THE HAAGERUP INVARIANT FOR TENSOR PRODUCTS OF OPERATOR SPACES

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1. Introduction

In [7, 8] Haagerup introduced two isomorphism invariants $\Lambda_1(\mathcal{A})$ and $\Lambda(\mathcal{M})$ for C^*-algebras $\mathcal{A}$ and von Neumann algebras $\mathcal{M}$, based on appropriate forms of the completely bounded approximation property defined below. These definitions have obvious extensions to operator spaces and dual operator spaces respectively, and in [16] we established the multiplicativity of Λ on the ultraweakly closed spatial tensor product of two dual operator spaces $\mathcal{X}$ and $\mathcal{Y}$:

$$\Lambda(\mathcal{X}\overline{\otimes}\mathcal{Y}) = \Lambda(\mathcal{X})\Lambda(\mathcal{Y}).$$

In this paper we consider the analogous formula

$$\Lambda_1(\mathcal{X} \otimes_\omega \mathcal{Y}) = \Lambda_1(\mathcal{X})\Lambda_1(\mathcal{Y})$$

where ω is an operator space tensor product norm on the tensor product of all pairs of operator spaces $\mathcal{X}$ and $\mathcal{Y}$. We show that this is valid under conditions on ω which are sufficiently mild that they are satisfied by many of the standard operator space tensor product norms. In particular, our results apply to the spatial, Haagerup, and operator space projective tensor products.

For a complex vector space $\mathcal{V}$, $\mathbb{M}_{p,q}(\mathcal{V})$ will denote the vector space of $p\times q$ rectangular matrices with entries from $\mathcal{V}$, abbreviated to $\mathbb{M}_p(\mathcal{V})$ if $p = q$. When $\mathcal{V}$ is $\mathbb{C}$ we will write simply $\mathbb{M}_{p,q}$ or $\mathbb{M}_p$ as appropriate. An operator space $\mathcal{E}$ is a subspace of $B(H)$ for some Hilbert space H and so there is a natural sequence of norms on $\mathbb{M}_n(\mathcal{E})$, induced by regarding $\mathbb{M}_n(\mathcal{E})$ as a subspace of $\mathbb{M}_n(B(H))$ and identifying this algebra with $B(H \oplus \cdots \oplus H)$. Each rectangular space $\mathbb{M}_{p,q}(\mathcal{E})$ embeds into a larger square one, and so we also obtain a norm on $\mathbb{M}_{p,q}(\mathcal{E})$ for any pair of integers p and q. It is simple to verify that the following conditions are satisfied:

(a) $\|ABC\| \leq \|A\|\,\|B\|\,\|C\|$
for $A \in \mathbb{M}_{r,p}, B \in \mathbb{M}_{p,q}(\mathcal{E}), C \in \mathbb{M}_{q,s}$;
(b) If $A \in \mathbb{M}_{p,q}(\mathcal{E})$ and $B \in \mathbb{M}_{r,s}(\mathcal{E})$ then

$$\|A \oplus B\| = \max\{\|A\|, \|B\|\}$$

where $A \oplus B$ denotes the matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ in $\mathbb{M}_{p+r,q+s}(\mathcal{E})$.

In [15] Ruan showed that if a set of norms on $\mathbb{M}_{p,q}(\mathcal{V})$ satisfy these two conditions then $\mathcal{V}$ is completely isometrically isomorphic to a subspace of some $B(H)$, and so (a) and (b) characterize operator spaces (we have given a formulation which is slightly different from, but equivalent to, the original one). This characterization followed the one for operator systems by Choi and Effros [3] and both have proved useful in showing that various constructions for operator spaces again yield operator spaces (see, for example, [2, 6, 13]).

