Intuitionistic fuzzy geometric interaction averaging operators and their application to multi-criteria decision making

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Abstract

This paper proposes some new geometric operations on intuitionistic fuzzy sets (IFSs) based on probability non-membership (PN) function operator, probability membership (PM) function operator and probability heterogeneous (PH) operator, which are constructed from the probability point of view. The geometric interpretations of these operations are given. Moreover, we develop some intuitionistic fuzzy geometric interaction averaging (IFGIA) operators. The properties of these aggregation operators are investigated. The key advantage of the IFGIA operators is that the interactions between non-membership function and membership function of different IFSs are considered. Finally, an approach to multiple attributes decision making is given based on the proposed aggregation operators under intuitionistic fuzzy environment, and an example is illustrated to show the validity and feasibility of the proposed approach.

1. Introduction

Intuitionistic fuzzy sets (IFSs), as a generalization of fuzzy sets [42], developed by Atanassov [3], is a powerful tool to deal with vagueness.

Since information aggregation is a pervasive activity in daily life, many researches had been done on this issue [7,15,24,27,32,37,34,35,30,39,38,18,26]. Among them, the weighted geometric averaging (WGA) operator [24] and the ordered weighted geometric averaging (OWGA) operator [34] are the most common operators. On a basis of the multiplication operation by Atanassov [4] and power operation by De and Biswas [12] on intuitionistic fuzzy sets, Xu and Yager [37] proposed some intuitionistic fuzzy geometric aggregation operators and applied them to multi-attribute decision making problems. After these pioneering works, more attentions have been paid to intuitionistic fuzzy multi-criteria decision making problems [1,2,5,6,10,11,13,14,19–23,25,29,33,36,41,43–51]. Dymova and Sevastjinov [13] presented a method to deal with the intuitionistic fuzzy multi-criteria decision making problems based on Dempster–Shefer theory of evidence. Ye [40] used entropy weight to get criteria weights and rank alternatives according to the correlation coefficients. Xu [33] developed intuitionistic fuzzy power aggregation operators. Xu and Xia [36] presented the induced generalized aggregation operators under interval-valued intuitionistic fuzzy environments. Zhu et al. [51] proposed hesitant fuzzy geometric Bonferroni means.

Li et al. [19] investigated the relationship between the similarity measure and the entropy of IFSs. Zhang et al. [45] presented

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the intuitionistic fuzzy rough approximation operators and discussed their connections with special intuitionistic fuzzy relations.

However, it is found that the operational laws and geometric aggregation operators on intuitionistic fuzzy sets in [4,37] are not suitable to be used in the special circumstances. For example, suppose that A and B are two intuitionistic fuzzy sets, $A = \langle u_A, v_A \rangle$ and $u_A = 0, B = \langle u_B, v_B \rangle$ and $u_B \neq 0$, then according to the multiplication operation by Atanassov [4], we have $u_{A \cdot B} = 0$. Obviously, $u_B$ is not accounted for at all, which is an undesirable feature of an averaging operation. Furthermore, the IFHGIA operator [37] has similar problems. For example, if $A_i = \langle u_{A_i}, v_{A_i} \rangle, i = 1, 2, \ldots, n, i \neq k$ are a collection of intuitionistic fuzzy sets, $A_k = \langle 0, v_{A_k} \rangle$, and $v_{A_k} \neq 0$, then we have $u_{IFHGIA(A_1, \ldots, A_k)} = 0$ by using aggregation law in [37]. It is obvious that $u_{A_i} (i = 1, 2, \ldots, n, i \neq k)$ have no effects on the aggregation result.

Motivated by the works of [4,37] and the idea of interactions between non-membership function and membership function of different intuitionistic fuzzy sets, we focus on developing some new geometric operations on intuitionistic fuzzy sets (IFSs) and giving the geometric interpretations of these operations. Based on the new operations, we propose some intuitionistic fuzzy geometric interaction aggregation operations, including the IFWGIA operator, the IFOWGIA operator and the IFHGIA operator, which are more practical for an averaging operator. By the comparison with the existing method, it is concluded that the method proposed in this paper is a good complement to the existing works on IFSs, especially when one of the membership degrees of intuitionistic fuzzy sets is zero.

The rest of the paper is organized as follows. Section 2 reviews some basic concepts. Section 3 introduces new geometric operations on intuitionistic fuzzy sets and gives the geometric interpretations of these operations. In Section 4, we develop the intuitionistic fuzzy geometric interaction averaging (IFGIA) operators, and investigate their properties. In Section 5, an approach to intuitionistic fuzzy multi-criteria decision making is given based on the proposed IFGIA operator. In Section 6, a numerical example is illustrated to show the feasibility and validity of the new approach, and the comparison between the work of this paper and other corresponding works is presented systematically. Finally, Section 7 concludes the paper.

2. Preliminaries

The concept of fuzzy sets (FSs) was introduced by Zadeh [42]. Let X be a universe of discourse in the following.

**Definition 1** [42]. A fuzzy set $F$ in $X$ is defined as follows: $F = \{(x, u_F(x))|x \in X\}$, where $u_F: X \rightarrow [0,1]$ is the membership function of the fuzzy set $F$, and $0 \leq u_F(x) \leq 1$.

Atanassov [3] generalized the fuzzy set to intuitionistic fuzzy set (IFS) by adding an hesitation degree.

**Definition 2** [3]. An intuitionistic fuzzy set in $X$ is an expression: $A = \{(x, u_A(x), v_A(x))|x \in X\}$, where the functions $u_A: X \rightarrow [0,1]$ and $v_A: X \rightarrow [0,1]$ define the degree of membership and the degree of nonmembership of the element $x \in X$ to A, and for every $x \in X$, $0 \leq u_A(x) + v_A(x) \leq 1$.

For each IFS $A$ in $X$, if $\pi_A(x) = 1 - u_A(x) - v_A(x)$, for all $x \in X$, then $\pi_A(x)$ is called the degree of indeterminacy of the element $x$ to the set $A$.

In practice, intuitionistic fuzzy numbers can be denoted as $A = (u, v)$ [32,37]. For convenience, the sets of all the intuitionistic fuzzy numbers are denoted by IFNs.

Some basic operations, such as multiplication operation [4] and power operation [12], were introduced under intuitionistic fuzzy environment.

**Definition 3** [4,12]. Suppose that $A = \langle u_A, v_A \rangle$ and $B = \langle u_B, v_B \rangle$ are two intuitionistic fuzzy sets, then

1. $A \otimes B = \langle u_A \cdot u_B \cdot v_A + v_B - v_A \cdot v_B \rangle$
2. $A^k = \langle u_A^k, 1 - (1 - u_A)^k \rangle, k > 0$

Chen and Tan [9] proposed a score function $S(A) = u_A - v_A$ to evaluate the degree of suitability that an alternative satisfies a decision maker’s requirement under intuitionistic fuzzy environment, where $A$ is an intuitionistic fuzzy set, and $A = \langle u_A, v_A \rangle$. The score of $A$ is directly related to the deviation between $u_A$ and $v_A$, i.e., the bigger the score of intuitionistic fuzzy set $A$, the larger the intuitionistic fuzzy set $A$.

Hong and Choi [16] presented an accuracy function $H(A) = u_A + v_A$ to evaluate the accuracy degree of the intuitionistic fuzzy set $A = \langle u_A, v_A \rangle$, where $0 \leq H(A) \leq 1$. The larger the value of $H(A)$, the higher the accuracy degree of intuitionistic fuzzy set $A$ [32].

Based on score function [9] and accuracy function [16], Xu [32,37] gave the comparison law for intuitionistic fuzzy sets as follows.

**Definition 4** [32,37]. Let $A = \langle u_A, v_A \rangle$ and $B = \langle u_B, v_B \rangle$ be two intuitionistic fuzzy sets. Then $A < B$ if and only if

(i) $S(A) < S(B)$, or
(ii) $S(A) = S(B)$ and $H(A) < H(B)$. 

3. New geometric operational laws on intuitionistic fuzzy sets

3.1. New geometric operational laws and corresponding geometric interpretations

Considering that the existing multiplication operation [4] on IFSs cannot be used in some cases from averaging point of view, we introduce new operational laws on intuitionistic fuzzy sets, including multiplication operation and power operation. The feature of new geometric operational laws is that we take the interactions into consideration between non-membership function and membership function of different intuitionistic fuzzy sets.

**Definition 5.** Suppose that \( A = (u_A, v_A) \) and \( B = (u_B, v_B) \) are two intuitionistic fuzzy numbers, then

1) \( A \odot B = \{(1 - v_A)(1 - v_B) - (1 - (u_A + v_A))(1 - (u_B + v_B)), 1 - (1 - v_A)(1 - v_B)\} \)

2) \( A^i = \{(1 - u_A)^i - (1 - (u_A + v_A))^i, 1 - (1 - u_A)^i\}, \lambda > 0 \)

The geometric meaning of new multiplication operation on intuitionistic fuzzy sets can be interpreted from three aspects as follows.

- The operational rule between non-membership function and non-membership function of different IFSs.
- The operational rule between membership function and membership function of different IFSs.
- The operational rule between non-membership function and membership function of different IFSs.

Let \( A = (u_A, v_A) \) and \( B = (u_B, v_B) \) be two intuitionistic fuzzy numbers.

(1) The operational rule between non-membership function and non-membership function can be explained geometrically as follows in Fig. 1.

