathbb{Z})$  spanned by the sequence  $(b_i(w))$  is free and has  $(b_i(w))$  as a  $\mathbb{Z}$ -basis. It remains to show that this is the whole of  $\text{Int}(\mathbb{Z})$ .

We need to show that a non-zero element in  $\text{Int}(\mathbb{Z})$  can be written as a  $\mathbb{Z}$ -linear combination of the  $b_i(w)$ . As the  $b_i(w)$  are a  $\mathbb{Q}$ -basis for  $\mathbb{Q}[w]$  we already know that any such element can be written as a  $\mathbb{Q}$ -linear combination of the  $b_i(w)$ . We just need to show that the coefficients have to be integers. Define the *length* of a non-zero polynomial  $f(w) \in \text{Int}(\mathbb{Z})$  to be the smallest integer  $k$  such that  $f(w)$  is a  $\mathbb{Q}$ -linear combination of  $\{b_i(w), \dots, b_{i+k}(w)\}$  for some  $i$ . Thus:

$$f(w) = c_i b_i(w) + \dots + c_{i+k} b_{i+k}(w)$$

for  $c_j \in \mathbb{Q}$  with  $c_i$  and  $c_{i+k}$  non-zero. We shall proceed by induction on this value.

For the initial step, let  $f(w) \in \text{Int}(\mathbb{Z})$  have length zero. Then there is some  $i$  and  $c_i \in \mathbb{Q}$  such that  $f(w) = c_i b_i(w)$ . Now  $b_i(i) = \binom{i}{i} = 1$  so evaluating this at  $i$  yields the identity  $c_i = f(i)$  which, as  $f(w) \in \text{Int}(\mathbb{Z})$ , lies in  $\mathbb{Z}$ . Hence  $f(w)$  is a  $\mathbb{Z}$ -linear combination of the  $b_i(w)$ .

Now suppose that whenever  $g(w) \in \text{Int}(\mathbb{Z})$  has length  $k$  then  $g(w)$  lies in the  $\mathbb{Z}$ -linear span of the  $b_i(w)$ . Let  $f(w) \in \text{Int}(\mathbb{Z})$  have length  $(k+1)$  and write:

$$f(w) = c_i b_i(w) + \dots + c_{i+k+1} b_{i+k+1}(w),$$

where  $i \geq 0$  is chosen appropriately. Evaluate this at  $i$ . On the left-hand side we have  $f(i)$  which is an integer. On the right-hand side the first term evaluates to  $c_i$  as  $b_i(i) = 1$ . The rest of the terms evaluate to zero as  $b_j(i) = \binom{i}{j} = 0$  for  $j > i$ . Hence  $c_i \in \mathbb{Z}$ . Let  $g(w) = f(w) - c_i b_i(w)$ . This lies in  $\text{Int}(\mathbb{Z})$  and has length  $k$  hence is in the  $\mathbb{Z}$ -linear span of the  $b_i(w)$ . Consequently, so does  $f(w)$ .  $\square$

This basis is particularly nice as it is filtered by degree. Thus if we have another sequence of polynomials,  $(f_i(w))$ , known to lie in  $\text{Int}(\mathbb{Z})$ , such that  $f_i(w)$  is of degree  $i$  then  $f_i(w)$  is a  $\mathbb{Z}$ -linear combination of  $\{b_0(w), \dots, b_i(w)\}$  only. Hence if we gather together all the coefficients into a matrix equation:

$$\begin{pmatrix} f_0(w) \\ f_1(w) \\ f_2(w) \\ \vdots \end{pmatrix} = (c_{ij}) \begin{pmatrix} b_0(w) \\ b_1(w) \\ b_2(w) \\ \vdots \end{pmatrix}$$

then the coefficient matrix is lower triangular. We can therefore truncate this at any  $k \geq 0$  to produce:

$$\begin{pmatrix} f_0(w) \\ f_1(w) \\ \vdots \\ f_k(w) \end{pmatrix} = (c_{ij}) \begin{pmatrix} b_0(w) \\ b_1(w) \\ \vdots \\ b_k(w) \end{pmatrix}.$$