If $v = (v_{ij}) \in \mathbb{M}_{p,q}(\mathcal{V})$ and $w = (w_{k\ell}) \in \mathbb{M}_{r,s}(\mathcal{W})$, then $v \odot w$ will denote the element of $\mathbb{M}_{pr,qs}(\mathcal{V} \otimes \mathcal{W})$ which, when viewed as an $i \times j$ matrix of $p \times q$ blocks, has $v_{ij} \otimes w_{k\ell}$ for the (k, ℓ) entry of the (i, j) block. Given operator spaces $\mathcal{X}$ and $\mathcal{Y}$ and norms ω on $\mathbb{M}_n(\mathcal{X} \otimes \mathcal{Y})$ (and thus on $\mathbb{M}_{p,q}(\mathcal{X} \otimes \mathcal{Y})$), $\mathcal{X} \otimes_\omega \mathcal{Y}$ will denote the completion of the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$ for the norm ω. Following the terminology of [2], we say that ω is an operator space cross norm if $\mathcal{X} \otimes_\omega \mathcal{Y}$ is an operator space and satisfies

(P1) $\|v \odot w\|_\omega = \|v\| \, \|w\|$

for all $v \in \mathbb{M}_{p,q}(\mathcal{X}), w \in \mathbb{M}_{r,s}(\mathcal{Y})$.

Given an operator space tensor product norm ω, there is a dual norm ω^* defined on the matrix spaces over $\mathcal{X}^* \otimes \mathcal{Y}^*$ as follows. For $\phi = (\phi_{ij}) \in \mathbb{M}_{p,q}(\mathcal{X}^* \otimes \mathcal{Y}^*)$ and $v = (v_{k\ell}) \in \mathbb{M}_{r,s}(\mathcal{X} \otimes_\omega \mathcal{Y})$, $\langle \phi, v \rangle$ denotes the scalar matrix in $\mathbb{M}_{pr,qs}$ whose entries are $\phi_{ij}(v_{k\ell})$, using the natural action of $\mathcal{X}^* \otimes \mathcal{Y}^*$ as linear functionals on $\mathcal{X} \otimes \mathcal{Y}$. Then ω^* is defined by

$$\|\phi\|_{\omega^*} = \sup\{\|\langle \phi, v \rangle\| \colon \|v\|_\omega \leq 1\}$$

where the dimensions of the matrix v are arbitrary [2]. If ω^* is also an operator space cross norm then it satisfies

(P2) $\|\phi \odot \psi\|_{\omega^*} = \|\phi\| \, \|\psi\|$

for all $\phi \in \mathbb{M}_{p,q}(\mathcal{X}^*), \psi \in \mathbb{M}_{r,s}(\mathcal{Y}^*)$.

We will require a third property of the operator space tensor product norms which we consider. If $S\colon \mathcal{X}_1 \to \mathcal{X}_2$ and $T\colon \mathcal{Y}_1 \to \mathcal{Y}_2$ are completely bounded maps between operator spaces then there is a map $S \otimes T$ on the algebraic tensor product $\mathcal{X}_1 \otimes \mathcal{Y}_1$ into $\mathcal{X}_2 \otimes \mathcal{Y}_2$.

Recalling that $\mathcal{X} \otimes_\omega \mathcal{Y}$ is defined for all pairs of operator spaces, it may be the case that $S \otimes T\colon \mathcal{X}_1 \otimes_\omega \mathcal{Y}_1 \to \mathcal{X}_2 \otimes_\omega \mathcal{Y}_2$ is completely bounded and satisfies

(P3) $\|S \otimes T\|_{cb} = \|S\|_{cb}\|T\|_{cb}$.

A stronger condition than (P3) was considered in [2]. In the terminology of that paper, if ω and ω^* are operator space cross norms and ω is also a uniform operator space tensor norm then (P1)–(P3) are satisfied. We will refer to an operator space tensor norm which satisfies these three properties as *admissible*, and we now give some examples.