\( v_A \) and \( v_B \) denote the non-membership degree of \( A \) and \( B \), respectively. From the probability point of view, we may as well regard \( v_A \) and \( v_B \) as two independent events. \( E(v_A, v_B) \) represents the probability of \( v_A \) and \( v_B \) occurring simultaneously. Thus, \( E(v_A, v_B) = v_A \cdot v_B \). Therefore, \( \nu_{A \odot B} = \nu_A + \nu_B - \nu_A \cdot \nu_B \). \( \nu_{A \odot B} \) is called the probability non-membership (PN) function operator, i.e.,

\[
PN(v_A, v_B) = v_A + v_B - v_A \cdot v_B
\]

(5)

(2) The operational rule between membership function and membership function can be explained geometrically as follows in Fig. 2.

\( u_A \) and \( u_B \) denote the membership degree of \( A \) and \( B \), respectively. We may as well regard \( u_A \) and \( u_B \) as two independent events, \( E(u_A, u_B) \) represents the probability of \( u_A \) and \( u_B \) occurring simultaneously. Thus, \( E(u_A, u_B) = u_A \cdot u_B \). Therefore, \( \nu_{A \odot B} = u_A + u_B - u_A \cdot u_B \). \( \nu_{A \odot B} \) is called the probability membership (PM) function operator, i.e.,

\[
PM(u_A, u_B) = u_A + u_B - u_A \cdot u_B
\]

(6)

(3) The operational rule between non-membership function and membership function can be explained geometrically as follows in Figs. 3 and 4.

\( v_A \) denotes the non-membership degree of \( A \), \( u_B \) denotes the membership degree of \( B \). Similarly, \( E(v_A, u_B) \) represents the probability of two independent events \( v_A \) and \( u_B \) occurring simultaneously. Thus, \( E(v_A, u_B) = v_A \cdot u_B \). \( E(v_A, u_B) \) is called probability heterogeneous (PH) function operator, i.e.,

\[
PH(v_A, u_B) = u_B \cdot v_A
\]

(7)

In similar way, \( E(v_B, u_A) \) denotes the probability of two independent events \( v_B \) and \( u_A \) occurring simultaneously. Thus, \( E(v_B, u_A) = v_B \cdot u_A \). \( E(v_B, u_A) \) is called the probability heterogeneous (PH) function operator, i.e.

\[
PH(v_B, u_A) = u_A \cdot v_B
\]

(8)

(4) The interaction between different intuitionistic fuzzy sets can be explained geometrically as follows in Fig. 5.
From Fig. 5 and Eqs. (5)–(8), the multiplication of different IFNs can be rewritten as follows.

\[ A \otimes B = \frac{PM(u_A, u_B) - PH(u_A, v_B) - PH(u_B, v_A)}{PH(v_A, u_B)} \]

From Fig. 5 and Eq. (9), it is obvious that the non-membership function of \( A \otimes B \) contains \( PH(v_A, u_B) \) and \( PH(v_B, u_A) \), while the membership function of \( A \otimes B \) does not contain \( PH(v_A, u_B) \) and \( PH(v_B, u_A) \).

In fact, if \( A, B \in \text{IFNs}, A = (u_A, v_A), u_A \neq 1 \) and \( u_A = 0, B = (u_B, v_B) \) and \( u_B = 0 \), then according to Eq. (9), we have

\[ PM(u_A, u_B) - PH(u_A, v_B) - PH(u_B, v_A) = u_B - u_B \cdot v_A 
\]

which shows that \( u_A \) cannot play a decisive role. As a result, the proposed Definition 5 is more practical for an averaging operator, which can be seen visually by the following example.

**Example 1.** Suppose that \( A = (u_A, v_A) = (0.6, 0.6) \) and \( B = (u_B, v_B) = (0.6, 0.2) \) are two intuitionistic fuzzy numbers, then by Definition 3, we have

\[ A \otimes B = (0 \cdot 0.6, 0.6 + 0.2 - 0.6 \cdot 0.2) = (0, 0.68). \]

According to Definition 5, we get that
\[ A \otimes B = (0 + 0.6 - 0.2 - 0.6 - 0.6, 0.6, 0.6 + 0.2 - 0.6, 0.2) = (0.24, 0.68). \]

i.e., \( u_{A \otimes B} = 0.24 \), which indicates that \( u_b \) does not play a decisive role.

It is noted that
\[
P_N(u_A, u_b) = v_A + v_b - v_A \cdot v_B = 1 - (1 - v_A) \cdot (1 - v_B),
\]
and
\[
P_M(u_A, u_B) = PH(u_A, v_B) - PH(u_B, v_A) = (u_A + u_B - u_A \cdot u_B) + (v_B - v_A \cdot v_B - (u_A \cdot v_B) - (u_B \cdot v_A)) - PN(v_A, v_B)
\]
\[
= (1 - (1 - (u_A + v_B)) \cdot (1 - (u_B + v_A))) - (1 - 1 - (1 - v_B) \cdot (1 - v_B))
\]
\[
= (1 - v_A) \cdot (1 - v_B) - (1 - (u_A + v_B)) \cdot (1 - (u_B + v_B)).
\]

Therefore, Eq. (9) is equivalent to Eq. (3).

Next, we explain the origin of the power operation.

By Eq. (3), we have
\[
A^2 = A \otimes A = (1 - v_A)^2 - (1 - (u_A + v_A))^2, 1 - (1 - v_A)^2.
\]

Similarly, we have
\[
A^3 = A^2 \otimes A = (1 - v_A)^3 - (1 - (u_A + v_A))^3, 1 - (1 - v_A)^3.
\]

By analogy, for any positive integer \( n \), we have
\[
A^n = A^{n-1} \otimes A = (1 - v_A)^n - (1 - (u_A + v_A))^n, 1 - (1 - v_A)^n.
\]

We extend positive integer \( n \) to any nonnegative real number \( \lambda \), and have
\[
A^\lambda = (1 - v_A)^\lambda - (1 - (u_A(x) + v_A(x)))^\lambda, 1 - (1 - v_A)^\lambda, \lambda > 0.
\]

3.2. The properties of the new geometric operations

In this subsection, we investigate some properties of the \( PN \) function operator, the \( PM \) function operator, the \( PH \) function operator, the new multiplication operation and power operation under intuitionistic fuzzy environment.

**Theorem 1.** Let \( A = (u_A, v_A) \), \( B = (u_B, v_B) \), \( C = (u_C, v_C) \), \( A' = (u_{A'}, v_{A'}) \) and \( B' = (u_{B'}, v_{B'}) \) be five intuitionistic fuzzy numbers, then

1. **Boundedness:** \( PN(1, 1) = 1, PN(0, 0) = 0, 0 \leq PN(v_A, v_B) \leq 1. \)
2. **Monotonicity:** if \( v_A \leq u_{A'} \) and \( v_B \leq u_{B'} \), then \( PN(v_A, v_B) \leq PN(v_{A'}, v_{B'}). \)
3. **Commutativity:** \( PN(v_A, v_B) = PN(v_B, v_A). \)
4. **Associativity:** \( PN(PN(v_A, v_B), v_C) = PN(PN(v_A, v_B), v_C). \)
5. **Zero element:** \( PN(0, v_B) = PN(v_A, 0) = v_B. \)

**Proof.** (1) According to Eq. (5), we have
\[
P_N(v_A, v_B) = v_A + v_B - v_A \cdot v_B = 1 - (1 - v_A)(1 - v_B)
\]

Therefore,
\[
PN(1, 1) = 1 + 1 - 1 \times 1 = 1, \quad PN(0, 0) = 1 - 1 = 0.
\]

It is noted that \( 0 \leq v_A \leq 1, 0 \leq v_B \leq 1. \) Thus, \( 0 \leq (1 - v_A)(1 - v_B) \leq 1 \), then according to Eq. (10), we have \( 0 \leq PN(v_A, v_B) \leq 1. \)

(2) Because \( v_A \leq u_{A'} \) and \( v_B \leq u_{B'} \), we have \( 1 - v_A \geq 1 - u_{A'}, \quad 1 - v_B \geq 1 - u_{B'} \).

According to Eq. (10), we obtain
\[
P_N(v_A, v_B) \leq PN(v_{A'}, v_{B'}).
\]

(4) According to Eq. (10), we get
\[
P_N(v_A, PN(v_B, v_C)) = 1 - (1 - v_A)(1 - PN(v_B, v_C))
\]

and
\[
1 - PN(v_B, v_C) = (1 - v_B)(1 - v_C).
\]

Thus, \( PN(v_A, PN(v_B, v_C)) = 1 - (1 - v_A)(1 - v_B)(1 - v_C). \)
Similarly, \( PN(PN(v_a, v_b), v_c) = 1 - (1 - v_a)(1 - v_b)(1 - v_c) \).

Therefore, \( PN(v_a, PN(v_b, v_c)) = PN(PN(v_a, v_b), v_c) \).

According to Eq. (5), we can get the results of (3) and (5) easily. So the proofs are omitted here. \( \square \)

**Theorem 2.** Let \( A = (u_a, v_a) \), \( B = (u_b, v_b) \), \( C = (u_c, v_c) \), \( A' = (u_{a'}, v_{a'}) \) and \( B' = (u_{b'}, v_{b'}) \) be five intuitionistic fuzzy numbers, then we have

1. **Boundedness:** \( PH(0, 0) = 0, PH(1, 1) = 1, 0 \leq PH(v_a, v_b) \leq 1 \).
2. **Monotonicity:** If \( u_a \leq u'_a \) and \( v_b \leq v'_b \), then \( PH(u_a, v_b) \leq PH(u'_a, v_b) \); and if \( v_a \leq v'_a \) and \( u_b \leq u'_b \), then \( PH(v_a, u_b) \leq PH(v'_a, u_b) \).
3. **Commutativity:** \( PH(u_a, v_b) = PH(v_b, u_a) \).
4. **Associativity:** \( PH(u_a, PH(u_b, u_c)) = PH(PH(u_a, u_b), u_c) \).
5. **Identify element:** \( PH(1, u_a) = PH(u_a, 1) = u_a \).