This holds as an equation for polynomials so it also holds if we evaluate at any point. Thus if we choose a finite sequence of integers  $(l_0, \dots, l_k)$  we see that:

$$\begin{pmatrix} f_0(l_0) & f_0(l_1) & \cdots & f_0(l_k) \\ f_1(l_0) & f_1(l_1) & \cdots & f_1(l_k) \\ \dots & \dots & \dots & \dots \\ f_k(l_0) & f_k(l_1) & \cdots & f_k(l_k) \end{pmatrix} = (c_{ij}) \begin{pmatrix} b_0(l_0) & b_0(l_1) & \cdots & b_0(l_k) \\ b_1(l_0) & b_1(l_1) & \cdots & b_1(l_k) \\ \dots & \dots & \dots & \dots \\ b_k(l_0) & b_k(l_1) & \cdots & b_k(l_k) \end{pmatrix}.$$

As the matrix  $(c_{ij})$  is lower triangular it is simple to read off the determinant of the matrix  $(f_i(l_j))$  in terms of the determinant of the matrix  $(b_i(l_j))$ .

**Theorem 2.2** Let  $(f_0(w), \dots, f_k(w))$  be a finite sequence of polynomials with the property that  $f_i(j) \in \mathbb{Z}$  for any  $j \in \mathbb{Z}$  and  $f_i(w)$  has degree  $i$ . Let  $(l_0, \dots, l_k)$  be a sequence of integers. Then the determinant of the matrix  $(f_i(l_j))$  is the product of the integer  $\prod_{j=0}^k f_j^{(j)}(0)$  and the determinant of the matrix  $(b_i(l_j))$ , where  $b_i(w)$  is the  $i$ th Newtonian polynomial.

*Proof.* All we need to show is that the diagonal of the coefficient matrix is the sequence  $(f_j^{(j)}(0))$ . The integer  $c_{jj}$  is the coefficient of  $b_j(w)$  in the expression of  $f_j(w)$  as a linear combination of  $\{b_0(w), \dots, b_j(w)\}$ . As  $b_i(w)$  has degree  $i$  we can kill off all the lower terms by taking the  $j$ th derivative on each side to leave  $f_j^{(j)}(w) = c_{jj}b_j^{(j)}(w)$ . A simple calculation reveals that  $b_j^{(j)}(w)$  is the constant polynomial 1 and hence  $f_j^{(j)}(w)$  is the constant polynomial  $c_{jj}$ . Equivalently,  $c_{jj} = f_j^{(j)}(l)$  for any number  $l$ ; in particular,  $c_{jj} = f_j^{(j)}(0)$ .  $\square$

Note that although we express the determinant in terms of the  $j$ th derivative of  $f_j(w)$  this is just a fancy way of talking about  $j!$  times by the coefficient of  $w^j$  in  $f_j(w)$ . It is often simplest to calculate this directly.

### 3 Shifting Pascal's Triangle

We now put in to theorem 2.2 various sequences of polynomials to determine the determinants of the corresponding matrices. The easiest sequence of integers at which to evaluate our polynomials is  $(0, 1, \dots, k)$ . This is because the matrix  $(b_i(j))_{0 \leq i, j \leq k}$  is upper triangular with leading diagonal all 1s and hence has determinant 1.

**Corollary 3.1** Let  $k \in \mathbb{N}$  and let  $l_1, \dots, l_k, n_0, \dots, n_k \in \mathbb{Z}$ . Let  $B$  be the  $(k+1) \times (k+1)$  matrix:

$$B = \begin{pmatrix} 1 & n_1 & \binom{n_2}{2} & \cdots & \binom{n_k}{k} \\ 1 & n_1 + l_1 & \binom{n_2 + l_2}{2} & \cdots & \binom{n_k + l_k}{k} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n_1 + l_1 k & \binom{n_2 + l_2 k}{2} & \cdots & \binom{n_k + l_k k}{k} \end{pmatrix},$$

then  $\det B = \prod_{i=1}^k l_i^i$ . In particular, if  $l_i = 1$  for all  $i$  then  $\det B = 1$  and if  $l_i = l$  for all  $i$  then  $\det B = l^{\frac{1}{2}k(k+1)}$ .

*Proof.* First we observe that the matrix  $(b_i(j))$  is upper triangular with leading diagonal 1s. Hence it has determinant 1.