Let $\mathcal{X} \subseteq B(H)$ and $\mathcal{Y} \subseteq B(K)$ be operator spaces. The minimal (or spatial) tensor product $\mathcal{X} \otimes_{\min} \mathcal{Y}$ is the norm closed subspace of $B(H \otimes K)$ generated by the operators $\{x \otimes y\colon\ x \in \mathcal{X}, y \in \mathcal{Y}\}$, and is clearly an operator space. Given matrices $A = (x_{ij}) \in \mathbb{M}_{p,r}(\mathcal{X})$ and $B = (y_{ij}) \in \mathbb{M}_{r,q}(\mathcal{Y})$, $(A \otimes I)(I \otimes B)$ denotes the matrix in $\mathbb{M}_{p,q}(\mathcal{X} \otimes \mathcal{Y})$ whose (i, j) entry is

$$\sum_k x_{ik} \otimes y_{kj}.$$

The Haagerup norm on $\mathbb{M}_{p,q}(\mathcal{X} \otimes \mathcal{Y})$ is defined by

$$\|U\|_h = \inf\{\|A\|\,\|B\|\colon\ U = (A \otimes I)(I \otimes B)\}$$

where $A \in \mathbb{M}_{p,r}(\mathcal{X}), B \in \mathbb{M}_{r,q}(\mathcal{Y})$ and r is arbitrary. This is an operator space denoted by $\mathcal{X} \otimes_h \mathcal{Y}$ [13]. Finally, the operator space projective tensor product norm is defined on $\mathbb{M}_{p,q}(\mathcal{X} \otimes \mathcal{Y})$ by

$$\|U\|_\wedge = \inf\{\|\alpha\|\,\|A\|\,\|B\|\|\beta\|\colon\ U = \alpha(A \odot B)\beta\}$$

where $A \in \mathbb{M}_{i,j}(\mathcal{X}), B \in \mathbb{M}_{k,\ell}(\mathcal{Y})$, $\alpha \in \mathbb{M}_{p,ik}$, $\beta \in \mathbb{M}_{j\ell,q}$. This gives the operator space projective tensor product $\mathcal{X}\widehat{\otimes}\mathcal{Y}$, and it is an operator space [2, 6]. Theorem 5.7 of [2] shows that ω satisfies (P1) and (P2) if and only if it lies between the minimal and the operator space projective tensor norms, while the three norms defined above are all admissible by [2, Proposition 5.11].

We now recall from [16] the definition of three approximation constants, the first of which is the Haagerup constant for a norm closed operator space. In order to define the third, we need to specify a distinguished collection $\mathcal{C}$ of finite dimensional subspaces of $\mathcal{X}$ satisfying two conditions:

(C1) given $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{C}$ there exists $\mathcal{E}_3 \in \mathcal{C}$ such that $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{E}_3$,

(C2) $\bigcup_{\mathcal{E} \in \mathcal{C}} \mathcal{E}$ is norm dense in $\mathcal{X}$.

The constants will be defined in terms of the following types of nets $\{T_\alpha\colon\ \mathcal{X} \to \mathcal{X}\}_{\alpha \in A}$ of finite rank completely bounded maps:

(T1) $\lim_\alpha \|T_\alpha x - x\| = 0$ for all $x \in \mathcal{X}$, and $\sup_\alpha \|T_\alpha\|_{cb} < \infty$,

(T2) for each finite dimensional subspace $\mathcal{F}$ of $\mathcal{X}$, there exists $\alpha_0 \in A$ such that $T_\alpha = I$ on $\mathcal{F}$ for $\alpha \geq \alpha_0$, and $\sup_\alpha \|T_\alpha\|_{cb} < \infty$,

(T3) the range of each T_α is contained in an element of $\mathcal{C}$, for each $\mathcal{F} \in \mathcal{C}$ there exists $\alpha_0 \in A$ such that $T_\alpha = I$ on $\mathcal{F}$ for $\alpha \geq \alpha_0$, and $\sup_\alpha \|T_\alpha\|_{cb} < \infty$.