**Proof.** According to Eq. (8), we can get Theorem 2 easily, so it is omitted here. \( \square \)

In practice, the operational results of geometric operational laws on intuitionistic fuzzy numbers are also intuitionistic fuzzy numbers.

**Theorem 3.** If \( A = (u_a, v_a) \in IFNs \) and \( B = (u_b, v_b) \in IFNs \), then \( C = (u_c, v_c) \), \( D = (u_d, v_d) \), then \( C, D \in IFNs \).

The proof of theorem 3 is similar to that in [37], so it is omitted here.

**Theorem 4.** Let \( A = (u_a, v_a) \) and \( B = (u_b, v_b) \) be two intuitionistic fuzzy numbers, \( \lambda, \lambda_1, \lambda_2 > 0 \). Then, we have

1. \( A \circ B = B \circ A \);
2. \( (A \circ B)^{1/2} = A^{1/2} \circ B^{1/2} \);
3. \( A^{(1/2)} \circ A^{(1/2)} = A \).

The proof of theorem 4 is similar to that in [37], so it is omitted here.

4. **Intuitionistic fuzzy geometric interaction averaging operators**

4.1. **Intuitionistic fuzzy weighted geometric averaging (IFWGA) operator**

Xu and Yager [37] generalized the WGA operators and the OWGA operators to intuitionistic fuzzy environment and proposed the intuitionistic fuzzy weighted geometric averaging (IFWGA) operator and the intuitionistic fuzzy ordered weighted geometric averaging (IFOWGA) operator.

**Definition 6** [37]. Let \( A_i = (u_{a_i}, v_{a_i})(i = 1, 2, \ldots, n) \) be a collection of intuitionistic fuzzy numbers. If the mapping

\[
IFWGA_w(A_1, A_2, \ldots, A_n) = A_1^{w_1} \circ A_2^{w_2} \circ \cdots \circ A_n^{w_n} = \left( \prod_{i=1}^{n} u_{a_i}^{w_i}, 1 - \prod_{i=1}^{n} (1 - v_{a_i})^{w_i} \right),
\]

then \( IFWGA_w \) is called intuitionistic fuzzy weighted geometric averaging operator with respect to a weighting vector \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighting vector of \( A_i \) with \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \).

**Definition 7** [37]. Let \( A_i = (u_{a_i}, v_{a_i})(i = 1, 2, \ldots, n) \) be a collection of intuitionistic fuzzy numbers. If the mapping

\[
IFOWGA_w(A_1, \ldots, A_n) = (A_{a(1)})^{w_1} \circ \cdots \circ (A_{a(n)})^{w_n} = \left( \prod_{i=1}^{n} u_{a(i)}^{w_i}, 1 - \prod_{i=1}^{n} (1 - v_{a(i)})^{w_i} \right),
\]

then \( IFOWGA_w \) is called intuitionistic fuzzy ordered weighted geometric averaging operator with respect to a weighting vector \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighting vector with \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \).

4.2. **Intuitionistic fuzzy weighted geometric interaction averaging (IFWGA) operator**

Motivated by the WGA operator [24] and the IFWGA operator [37], we develop intuitionistic fuzzy weighted geometric interaction averaging (IFWGA) operator based on the proposed operational laws.
**Definition 8.** Let \( A_i = (u_{A_i}, v_{A_i})(i = 1, \ldots, n) \) be a collection of intuitionistic fuzzy numbers. If the mapping

\[
IFWGA_w(A_1, \ldots, A_n) = \bigotimes_{i=1}^{n} A_i^{w_i},
\]

then \( IFWGA_w \) is called intuitionistic fuzzy weighted geometric interaction averaging (IFWGIA) operator, where \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighting vector of \( A_i \) with \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \).

**Theorem 5.** Let \( A_i = (u_{A_i}, v_{A_i})(i = 1, \ldots, n) \) be a collection of intuitionistic fuzzy numbers. Then,

\[
IFWGA_w(A_1, \ldots, A_n) = \left\{ \prod_{i=1}^{n} (1 - v_{A_i})^{w_i} - \prod_{i=1}^{n} (1 - (u_{A_i} + v_{A_i}))^{w_i}, 1 - \prod_{i=1}^{n} (1 - v_{A_i})^{w_i} \right\}.
\]

**Proof.** We prove Eq. (12) by using mathematical induction on \( n \).

1. When \( n = 1, w_1 = 1 \), we have \( IFWGA_w(A_1) = A_1^{w_1} = (u_{A_1}, v_{A_1}) = (1 - v_{A_1})^{-1} - (1 - (u_{A_1} + v_{A_1}))^{-1}, 1 - (1 - v_{A_1})^{-1} \).
   Thus, Eq. (12) holds for \( n = 1 \).

2. If Eq. (12) holds for \( n = k \), i.e.,

\[
IFWGA_w(A_1, A_2, \ldots, A_k) = \left\{ \prod_{i=1}^{k} (1 - v_{A_i})^{w_i} - \prod_{i=1}^{k} (1 - (u_{A_i} + v_{A_i}))^{w_i}, 1 - \prod_{i=1}^{k} (1 - v_{A_i})^{w_i} \right\}.
\]

Then, when \( n = k + 1 \), by the operational laws in **Definition 5**, we have

\[
IFWGA_w(A_1, A_2, \ldots, A_{k+1}) = \bigotimes_{i=1}^{k+1} A_i^{w_i}
= IFWGA_w(A_1, A_2, \ldots, A_k) \bigotimes (A_{k+1})^{w_{k+1}}
= \left\{ \prod_{i=1}^{k} (1 - v_{A_i})^{w_i} - \prod_{i=1}^{k} (1 - (u_{A_i} + v_{A_i}))^{w_i}, 1 - \prod_{i=1}^{k} (1 - v_{A_i})^{w_i} \right\}
\bigotimes (1 - v_{A_{k+1}})^{w_{k+1}} - (1 - (u_{A_{k+1}} + v_{A_{k+1}}))^{w_{k+1}}, 1 - (1 - v_{A_{k+1}})^{w_{k+1}}
\]

\[
\bigotimes (1 - v_{A_{k+1}})^{w_{k+1}} - (1 - (u_{A_{k+1}} + v_{A_{k+1}}))^{w_{k+1}}, 1 - (1 - v_{A_{k+1}})^{w_{k+1}}
\]

i.e. Eq. (12) holds for \( n = k + 1 \).

Therefore, by using mathematical induction on \( n \), Eq. (12) holds for all \( n \). \( \square \)

**Theorem 6.** If \( A_i = (u_{A_i}, v_{A_i}) \in IFNs, i = 1, 2, \ldots, n \), then the aggregated value by using the IFWGIA operator is also an intuitionistic fuzzy number, i.e. \( IFWGA(A_1, \ldots, A_n) \in IFNs \).

**Proof.** Since \( A_i = (u_{A_i}, v_{A_i}) \in IFNs, i = 1, 2, \ldots, n \), by Definition 2, we have

\[ 0 \leq u_{A_i}, v_{A_i} \leq 1 \text{ and } 0 \leq u_{A_i} + v_{A_i} \leq 1, \]

then

\[ 0 \leq 1 - \prod_{i=1}^{n} (1 - v_{A_i})^{w_i} \leq 1, \quad 0 \leq \prod_{i=1}^{n} (1 - v_{A_i})^{w_i} - \prod_{i=1}^{n} (1 - (u_{A_i} + v_{A_i}))^{w_i} \leq 1 \]

and

\[
\left( 1 - \prod_{i=1}^{n} (1 - v_{A_i})^{w_i} \right) + \left( \prod_{i=1}^{n} (1 - v_{A_i})^{w_i} - \prod_{i=1}^{n} (1 - (u_{A_i} + v_{A_i}))^{w_i} \right) = 1 - \prod_{i=1}^{n} (1 - (u_{A_i} + v_{A_i}))^{w_i} \in [0, 1].
\]

Thus, \( IFWGA(A_1, \ldots, A_n) \in IFNs \). \( \square \)

**Theorem 7.** Let \( A_i = (u_{A_i}, v_{A_i})(i = 1, \ldots, n) \) and \( B_i = (u_{B_i}, v_{B_i})(i = 1, \ldots, n) \) be two collections of intuitionistic fuzzy numbers and \( w = (w_1, w_2, \ldots, w_n)^T \) is the associated weighting vector satisfying \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \).
(1) **Idempotency**: If \( A_i = A_0 = (u_{A_i}, v_{A_i}) \) for all \( i \), then \( \text{IFWGIA}_w(A_1, A_2, \ldots, A_n) = A_0 \).

(2) **Boundedness**: Let \( A^+ = (\max(0, (\min(u_{A_i} + v_{A_i}) - \max(v_{A_i}))), \max(v_{A_i})) \).

Then we have \( A^+ \leq \text{IFWGIA}_w(A_1, A_2, \ldots, A_n) \leq A^* \).

(3) **Monotonicity**: When \( v_{A_i} = v_{B_i} \), \( u_{A_i} + v_{A_i} \leq u_{B_i} + v_{B_i} \) for all \( i \), we have \( \text{IFWGIA}_w(A_1, A_2, \ldots, A_n) \leq \text{IFWGIA}_w(B_1, B_2, \ldots, B_n) \).