Let  $f_0(w) = 1$  and for  $1 \leq i \leq k$ , let  $f_i(w) = b_i(l_i w + n_i)$ . As  $\text{Int}(\mathbb{Z})$  is closed under composition,  $f_i(w) \in \text{Int}(\mathbb{Z})$ . Clearly  $f_i(w)$  is of degree  $i$  and, by the

chain rule,  $f_i^{(i)}(0) = l_i^i$ . Hence the matrix  $(f_i(j))_{0 \leq i, j \leq k}$  has determinant  $\prod l_i^i$ . By construction,  $f_i(j) = \binom{n_i + l_i j}{i}$ . The matrix  $B$  is the transpose of this.  $\square$

The column of 1s in this matrix is mildly irritating. Using the fact that the  $b_i(w)$  for  $i \geq 1$  have zero constant term we can impose a mild condition on the  $f_i(w)$  in theorem 2.2 to remove the first column. In fact, we can go somewhat further and use the fact that  $b_i(j) = 0$  for  $i > j$  to remove the first  $l$  columns under a vanishing assumption.

**Proposition 3.2** *Let  $0 \leq m < n$  be integers. Let  $\{f_i(w), m \leq i \leq n\} \subseteq \text{Int}(\mathbb{Z})$  be such that  $\deg f_i(w) = i$  and  $f_i(j) = 0$  for  $j < m$ . Then the integral matrix  $(f_i(j))_{m \leq i, j \leq n}$  has determinant  $\prod f_i^{(i)}(w)$ .*

*Proof.* We augment our family by throwing in  $f_i(w) = b_i(w)$  for  $i < m$  and apply theorem 2.2. The assumption on the  $f_i(w)$  for  $i \geq m$  means that the first  $m$  columns of the matrix  $(f_i(j))_{0 \leq i, j \leq k}$  are:

$$\begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Thus its determinant is the same as the submatrix formed by removing the first  $m$  rows and columns, which is  $(f_i(j))_{m \leq i, j \leq n}$ .  $\square$

We can of course alter any family to this form by replacing  $f_i(w)$  by  $f_i(w) - \sum_{j=0}^{l-1} c_{i,j} b_j(w)$ . For  $l = 1$  we get  $f_i(w) - f_i(0)$ . Applying this to the family in corollary 3.1 leaves the matrix:

$$\begin{pmatrix} l_1 & \binom{n_2 + l_2}{2} - \binom{n_2}{2} & \dots & \binom{n_k + l_k}{k} - \binom{n_k}{k} \\ 2l_1 & \binom{n_2 + 2l_2}{2} - \binom{n_2}{2} & \dots & \binom{n_k + 2l_k}{k} - \binom{n_k}{k} \\ \dots & \dots & \dots & \dots \\ kl_1 & \binom{n_2 + kl_2}{2} - \binom{n_2}{2} & \dots & \binom{n_k + kl_k}{k} - \binom{n_k}{k} \end{pmatrix}.$$

Notice that  $n_1$  is now redundant. We have, essentially, done the first obvious row reduction on the original matrix.

Another interesting sequence to apply this to, although it does not yield a matrix with determinant  $\pm 1$ , comes from taking products of the  $b_i(w)$ . Let  $f_i(w) = b_{i-m}(w)b_m(w)$  for some fixed  $m$  and for  $m \leq i \leq n$ . This satisfies the required conditions since  $b_m(j) = 0$  for  $j < m$ . The determinant of the matrix  $(f_i(j))_{m \leq i, j \leq n}$  is:

$$\prod_{j=m}^n \binom{j}{m}.$$

More generally, given a sequence  $(k_m, \dots, k_n)$  with  $m \leq k_i \leq i$  define  $f_i(w) := b_{i-k_m}(w)b_{k_m}(w)$ . The matrix  $(f_i(j))_{m \leq i, j \leq n}$  has determinant:

$$\prod_{j=m}^n \binom{j}{k_j}.$$

In a similar vein we can consider compositions of the  $b_i(w)$ . Let  $(k_m, \dots, k_n)$  and  $(l_m, \dots, l_n)$  be finite sequences of positive integers such that for each  $j$ ,  $k_j l_j = j$  and  $l_j \geq m$ . Let  $f_i(w) = b_{k_i}(b_{l_i}(w))$ . This satisfies the required conditions since  $b_{l_i}(j) = 0$  for  $j < m$  and, if  $m > 0$ ,  $b_{k_i}(0) = 0$ . The matrix  $(f_i(j))_{m \leq i, j \leq n}$  has determinant:

$$\prod_{j=m}^n \frac{j!}{k_j!} \left( \frac{1}{l_j!} \right)^{k_j}.$$