For $1 \leq i \leq 3$ we define $\Lambda_i(\mathcal{X})$ to be ∞ if no net of type (Ti) exists, otherwise

$$\Lambda_i(\mathcal{X}) = \inf\{\lambda\colon\ \text{ there exists a net } \{T_\alpha\}_{\alpha \in A} \text{ of type } (Ti) \text{ and } \sup_\alpha \|T_\alpha\|_{cb} \leq \lambda\}.$$

It was shown in [16] that these constants are equal, so that $\Lambda_2(\mathcal{X})$ and $\Lambda_3(\mathcal{X})$ are to be regarded as convenient reformulations of $\Lambda_1(\mathcal{X})$. In particular, $\Lambda_3(\mathcal{X})$ is independent of the collection $\mathcal{C}$ of subspaces used to define it, and this will allow us to make a suitable choice for $\mathcal{C}$ in Theorem 2.2 below.

We refer the reader to [4, 5, 7, 8, 9, 16] for further information on the Haagerup invariant, and to [12] for the theory of completely bounded maps.

2. The main result

Let $\mathcal{F} \subseteq \mathcal{E} \subseteq \mathcal{X}$ be finite dimensional subspaces of an operator space $\mathcal{X}$, where we view $\mathcal{F}$ as fixed and $\mathcal{E}$ as variable. In general the operator space projective tensor product is not injective, and so the norm $\|\cdot\|_\wedge$ on $\mathcal{F}\widehat{\otimes}\mathcal{E}^*$ will not coincide with the restriction to $\mathcal{F}\otimes\mathcal{E}^*$ of $\|\cdot\|_\wedge$ on $\mathcal{X}\widehat{\otimes}\mathcal{E}^*$. The restriction of this latter norm to $\mathcal{F}\otimes\mathcal{E}^*$ will be denoted by $\|\cdot\|_{\mathcal{E}}$. The trace functional τ on $(\mathcal{F}\otimes\mathcal{E}^*, \|\cdot\|_{\mathcal{E}})$ is defined by

$$\tau(f\otimes\phi) = \phi(f), \qquad f\in\mathcal{F}, \phi\in\mathcal{E}^*.$$

The significance of this norm and trace is exhibited by the following result from [16].

Proposition 2.1. *If $\mathcal{F}\subseteq\mathcal{E}$ are finite dimensional subspaces of an operator space $\mathcal{X}$ and $\lambda > 0$, then there exists a map $T\colon\ \mathcal{X}\to\mathcal{E}$ satisfying*

$$\|T\|_{cb} \le \lambda, \quad T = I \text{ on } \mathcal{F}, \tag{2.1}$$

if and only if

$$|\tau(u)| \le \lambda\|u\|_{\mathcal{E}}, \quad u\in\mathcal{F}\otimes\mathcal{E}^*. \tag{2.2}$$

The following is the main result of the paper. The idea of the proof comes from [14].

Theorem 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be operator spaces and let ω be an admissible operator space tensor product norm. Then*

$$\Lambda_1(\mathcal{X}\otimes_\omega\mathcal{Y}) = \Lambda_1(\mathcal{X})\Lambda_1(\mathcal{Y}).$$

Proof. Suppose that $\Lambda_1(\mathcal{X})$ and $\Lambda_1(\mathcal{Y})$ are both finite. Given $\varepsilon > 0$, choose nets $\{S_\alpha\colon \mathcal{X}\to\mathcal{X}\}_{\alpha\in A}$, $\{T_\beta\colon\ \mathcal{Y}\to\mathcal{Y}\}_{\beta\in B}$ of type (T1) satisfying

$$\sup_\alpha \|S_\alpha\|_{cb} \le \Lambda_1(\mathcal{X})+\varepsilon,\ \sup_{\beta\in B}\|T_\beta\| \le \Lambda_1(\mathcal{Y})+\varepsilon.$$

By property (P3) the net

$$\{S_\alpha\otimes T_\beta\colon\ \mathcal{X}\otimes_\omega\mathcal{Y}\to\mathcal{X}\otimes_\omega\mathcal{Y}\}_{\alpha\in A,\beta\in B}$$

satisfies

$$\sup_{\substack{\alpha\in A\\ \beta\in B}} \|S_\alpha\otimes T_\beta\|_{cb} \le (\Lambda_1(\mathcal{X})+\varepsilon)(\Lambda_1(\mathcal{Y})+\varepsilon)$$

and is of type (T1). Since $\varepsilon > 0$ was arbitrary, it follows that

$$\Lambda_1(\mathcal{X} \otimes_\omega \mathcal{Y}) \leq \Lambda_1(\mathcal{X})\Lambda_1(\mathcal{Y}). \tag{2.3}$$

This inequality is immediate if either of $\Lambda_1(\mathcal{X})$ and $\Lambda_1(\mathcal{Y})$ is infinite.