**Proof.** (1) Since \( A_i = A_0 = (u_{A_i}, v_{A_i}) (i = 1, \ldots, n) \) and \( \sum_{i=1}^n w_i = 1 \), by Theorem 5, we have

\[
\text{IFWGIA}_w(A_1, A_2, \ldots, A_n) = \left( \prod_{i=1}^n (1 - u_{A_i})^{w_i} - \prod_{i=1}^n (1 - (u_{A_i} + v_{A_i}))^{w_i}, 1 - \prod_{i=1}^n (1 - v_{A_i})^{w_i} \right)
= \left( (1 - u_{A_0})^{\sum w_i} - (1 - (u_{A_0} + v_{A_0}))^{\sum w_i}, 1 - (1 - v_{A_0})^{\sum w_i} \right)
= (u_{A_0}, v_{A_0}) = A_0.
\]

(2) Since \( \max(v_{A_i}) = 1 - (1 - \max(v_{A_i}))^{\sum_{i=1}^n w_i} \geq 1 - \prod_{i=1}^n (1 - v_{A_i})^{w_i} \), and

\[
1 - \prod_{i=1}^n (1 - u_{A_i})^{w_i} \geq 1 - \prod_{i=1}^n (1 - \min(v_{A_i}))^{w_i} = 1 - (1 - \min(v_{A_i}))^{\sum_{i=1}^n w_i} = \min(v_{A_i}),
\]

\[
\max(u_{A_i} + v_{A_i}) - \min(v_{A_i}) = (1 - \min(v_{A_i}))^{\sum_{i=1}^n w_i} - (1 - \max(u_{A_i} + v_{A_i}))^{\sum_{i=1}^n w_i}
= (1 - \max(v_{A_i}))^{\sum_{i=1}^n w_i} - (1 - \min(u_{A_i} + v_{A_i}))^{\sum_{i=1}^n w_i}
= \min(u_{A_i} + v_{A_i}) - \max(v_{A_i}).
\]

According to Theorem 6, \( \text{IFWGIA}(A_1, \ldots, A_n) \in \text{IFNs} \), we have

\[
\prod_{i=1}^n (1 - u_{A_i})^{w_i} - \prod_{i=1}^n (1 - (u_{A_i} + v_{A_i}))^{w_i} \geq 0.
\]

Therefore,

\[
\prod_{i=1}^n (1 - u_{A_i})^{w_i} - \prod_{i=1}^n (1 - (u_{A_i} + v_{A_i}))^{w_i} \geq \max(0, (\min(u_{A_i} + v_{A_i}) - \max(v_{A_i}))).
\]

Then, according to Definition 4, we obtain that

\[
A^- \leq \text{IFWGIA}_w(A_1, A_2 \ldots A_n) \leq A^+.
\]

(3) Since \( v_{A_i} = v_{B_i} \), we have \( 1 - \prod_{i=1}^n (1 - v_{A_i})^{w_i} \leq 1 - \prod_{i=1}^n (1 - v_{B_i})^{w_i} \).

Because \( v_{A_i} \geq v_{B_i} (i = 1, 2, \ldots, n) \) and \( u_{A_i} + v_{A_i} \leq u_{B_i} + v_{B_i} (i = 1, 2, \ldots, n) \), we have

\[
\prod_{i=1}^n (1 - v_{A_i})^{w_i} - \prod_{i=1}^n (1 - (u_{A_i} + v_{A_i}))^{w_i} \leq \prod_{i=1}^n (1 - v_{B_i})^{w_i} - \prod_{i=1}^n (1 - (u_{B_i} + v_{B_i}))^{w_i}.
\]

Therefore, according to Definition 4, we get

\[
\text{IFWGIA}_w(A_1, A_2, \ldots, A_n) \leq \text{IFWGIA}_w(B_1, B_2, \ldots, B_n).
\]

In some special cases, the IFWGIA operator reduces to the IFWG operator. \( \square \)

**Remark 1.** If \( w = (1, 0, \ldots, 0)^T \), then \( \text{IFWGIA}_w(A_1, A_2, \ldots, A_n) = \text{IFWGIA}_w(A_1, A_2, \ldots, A_n) \).

**Remark 2.** If \( w = (0, 0, \ldots, 1)^T \), then \( \text{IFWGIA}_w(A_1, A_2, \ldots, A_n) = \text{IFWGIA}_w(A_1, A_2, \ldots, A_n) \).
Remark 3. If \( w_i = 1, w_j = 0, \) and \( j \neq i, \) then we have \( IFOWGIA_w(A_1, A_2, \ldots, A_n) = IFOWGIA_w(A_1, A_2, \ldots, A_n) \).

4.3. Intuitionistic fuzzy ordered weighted geometric interaction averaging (IFOWGIA) operator

Inspired by the OWGA operator [34] and the IFOWGA operator [37], we propose intuitionistic fuzzy ordered weighted geometric interaction averaging (IFOWGIA) operator.

Definition 9. Let \( A_i = (u_{A_i}, v_{A_i}) (i = 1, \ldots, n) \) be a collection of intuitionistic fuzzy numbers. If

\[
IFOWGIA_w(A_1, \ldots, A_n) = \odot_{i=1}^{n} (A_{\sigma(i)})^w,
\]

then \( IFOWGIA_w \) is called intuitionistic fuzzy ordered weighted geometric interaction averaging operator, where \( A_{\sigma(i)} \) is the ith largest value of \( A_i (i = 1, \ldots, n) \), \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighting vector of the \( IFOWGIA \) operator, with \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \).

Theorem 8. Let \( A_i = (u_{A_i}, v_{A_i}), i = 1, 2, \ldots, n \) be a collection of intuitionistic fuzzy numbers. Then,

\[
IFOWGIA_w(A_1, \ldots, A_n) = \left\langle \prod_{i=1}^{n} \left( 1 - v_{A_{\sigma(i)}} \right)^{w_i} - \prod_{i=1}^{n} \left( 1 - (u_{A_{\sigma(i)}} + v_{A_{\sigma(i)}}) \right)^{w_i} \right\rangle,
\]

(13)

The proof is similar to Theorem 5, so it is omitted here.

Theorem 9. Let \( A_i = (u_{A_i}, v_{A_i}) \in \text{IFNs}, i = 1, 2, \ldots, n \). Then the aggregated value by using the \( IFOWGIA \) operator is also an intuitionistic fuzzy number, i.e.,

\[
IFOWGIA(A_1, \ldots, A_n) \in \text{IFNs}.
\]

Proof. Since \( A_{\sigma(i)} = (u_{A_{\sigma(i)}}, v_{A_{\sigma(i)}}) \in \text{IFN}, i = 1, 2, \ldots, n \), according to Definition 2, we have

\[
0 \leq u_{A_{\sigma(i)}} + v_{A_{\sigma(i)}} \leq 1,\quad 0 \leq u_{A_{\sigma(i)}} \leq 1,\quad 0 \leq v_{A_{\sigma(i)}} \leq 1.
\]

Therefore,

\[
0 \leq 1 - \prod_{i=1}^{n} \left( 1 - v_{A_{\sigma(i)}} \right)^{w_i} \leq 1,\quad 0 \leq \prod_{i=1}^{n} \left( 1 - u_{A_{\sigma(i)}} \right)^{w_i} \leq 1
\]

and

\[
1 - \prod_{i=1}^{n} \left( 1 - u_{A_{\sigma(i)}} \right)^{w_i} - \prod_{i=1}^{n} \left( 1 - u_{A_{\sigma(i)}} + v_{A_{\sigma(i)}} \right)^{w_i}
\]

\[
= 1 - \prod_{i=1}^{n} \left( 1 - u_{A_{\sigma(i)}} + v_{A_{\sigma(i)}} \right)^{w_i} \in [0, 1].
\]

Thus, \( IFOWGIA(A_1, \ldots, A_n) \in \text{IFNs.} \)

Theorem 10. Suppose that \( A_i = (u_{A_i}, v_{A_i}) \) and \( B_i = (u_{B_i}, v_{B_i}) \) are two intuitionistic fuzzy numbers, \( i = 1, \ldots, n \). And \( w = (w_1, w_2, \ldots, w_n)^T \) is the weighting vector of the \( IFOWGIA \) operator, satisfying \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \).