To arrive at a contradiction, we now suppose that strict inequality holds in (2.3), which we then rewrite as

$$\Lambda_3(\mathcal{X} \otimes_\omega \mathcal{Y}) < \Lambda_2(\mathcal{X})\Lambda_2(\mathcal{Y}). \tag{2.4}$$

Then choose $\lambda, \mu \in \mathbb{R}^+$ such that

$$\lambda < \Lambda_2(\mathcal{X}), \quad \mu < \Lambda_2(\mathcal{Y}), \quad \lambda\mu > \Lambda_3(\mathcal{X} \otimes_\omega \mathcal{Y}).$$

By definition of $\Lambda_2(\mathcal{X})$, we may find a finite dimensional subspace $\mathcal{F}_1$ of $\mathcal{X}$ such that, for any finite dimensional subspace $\mathcal{E}_1$ containing $\mathcal{F}_1$, there cannot exist $S\colon\ \mathcal{X} \to \mathcal{E}_1$ satisfying

$$\|S\|_{cb} \leq \lambda, \quad S = I \text{ on } \mathcal{F}_1. \tag{2.5}$$

In the same way we may find a finite dimensional subspace $\mathcal{F}_2$ of $\mathcal{Y}$ such that for any larger finite dimensional subspace $\mathcal{E}_2$, there does not exist $T\colon\ \mathcal{Y} \to \mathcal{E}_2$ satisfying

$$\|T\|_{cb} \leq \mu, \quad T = I \text{ on } \mathcal{F}_2. \tag{2.6}$$

For the definition of $\Lambda_3(\mathcal{X} \otimes_\omega \mathcal{Y})$, choose $\mathcal{C}$ to be the collection of finite dimensional subspaces of the form $\mathcal{E}_1 \otimes \mathcal{E}_2$. Then, since $\Lambda_3(\mathcal{X} \otimes_\omega \mathcal{Y}) < \lambda\mu$, there exists a finite dimensional subspace $\mathcal{E}_1 \otimes \mathcal{E}_2$ containing $\mathcal{F}_1 \otimes \mathcal{F}_2$ and a map

$$R\colon\ \mathcal{X} \otimes_\omega \mathcal{Y} \to \mathcal{E}_1 \otimes \mathcal{E}_2 \subseteq \mathcal{X} \otimes_\omega \mathcal{Y}$$

satisfying

$$\|R\|_{cb} < \lambda\mu, \quad R = I \text{ on } \mathcal{F}_1 \otimes \mathcal{F}_2. \tag{2.7}$$

Let τ_1, τ_2 and τ denote the traces on $\mathcal{F}_1 \otimes \mathcal{E}_1^*$, $\mathcal{F}_2 \otimes \mathcal{E}_2^*$ and $(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes (\mathcal{E}_1 \otimes \mathcal{E}_2)^*$ respectively. By Proposition 2.1, (2.5) and (2.6), we may find elements $u_i \in \mathcal{F}_i \otimes \mathcal{E}_i^*$ $(i = 1, 2)$ such that

$$\|u_i\|_{\mathcal{E}_i} < 1, (i = 1, 2), \quad |\tau_1(u_1)| > \lambda, \quad |\tau_2(u_2)| > \mu.$$