1. Idempotency: If \( A_i = A_0 = (u_{A_0}, v_{A_0}) \) for all \( i, \) then \( IFOWGIA_w(A_1, A_2, \ldots, A_n) = A_0. \)

2. Boundedness: If \( A^+ = (\max(0, (\min(u_{A_i} + v_{A_i}) - \max(v_{A_i}))), \max(v_{A_i})), \)

\[
A^+ = (\max(u_{A_i} + v_{A_i}) - \min(v_{A_i}), \min(u_{A_i})), \text{ then}
\]

\[
A^+ \leq IFOWGIA_w(A_1, A_2, \ldots, A_n) \leq A^+.
\]

3. Commutativity: Suppose that \( A_i = (u_{A_i}, v_{A_i}) (i = 1, \ldots, n) \) is any permutation of \( A_i = (u_{A_i}, v_{A_i}) (i = 1, \ldots, n) \), then

\[
IFOWGIA_w(A_1, A_2, \ldots, A_n) = IFOWGIA_w(A_1, A_2, \ldots, A_n).
\]
Proof. (1) Since \( A_i = A_0 \) for all \( i \), and \( \sum_{i=1}^{n} w_i = 1 \), according to Theorem 8, we have

\[
IFOWGIA_w(A_1, \ldots, A_n) = \left( \prod_{i=1}^{n} (1 - v_{A_i})^w, \prod_{i=1}^{n} (1 - (u_{A_i} + v_{A_i}))^w, 1 - \prod_{i=1}^{n} (1 - v_{A_i})^w \right)
\]

\[
= \left( \prod_{i=1}^{n} (1 - u_{A_i})^w - \prod_{i=1}^{n} (1 - (u_{A_i} + v_{A_i}))^w, 1 - \prod_{i=1}^{n} (1 - u_{A_i})^w \right)
\]

\[
= \left( 1 - \sum_{i=1}^{n} v_{A_i}^w - \sum_{i=1}^{n} (u_{A_i} + v_{A_i})^w, 1 - \sum_{i=1}^{n} (1 - u_{A_i})^w \right)
\]

\[
= (u_{A_0}, v_{A_0}) = A_0.
\]

(2) Obviously,

\[
\max(v_A) = 1 - (1 - \max(v_{A_0})) \sum_{i=1}^{n} w_i \geq 1 - \prod_{i=1}^{n} (1 - v_{A_0})^w
\]

\[
= 1 - \prod_{i=1}^{n} (1 - v_A)^w \geq 1 - \prod_{i=1}^{n} (1 - \min(v_{A_0}))^w = 1 - (1 - \min(v_{A_0})) \sum_{i=1}^{n} w_i = \min(v_A),
\]

i.e.,

\[
\max(v_A) \geq 1 - \prod_{i=1}^{n} (1 - v_{A_0})^w \geq \min(v_A).
\]

Similarly, we have

\[
\max(u_A + v_A) - \min(v_A) = (1 - \min(v_{A_0})) \sum_{i=1}^{n} w_i - (1 - \max(u_{A_0} + v_{A_0})) \sum_{i=1}^{n} w_i
\]

\[
\geq \prod_{i=1}^{n} (1 - v_{A_0})^w - \prod_{i=1}^{n} (1 - (u_{A_0} + v_{A_0}))^w \geq (1 - \max(v_{A_0})) \sum_{i=1}^{n} w_i - (1 - \min(u_{A_0} + v_{A_0})) \sum_{i=1}^{n} w_i
\]

\[
= \min(u_A + v_A) - \max(v_A),
\]

i.e.,

\[
\max(u_A + v_A) - \min(v_A) \geq \prod_{i=1}^{n} (1 - v_{A_0})^w - \prod_{i=1}^{n} (1 - (u_{A_0} + v_{A_0}))^w \geq \min(u_A + v_A) - \max(v_A)
\]

By Theorem 9, \( IFOWGIA(A_1, \ldots, A_n) \in IFNs \), we have

\[
\prod_{i=1}^{n} (1 - v_{A_0})^w - \prod_{i=1}^{n} (1 - (u_{A_0} + v_{A_0}))^w \geq 0.
\]

Therefore, \( \prod_{i=1}^{n} (1 - v_{A_0})^w - \prod_{i=1}^{n} (1 - (u_{A_0} + v_{A_0}))^w \geq \max\{0, (\min(u_A + v_A) - \max(v_A))\} \).

Then, according to Definition 4, we have

\[
A' \leq IFOWGIA_w(A_1, A_2, \ldots, A_n) \leq A'.
\]

(3) According to Eq. (13), we get

\[
IFOWGIA_w(A_1, \ldots, A_n) = \left( \prod_{i=1}^{n} (1 - u_{A_i})^w, \prod_{i=1}^{n} (1 - (u_{A_i} + v_{A_i}))^w, 1 - \prod_{i=1}^{n} (1 - u_{A_i})^w \right).
\]

\[
IFOWGIA_w(A_1, A_2, \ldots, A_n) = \left( \prod_{i=1}^{n} (1 - v_{A_i})^w, \prod_{i=1}^{n} (1 - (u_{A_i} + v_{A_i}))^w, 1 - \prod_{i=1}^{n} (1 - v_{A_i})^w \right).
\]

Since \( A_i = (u_{A_i}, v_{A_i}) \) \((i = 1, \ldots, n)\) is any permutation of \( A_i = (u_{A_i}, v_{A_i}) \) \(i = 1, \ldots, n\), then we have \( A_{\sigma(i)} = A_{\sigma(i)} \) \((i = 1, \ldots, n)\). Thus,

\[
IFOWGIA_w(A_1', A_2', \ldots, A_n') = IFOWGIA_w(A_1, A_2, \ldots, A_n).
\]

The \( IFOWGIA \) operator reduces to the \( IFOWGA \) operator when we take special cases of the weighting vector \( w \). \( \square \)

Remark 4. If \( w = (1, 0, \ldots, 0)^T \), then \( IFOWGIA_w(A_1, A_2, \ldots, A_n) = IFOWGA_w(A_1, A_2, \ldots, A_n) \).
Remark 5. If \( w = (0, 0, \ldots, 1)^T \), then \( IFOWGIA_w(A_1, A_2, \ldots, A_n) = IFOWGIA_w(A_1, A_2, \ldots, A_n) \).

Remark 6. If \( w_i = 1, w_j = 0 \), and \( j \neq i \), then
\[
IFOWGIA_w(A_1, A_2 \ldots A_n) = IFOWGIA_w(A_1, A_2 \ldots A_n) = A_{\sigma(i)},
\]
where \( A_{\sigma(i)} \) is the \( i \)th largest of \( A_i (i = 1, \ldots, n) \).

4.4. Intuitionistic fuzzy hybrid geometric interaction averaging Operator

Considering both the given intuitionistic fuzzy value and its ordered position, Xu and Yager [37] developed the intuitionistic fuzzy hybrid geometric averaging (IFHA) operator.

Definition 10. Let \( A_i = (u_{Ai}, v_{Ai}) (i = 1, \ldots, n) \) be a collection of intuitionistic fuzzy numbers. The IFHA operator with respect to weighting vector \( w = (w_1, w_2, \ldots, w_n) \) is defined as
\[
IFHGA_w(A_1, A_2, \ldots, A_n) = \left( \tilde{A}_{\sigma(1)} \right)^{w_1} \otimes \ldots \otimes \left( \tilde{A}_{\sigma(n)} \right)^{w_n} = \left( \prod_{i=1}^{n} w_i^{u_{Ai}}, 1 - \prod_{i=1}^{n} (1 - v_{Ai})^{w_i} \right),
\]
(14)
where \( \tilde{A}_{\sigma(i)} \) is the \( i \)th largest of the intuitionistic fuzzy values \( A_i = A_{\sigma(i)}, i = 1, \ldots, n \), satisfying \( \omega_i \in [0, 1] \) and \( \sum_{i=1}^{n} \omega_i = 1 \), \( n \) is the balancing coefficient.

Suppose that \( A_i = (u_{Ai}, v_{Ai}) \in IFNs, i = 1, 2, \ldots, n \). If \( u_{Ai} = 0 \) and \( u_{Ak} \neq 0 (i \neq k) \), then according to Definition 10, we have \( IFHGA_w(A_1, A_2, \ldots, A_n) = 0 \). Obviously, \( u_{Ai} (i \neq k), i = 1, 2, \ldots, n \) are not accounted for at all, which is an undesirable feature of an averaging operation. As a result, we propose the intuitionistic fuzzy hybrid geometric interaction averaging operator, taking interactions into consideration between membership function and non-membership function of different intuitionistic fuzzy sets.

Definition 11. Let \( A_i = (u_{Ai}, v_{Ai}) (i = 1, \ldots, n) \) be a collection of intuitionistic fuzzy numbers, \( \Omega \) be the set of all intuitionistic fuzzy numbers. The intuitionistic fuzzy hybrid geometric interaction averaging (IFHGA) operator of dimension \( n \) is a mapping \( IFHGA: \Omega^n \rightarrow \Omega \) which has an associated vector \( w = (w_1, w_2, \ldots, w_n)^T \), satisfying \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \) such that
\[
IFHGA_w(A_1, A_2, \ldots, A_n) = \bigotimes_{i=1}^{n} \left( \tilde{A}_{\sigma(i)} \right)^{w_i},
\]
where \( \tilde{A}_{\sigma(i)} \) is the \( i \)th largest of the intuitionistic fuzzy values \( A_i = A_{\sigma(i)}, i = 1, \ldots, n \), satisfying \( \omega_i \in [0, 1] \) and \( \sum_{i=1}^{n} \omega_i = 1 \), \( n \) is the balancing coefficient.

Theorem 11. Let \( A_i = (u_{Ai}, v_{Ai}) (i = 1, \ldots, n) \) be a collection of intuitionistic fuzzy numbers. Then
\[
IFHGA_w(A_1, A_2, \ldots, A_n) = \left\langle \prod_{i=1}^{n} [1 - v_{Ai}]^{w_i} - \prod_{i=1}^{n} [1 - (u_{Ai} + v_{Ai})]^{w_i}, 1 - \prod_{i=1}^{n} (1 - v_{Ai})^{w_i} \right\rangle.
\]
(15)
The proof is similar to Theorem 5, so it is omitted here.

The IFHGA operator can be interpreted from four aspects as follows.

1. It not only includes the interactions of non-membership function of different IFSSs, and that of membership function of different IFSSs, but also the interactions are involved between non-membership function and membership function of different IFSSs.
2. It weights the intuitionistic fuzzy numbers \( A_i = (u_{Ai}, v_{Ai}) (i = 1, \ldots, n) \) by the associated weights \( \omega = (\omega_1, \ldots, \omega_n)^T \) and a balancing coefficient \( n \), and then we can get the weighted intuitionistic fuzzy numbers \( A_{\sigma(1)}^{\omega(i)} (i = 1, \ldots, n) \).
3. It reorders the weighted intuitionistic fuzzy values \( A_{\sigma(1)}^{\omega(i)} (i = 1, \ldots, n) \) in descending order \( (\tilde{A}_{\sigma(1)}, \tilde{A}_{\sigma(2)}, \ldots, \tilde{A}_{\sigma(n)}) \), where \( \tilde{A}_{\sigma(i)} \) is the \( i \)th largest of the weighted intuitionistic fuzzy values \( A_i = A_{\sigma(i)}^{\omega(i)}, i = 1, \ldots, n \).
4. It considers the given intuitionistic fuzzy numbers and the ordered positions, and then the intuitionistic fuzzy numbers \( (\tilde{A}_{\sigma(i)})^{w_i} (i = 1, 2, \ldots, n) \) are aggregated into a collective one.

Suppose that \( A_i = (u_{Ai}, v_{Ai}) (i = 1, \ldots, n) \) is a collection of IFNs. If \( u_{Ai} = 0, u_{Ak} \neq 0 (i \neq k), i = 1, 2, \ldots, n \), then by Definition 11, we have \( IFHGA_w(A_1, A_2, \ldots, A_n) = 0 \). Thus, \( u_{Ai} \) does not play a decisive role. Therefore, the proposed IFHGA operator can be seen as a good complement to the existing IFHGIA operator, which can be seen clearly in the following example.

Example 2. Let \( A_1 = (0, 0.5), A_2 = (0.4, 0.2), A_3 = (0.5, 0.4), A_4 = (0.3, 0.3), A_5 = (0.7, 0.1) \) be five intuitionistic fuzzy numbers. \( \omega = (0.25, 0.20, 0.15, 0.18, 0.22)^T \) is the weight vector of \( A_i (i = 1, 2, \ldots, 5) \).

Then according to the operational law of Definition 3, we have
\[ \tilde{A}_1 = (0.5^{0.25}, 1 - (1 - 0.5)^{5^{0.25}}) = (0.0.5796), \quad \tilde{A}_2 = (0.4^{5^{0.20}}, 1 - (1 - 0.2)^{5^{0.20}}) = (0.4000, 0.2000), \]
\[ \tilde{A}_3 = (0.5^{5^{0.15}}, 1 - (1 - 0.4)^{5^{0.15}}) = (0.504, 0.318), \quad \tilde{A}_4 = (0.3^{5^{0.18}}, 1 - (1 - 0.3)^{5^{0.18}}) = (0.287, 0.275), \]
\[ \tilde{A}_5 = (1 - (1 - 0.1)^{5^{0.22}}, 0.7^{5^{0.22}}) = (0.7203, 0.1094). \]

According to Definition 3, we obtain that
\[ S(\tilde{A}_1) = -0.5796, \quad S(\tilde{A}_2) = 0.2000, \quad S(\tilde{A}_3) = 0.2763, \quad S(\tilde{A}_4) = 0.0638, \quad S(\tilde{A}_5) = 0.5660. \]

Obviously,
\[ S(\tilde{A}_3) > S(\tilde{A}_1) > S(\tilde{A}_2) > S(\tilde{A}_4) > S(\tilde{A}_5). \]

Therefore,
\[ \tilde{A}_{\sigma(1)} = (0.6755, 0.1094), \quad \tilde{A}_{\sigma(2)} = (0.5946, 0.3183), \quad \tilde{A}_{\sigma(3)} = (0.4000, 0.2000), \quad \tilde{A}_{\sigma(4)} = (0.3384, 0.2746), \]
\[ \tilde{A}_{\sigma(5)} = (0.0.5796). \]

Supposing that \( w = (w_1, w_2, \ldots, w_5)^T \) is determined by the normal distribution based method [31], then \( w = (0.112, 0.236, 0.304, 0.236, 0.112). \) By Definition 10, we get
\[ A = \text{IFHGIA}_{w,\omega}(A_1, A_2, A_3, A_4, A_5) = \left( \prod_{j=1}^{5} (u_{A_j})^{n_j}, 1 - \prod_{j=1}^{5} (1 - u_{A_j})^{n_j} \right) \]
\[ = (0.6755^{0.112} \cdot 0.5946^{0.236} \cdot 0.4^{0.304} \cdot 0.3384^{0.236} \cdot 0.0000^{0.304}, \]
\[ 1 - (1 - 0.1094)^{0.112} \cdot (1 - 0.3183)^{0.236} \cdot (1 - 0.2)^{0.304} \cdot (1 - 0.2746)^{0.236} \cdot (1 - 0.5796)^{0.112} \]
\[ = (0.0.2911). \]

While, by Definition 11 and Theorem 11, we have
\[ \tilde{A}_1 = \left( (1 - 0.5)^{5^{0.25}}, 1 - (1 - 0.5)^{5^{0.25}} \right) = (0.0.5796), \]
\[ \tilde{A}_2 = \left( (1 - 0.2)^{5^{0.20}}, 1 - (1 - 0.2)^{5^{0.20}} \right) = (0.4000, 0.2000), \]
\[ \tilde{A}_3 = \left( (1 - 0.4)^{5^{0.15}}, 1 - (1 - 0.4)^{5^{0.15}} \right) = (0.5039, 0.3183), \]
\[ \tilde{A}_4 = \left( (1 - 0.3)^{5^{0.18}}, 1 - (1 - 0.3)^{5^{0.18}} \right) = (0.2870, 0.2746), \]
\[ \tilde{A}_5 = \left( (1 - 0.1)^{5^{0.22}}, 1 - (1 - 0.1)^{5^{0.22}} \right) = (0.7203, 0.1094). \]

According to Definition 5, we have
\[ S(\tilde{A}_1) = -0.5796, \quad S(\tilde{A}_2) = 0.2000, \quad S(\tilde{A}_3) = 0.1856, S(\tilde{A}_4) = 0.0125, S(\tilde{A}_5) = 0.6109. \]

Obviously,
\[ S(\tilde{A}_3) > S(\tilde{A}_1) > S(\tilde{A}_2) > S(\tilde{A}_4) > S(\tilde{A}_5). \]

As a result, we have
\[ \tilde{A}_{\sigma(1)} = (0.7203, 0.1094), \quad \tilde{A}_{\sigma(2)} = (0.4000, 0.2000), \quad \tilde{A}_{\sigma(3)} = (0.5039, 0.3183), \quad \tilde{A}_{\sigma(4)} = (0.2870, 0.2746), \]
\[ \tilde{A}_{\sigma(5)} = (0.0.5796). \]

By Theorem 11, it follows that
\[ A = \text{IFHGIA}_{\omega,w}(A_1, A_2, A_3, A_4, A_5) \]
\[ = \left( \prod_{j=1}^{5} (1 - v_{A_j})^{n_j}, 1 - \prod_{j=1}^{5} (1 - v_{A_j})^{n_j} \right) = (0.4093, 0.2919). \]

It is evident that \( u_{\text{IFHGIA}_{\omega,w}(A_1, A_5)} = 0.4093 \neq 0. \) Therefore, \( u_{A_k} \) does not play a decisive role.

Practically, the \( \text{IFHGIA} \) operator is a generalization of the \( \text{IFWIGIA} \) operator or the \( \text{IFOWIGIA} \) operator.

**Remark 7.** If \( w = (w_1, w_2, \ldots, w_5)^T = \left( \frac{1}{5}, \ldots, \frac{1}{5} \right)^T \), the \( \text{IFHGIA} \) operator reduces to the \( \text{IFWIGIA} \) operator.

**Remark 8.** If \( \omega = (\omega_1, \omega_2, \ldots, \omega_5)^T = \left( \frac{1}{5}, \ldots, \frac{1}{5} \right)^T \), the \( \text{IFHGIA} \) operator reduces to the \( \text{IFOWIGIA} \) operator.
5. Multi-criteria decision making with the IFHGIA operator

For a multi-criteria decision making problem, it is assumed that \( X = \{x_1, x_2, \ldots, x_n\} \) is a set of alternatives, \( G = \{G_1, G_2, \ldots, G_m\} \) is a set of attributes with the associated weighting vector \( w = (w_1, w_2, \ldots, w_m)^T \), satisfying \( w_i \in [0, 1] \) and \( \sum_{i=1}^{m} w_i = 1 \).

Suppose that the characteristics of the alternatives \( x_i \) \( (i = 1, 2, \ldots, n) \) are represented by intuitionistic fuzzy sets \( A_{ij} = (u_{ij}, d_{ij}) \) \( (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m) \) in Table 1, where \( u_{ij} \) indicates the degree that the alternative \( x_i \) satisfies the attribute \( G_j \), \( d_{ij} \) denotes the degree that the alternative \( x_i \) does not satisfy the attribute \( G_j \), \( i = 1, 2, \ldots, n; j = 1, 2, \ldots, m \).

Then we have the following decision making method.

**Step 1:** Assume that \( \omega = (\omega_1, \omega_2, \ldots, \omega_m)^T \) is the weighting vector of \( A_{11}, A_{12}, \ldots, A_{im}, i = 1, 2, \ldots, n \), satisfying \( \omega_j \in [0, 1] \) and \( \sum_{j=1}^{m} \omega_j = 1 \), \( m \) is the balancing coefficient. By Definition 5, we get \( A_{ij} = A_{ij}^{\omega_j} \) \( (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m) \) in Table 2.

**Step 2:** By Definition 4, we get \( A_{i(j)} = A_{i(j)}^{m+1} \) \( (i = 1, 2, \ldots, n) \), where \( A_{i(j)}^{m+1} \) is the \( j \)th largest of \( A_{i1}, A_{i2}, \ldots, A_{im} \), \( i = 1, 2, \ldots, n \).

**Step 3:** According to the aggregation operators in Definition 11, we get the final intuitionistic fuzzy sets \( A_i \) \( (i = 1, 2, \ldots, n) \) by using the normal distribution based method [31] to determine weights of the IFOWGIA operator.

**Step 4:** According to Definition 4, we get the order of the final intuitionistic fuzzy sets \( A_i \) \( (i = 1, 2, \ldots, n) \).

**Step 5:** Rank all the alternatives \( x_i \) \( (i = 1, 2, \ldots, n) \) and select the best one (s).

6. Numerical example and systematic comparison

6.1. Numerical example

Assume that an investment company wants to invest a sum of money in the best option. Three possible alternatives are to be considered by analyzing the market.

- \( x_1 \) is a car company.
- \( x_2 \) is a food company.
- \( x_3 \) a computer company.

In order to assess these alternatives, the investors have brought panel data. After careful review of the information, they summarize the ability of companies with five attributes \( G = \{G_1, G_2, G_3, G_4, G_5\} \).

- \( G_1 \): The risk analysis.
- \( G_2 \): The growth analysis.
- \( G_3 \): The social-political impact analysis.
- \( G_4 \): The environmental impact analysis.
- \( G_5 \): The development of the society.

The three possible alternatives are to be evaluated by the intuitionistic fuzzy information in Table 3 under above five attributes.

| Intuitionistic fuzzy matrix \( (A_{ij})_{m \times m} \) |
|-----------------|----------------|----------------|-----------------|----------------|
| \( x_1 \) | \( A_{11} \) | \( A_{12} \) | \( \ldots \) | \( A_{im} \) |
| \( x_2 \) | \( A_{21} \) | \( A_{22} \) | \( \ldots \) | \( A_{2m} \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( x_n \) | \( A_{n1} \) | \( A_{n2} \) | \( \ldots \) | \( A_{nm} \) |

| Intuitionistic fuzzy matrix \( (A_{i(j)})_{m \times m} \) |
|-----------------|----------------|----------------|-----------------|----------------|
| \( x_1 \) | \( \hat{A}_{11} \) | \( \hat{A}_{12} \) | \( \ldots \) | \( \hat{A}_{1m} \) |
| \( x_2 \) | \( \hat{A}_{21} \) | \( \hat{A}_{22} \) | \( \ldots \) | \( \hat{A}_{2m} \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( x_n \) | \( \hat{A}_{n1} \) | \( \hat{A}_{n2} \) | \( \ldots \) | \( \hat{A}_{nm} \) |
Step 1: According to operational law in Definition 5, we obtain \( \tilde{A}_i = A_i^{000} \), where the balancing coefficient \( n = 5 \), \( \omega = (0.25, 0.20, 0.15, 0.18, 0.22) \) is the weighting vector of \( A_{i1}, A_{i2}, \ldots, A_{i5}, i = 1, 2, 3; j = 1, 2, \ldots, 5 \).

Step 2: By Definition 4, we get the score matrix \( (S(\tilde{A}_i))_{1,5} \) in Table 4.

Obviously,

\[
S(\tilde{A}_{15}) > S(\tilde{A}_{12}) > S(\tilde{A}_{13}) > S(\tilde{A}_{14}) > S(\tilde{A}_{11}),
\]

\[
S(\tilde{A}_{25}) > S(\tilde{A}_{22}) > S(\tilde{A}_{23}) > S(\tilde{A}_{24}) > S(\tilde{A}_{21})
\]

and

\[
S(\tilde{A}_{35}) > S(\tilde{A}_{32}) > S(\tilde{A}_{33}) > S(\tilde{A}_{34}) > S(\tilde{A}_{31}).
\]

Then, we get the intuitionistic fuzzy ordered matrix \( (\tilde{A}_{i}\sigma(i))_{1,5} \) in Table 5, where \( \tilde{A}_{i}\sigma(i)(i = 1, 2, 3; j = 1, 2, \ldots, 5) \) is the jth largest of \( A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}, i = 1, 2, 3 \).

Step 3: The weights \( w = (0.112, 0.236, 0.304, 0.236, 0.112) \) are determined by the normal distribution based method [31].

According to aggregation operators in Definition 9, we obtain the final intuitionistic fuzzy sets \( A_i(i = 1, 2, 3) \):

\[
A_1 = IFHGI_{w,0}(A_{11}, A_{12}, A_{13}, A_{14}, A_{15}) = \left( \prod_{j=1}^{5} \left[ 1 - (1 - \frac{1}{A_{i,j}})^{w_j} \right] \right) - \left( \prod_{j=1}^{5} \left[ 1 - (1 - \frac{1}{A_{i,j}})^{w_j} \right] \right),
\]

\[
= (0.4294, 0.2988),
\]

\[
A_2 = IFHGI_{w,0}(A_{21}, A_{22}, A_{23}, A_{24}, A_{25}) = \left( \prod_{j=1}^{5} \left[ 1 - (1 - \frac{1}{A_{i,j}})^{w_j} \right] \right) - \left( \prod_{j=1}^{5} \left[ 1 - (1 - \frac{1}{A_{i,j}})^{w_j} \right] \right),
\]

\[
= (0.4383, 0.3659),
\]

\[
A_3 = IFHGI_{w,0}(A_{31}, A_{32}, A_{33}, A_{34}, A_{35}) = \left( \prod_{j=1}^{5} \left[ 1 - (1 - \frac{1}{A_{i,j}})^{w_j} \right] \right) - \left( \prod_{j=1}^{5} \left[ 1 - (1 - \frac{1}{A_{i,j}})^{w_j} \right] \right),
\]

\[
= (0.3927, 0.4212).
\]

Step 4: According to Definition 4, we have

\[
S(A_1) = 0.1306, S(A_2) = 0.0724, S(A_3) = -0.0285.
\]

Thus, \( S(A_1) > S(A_2) > S(A_3) \).

Step 5: According to the scores in Step 4, we have

\[
S(A_1) > S(A_2) > S(A_3).
\]
Table 6
Score matrix \((S(\tilde{A}_i))_{3,5}\).

<table>
<thead>
<tr>
<th></th>
<th>(j = 1)</th>
<th>(j = 2)</th>
<th>(j = 3)</th>
<th>(j = 4)</th>
<th>(j = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i = 1)</td>
<td>-0.5796</td>
<td>0.2000</td>
<td>0.2763</td>
<td>0.0638</td>
<td>0.5660</td>
</tr>
<tr>
<td>(i = 2)</td>
<td>-0.6442</td>
<td>0.3000</td>
<td>0.2683</td>
<td>0.0698</td>
<td>0.4607</td>
</tr>
<tr>
<td>(i = 3)</td>
<td>-0.6442</td>
<td>0.2000</td>
<td>0.0976</td>
<td>-0.0302</td>
<td>0.3525</td>
</tr>
</tbody>
</table>

Table 7
Intuitionistic fuzzy ordered matrix \((\tilde{A}_{\omega(i))})_{3,5}\).

<table>
<thead>
<tr>
<th></th>
<th>(j = 1)</th>
<th>(j = 2)</th>
<th>(j = 3)</th>
<th>(j = 4)</th>
<th>(j = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i = 1)</td>
<td>(0.6755, 0.1094)</td>
<td>(0.5946, 0.3183)</td>
<td>(0.4000, 0.2000)</td>
<td>(0.3384, 0.2746)</td>
<td>(0.05796)</td>
</tr>
<tr>
<td>(i = 2)</td>
<td>(0.5701, 0.1094)</td>
<td>(0.6000, 0.3000)</td>
<td>(0.5030, 0.2347)</td>
<td>(0.4384, 0.3686)</td>
<td>(0.1337, 0.7780)</td>
</tr>
<tr>
<td>(i = 3)</td>
<td>(0.5010, 0.2176)</td>
<td>(0.5000, 0.3000)</td>
<td>(0.5050, 0.4054)</td>
<td>(0.3384, 0.3686)</td>
<td>(0.1337, 0.7780)</td>
</tr>
</tbody>
</table>

Table 8
Score matrix \((S(\tilde{A}_i))_{3,5}\).

<table>
<thead>
<tr>
<th></th>
<th>(j = 1)</th>
<th>(j = 2)</th>
<th>(j = 3)</th>
<th>(j = 4)</th>
<th>(j = 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i = 1)</td>
<td>-0.5796</td>
<td>0.2000</td>
<td>0.1856</td>
<td>0.0125</td>
<td>0.6109</td>
</tr>
<tr>
<td>(i = 2)</td>
<td>-0.6122</td>
<td>0.3000</td>
<td>0.1252</td>
<td>0.0280</td>
<td>0.5152</td>
</tr>
<tr>
<td>(i = 3)</td>
<td>-0.6122</td>
<td>0.2000</td>
<td>0.0144</td>
<td>-0.0755</td>
<td>0.3944</td>
</tr>
</tbody>
</table>

\(x_1 > x_2 > x_3\).

Therefore, the optimal alternative is \(x_1\).

If we use the method with the operator proposed in [37], we also have \(x_1 > x_2 > x_3\), i.e., the best alternative is \(x_1\).

Thus, the results of the above two methods are same. Therefore, the method proposed in this paper is effective and valid.

6.2. Systematic comparison with corresponding work of other papers

However, the method developed in [37] has some differences from the proposed method in this paper. For example, if \(A_{11} = A'_{11} = (0.0.5)\), we have the following steps using the method in [37].

Step 1: By the operational law of Definition 5, \(\tilde{A}_i = \tilde{A}_{\omega(i)}i = 1, 2, 3; j = 1, 2, \ldots, 5\), where \(\omega = (0.25, 0.20, 0.15, 0.18, 0.22)^T\) is the weighting vector of \(A_{11}, A_{12}, \ldots, A_{15}\).

Step 2: By Definition 4 we obtain the score matrix \((S(\tilde{A}_i))_{3,5}\) in Table 6.

\(\hat{S}(\tilde{A}_{15}) > \hat{S}(\tilde{A}_{13}) > \hat{S}(\tilde{A}_{12}) > \hat{S}(\tilde{A}_{14}) > \hat{S}(\tilde{A}_{11})\),

\(\hat{S}(\tilde{A}_{15}) > \hat{S}(\tilde{A}_{25}) > \hat{S}(\tilde{A}_{22}) > \hat{S}(\tilde{A}_{24}) > \hat{S}(\tilde{A}_{21})\)

and

\(\hat{S}(\tilde{A}_{15}) > \hat{S}(\tilde{A}_{35}) > \hat{S}(\tilde{A}_{34}) > \hat{S}(\tilde{A}_{33}) > \hat{S}(\tilde{A}_{31})\).

Then, we get the intuitionistic fuzzy ordered matrix \((\tilde{A}_{\omega(i))})_{3,5}\) in Table 7, where \(\tilde{A}_{\omega(i)}\) is the \(j\)th largest of \(\tilde{A}_{11}, \tilde{A}_{12}, \ldots, \tilde{A}_{15}\).

Step 3: By the aggregation operator in Definition 8, and \(w = (0.112, 0.236, 0.304, 0.236, 0.112)\). It follows that

\(A_1 = IFHG\omega(A_{11}, A_{12}, A_{13}, A_{14}, A_{15}) = \left(\prod_{j=1}^{5} (u_{A_{\omega(i)}})^{w_j} - \prod_{j=1}^{5} (1 - v_{A_{\omega(i)}})^{w_j}\right) = (0.02911)\),

Similarly, \(A_2 = (0.4438, 0.3659)\), \(A_3 = (0.3999, 0.4211)\).

Step 4: By Definition 4, \(S(A_1) = 0.2911, S(A_2) = 0.0779, S(A_3) = -0.0212\).

Thus, \(S(A_2) = S(A_3) > S(A_1)\).

Step 5: Therefore, \(x_2 > x_3 > x_1\)

Therefore, the optimal alternative is \(x_2\).
By Table 6, we know that $S(\tilde{A}_{ij}) > S(\tilde{A}_{ij})$ ($j = 1, 2, \ldots, 5$). By Table 7 and Definition 4, we have that $S(\tilde{A}_{(i)}) > S(\tilde{A}_{(j)})$ ($j = 1, 2, \ldots, 5$). Thus, it is more acceptable for us to get the result that $x_1 > x_3$ from monotonicity point of view. Therefore, the method proposed in [37] is not very stable.

However, if we use the method developed by this paper, we have following steps.

Step 1: According to operational law in Definition 5, we get $\tilde{A}_{ij} = A_{ij}^{\omega_1}$ $i = 1, 2, 3; j = 1, 2, \ldots, 5$, where $\omega = (0.25, 0.20, 0.15, 0.18, 0.22)^T$ is the weighting vector of $A_{i1}, A_{i2}, \ldots, A_{i5}$, $i = 1, 2, 3$.

Step 2: By Definition 4, we get the score matrix $(S(\tilde{A}_{ij}))_{3 \times 5}$ in Table 8. Obviously,

$$S(\tilde{A}_{15}) > S(\tilde{A}_{12}) > S(\tilde{A}_{13}) > S(\tilde{A}_{14}) > S(\tilde{A}_{11}),$$

$$S(\tilde{A}_{25}) > S(\tilde{A}_{22}) > S(\tilde{A}_{23}) > S(\tilde{A}_{24}) > S(\tilde{A}_{21}),$$

and

$$S(\tilde{A}_{35}) > S(\tilde{A}_{32}) > S(\tilde{A}_{33}) > S(\tilde{A}_{34}) > S(\tilde{A}_{31}).$$

Then, we get the intuitionistic fuzzy ordered matrix $(\tilde{A}_{(i)})_{3 \times 5}$ in Table 9, where $\tilde{A}_{(i)}$ is the jth largest of $A_{i1}, A_{i2}, \ldots, A_{i5}$, $i = 1, 2, 3$.

Step 3: By the aggregation operator in Definition 11 and Theorem 11, and $w = (0.112, 0.236, 0.304, 0.236, 0.112)$, we get that

$$A_1 = IFHGIA_{w, o}(A_{11}, A_{12}, A_{13}, A_{14}, A_{15}) = \left( \prod_{j=1}^{5} \left[ 1 - v_{x_{(i)}}^{\omega_1} \right] \right)_{i=1}^{w_i} - \left( \prod_{j=1}^{5} \left[ 1 - (u_{x_{(i)}}^{\omega_1} + v_{x_{(i)}}^{\omega_1}) \right] \right)_{i=1}^{w_i} - \left( \prod_{j=1}^{5} (1 - v_{x_{(i)}}^{\omega_1}) \right)_{i=1}^{w_i},$$

$$= (0.4093, 0.2919),$$

Similarly, $A_2 = (0.4383, 0.3659), A_3 = (0.3927, 0.4212)$.

Step 4: By Definition 4, we have

$$S(A_1) = 0.1105, \quad S(A_2) = 0.0724 \quad \text{and} \quad S(A_3) = -0.0285.$$

Therefore, $S(A_1) > S(A_2) > S(A_3)$.

Step 5: According to the scores $S(A_i)$ ($i = 1, 2, 3$), we rank all the alternatives $x_i$ ($i = 1, 2, 3$) and obtain that

$x_1 > x_2 > x_3$.

Therefore, the optimal alternative is $x_1$.

By Table 8, we have

$S(\tilde{A}_{ij}) \geq S(\tilde{A}_{ij}), j = 1, \ldots, 5.$

By Table 9 and Definition 4, we have

$S(\tilde{A}_{(i)}) \geq S(\tilde{A}_{(j)}), j = 1, \ldots, 5.$

Thus, it is acceptable for us to get the result that $x_1 > x_3$ from monotonicity point of view. That is to say, the result of this paper is acceptable, which means the proposed method is a good complement to the existing works on IFSs in the case that one of the membership degree is zero.

We have summarized the systematic comparison in four aspects:

(1) Atanassov [4] originally defined the multiplication operation on IFSs, De et al. [12] defined the power operation by reasoning from the multiplication operation. In this paper, we define the new geometric operations of IFSs from the probability point of view and give the geometric interpretations of these geometric operations. We propose the new multiplication operation on IFSs based on probability non-membership (PN) function operator, probability membership (PM) function operator and probability heterogeneous (PH) operator. And the geometric meanings of these new operations are interpreted in Figs. 1–5.

(2) The existing geometric operations just consider the effects of membership degree or non-membership degree of different IFSs, while the new geometric operations also take the interactions of non-membership degree and membership de-
gree of different IFSs into consideration, which could be used and explained reasonably in more case, especially when one of the membership degree is zero. It can be seen clearly in Example 1.

(3) By the aggregation operators in [37], when there exists only one membership degree of IFS equals to zero, the membership degree of aggregation result of n IFSs is zero even if the membership degrees of n – 1 IFSs are not zero, which is the weakness of the aggregation operators in [37]. In the above cases, if we use the operators presented in this paper, the aggregation result can be explained reasonably, which could be seen clearly by Example 2.

(4) In the process of intuitionistic fuzzy multi-criteria decision making, the evaluated values of candidate alternatives under attributes are IFSs. In Section 6.1, we know that if none of the membership degree of IFSs is zero, we can get the same optimal alternative by the approaches developed in [37], and the ranking result is same. But when only one membership degree of IFSs is zero, the result gotten by the proposed approach in this paper is acceptable from monotonicity point of view.

7. Conclusions

Intuitionistic fuzzy information aggregation is an interesting research field of the IFS theory. The rational operational laws on IFSs are very important to aggregate the intuitionistic fuzzy information. In this paper, we propose the new operational laws on IFSs and develop the intuitionistic fuzzy geometric interaction averaging (IFGIA) operator, which provide a good complement to the existing works on IFSs.

In the succeeding work, we plan to extend our research to the interval-valued intuitionistic fuzzy environment, we will develop some new operational laws on interval-valued intuitionistic fuzzy sets and propose the corresponding interval-valued intuitionistic fuzzy geometric interaction averaging (IVIFGIA) operators. We intend to apply these approaches to multi-attribute decision making, pattern recognition, data mining, clustering and medical diagnosis.

In decision-making, the generalization capability of extracting fuzzy rules is the key index. Recently, some new refinement techniques [28,44] related to maximize the uncertainty or to combine multiple classifiers have been proposed to improve the generalization of the decision rules. The fuzzy information systems is related to extracting rules because it contains conditional attributes and decision attributes. We will conduct some numerical experiments on comparing with some existing fuzzy rule extraction methods [17,43] in the future research under the intuitionistic fuzzy environment.

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