Let

$$u_1 = \sum_{i=1}^{m} x_i \otimes \phi_i, \qquad u_2 = \sum_{j=1}^{n} y_j \otimes \psi_j,$$

where $x_i \in \mathcal{F}_1, \phi_i \in \mathcal{E}_1^*, y_j \in \mathcal{F}_2$, $\psi_j \in \mathcal{E}_2^*$. Then define $u \in (\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes (\mathcal{E}_1 \otimes \mathcal{E}_2)^*$ by

$$u = \sum_{i,j} (x_i \otimes y_j) \otimes (\phi_i \otimes \psi_j),$$

identifying $(\mathcal{E}_1 \otimes \mathcal{E}_2)^*$ with $\mathcal{E}_1^* \otimes \mathcal{E}_2^*$ algebraically. Then

$$\tau(u) = \sum_{i,j} \phi_i(x_i)\psi_j(y_j) = \tau_1(u_1)\tau_2(u_2).$$

¿From the choice of u_1, u_2, this gives

$$|\tau(u)| > \lambda\mu. \tag{2.8}$$

By definition of $\|\cdot\|_{\mathcal{E}_k}$ we may find representations

$$u_k = \alpha_k(v_k \odot w_k)\beta_k \qquad (k = 1, 2)$$

where

$$\alpha_1 \in \mathbb{M}_{1,pq},\ v_1 \in \mathbb{M}_p(\mathcal{X}), \quad w_1 \in \mathbb{M}_q(\mathcal{E}_1^*),\ \beta_1 \in \mathbb{M}_{pq,1},$$

$$\alpha_2 \in \mathbb{M}_{1,rs},\ v_2 \in \mathbb{M}_r(\mathcal{Y}), \quad w_2 \in \mathbb{M}_s(\mathcal{E}_2^*),\ \beta_2 \in \mathbb{M}_{rs,1},$$

and all have norms strictly less than one. Write $\alpha_{ij}^{(k)}$ for the typical entries of α_k, with similar notation for the other matrices. Then

$$u_k = \sum_{ijmn} \alpha_{im}^{(k)} (v_{ij}^{(k)} \otimes w_{mn}^{(k)}) \beta_{jn}^{(k)},$$

for $k = 1, 2$. Thus u is represented by

$$u = \sum_{\substack{abcd \\ efgh}} \alpha_{ae}^{(1)} \alpha_{cg}^{(2)} (v_{ab}^{(1)} \otimes v_{cd}^{(2)}) \otimes (w_{ef}^{(1)} \otimes w_{gh}^{(2)}) \beta_{bf}^{(1)} \beta_{dh}^{(2)},$$

and this is of the form

$$\gamma((v_1 \odot v_2) \odot (w_1 \odot w_2))\delta,$$

where the entries of $\gamma \in \mathbb{M}_{1,pqrs}$ and $\delta \in \mathbb{M}_{pqrs,1}$ are permutations of the entries of $\alpha_1 \odot \alpha_2$ and $\beta_1 \odot \beta_2$ respectively. Then it is clear that $\|\gamma\|, \|\delta\| < 1$. The inequality

$$\|v_1 \odot v_2\|_\omega = \|v_1\|\,\|v_2\| < 1$$

is a consequence of property (P1). Now $w_1 \in \mathbb{M}_q(\mathcal{E}_1^*)$ corresponds to an element $T_1 \in CB(\mathcal{E}_1, \mathbb{M}_q)$ by

$$T_1(e) = (w_{ij}^{(1)}(e)), \qquad e \in \mathcal{E}_1,$$

and $\|T_1\|_{cb} < 1$ [2]. Then extend to $\widetilde{T}_1 \in CB(\mathcal{X}, \mathcal{M}_q)$ satisfying $\|\widetilde{T}_1\|_{cb} < 1$, by Wittstock's theorem [17]. This in turn corresponds to $\tilde{w}_1 \in \mathbb{M}_q(\mathcal{X}^*)$, $\|\tilde{w}_1\| < 1$. Similarly w_2 extends to $\tilde{w}_2 \in \mathbb{M}_s(\mathcal{Y}^*)$, $\|\tilde{w}_2\| < 1$. By property (P2)

$$\|\tilde{w}_1 \odot \tilde{w}_2\|_{\omega^*} < 1$$

and $w_1 \odot w_2$ is the restriction of $\tilde{w}_1 \odot \tilde{w}_2$ to $\mathcal{E}_1 \otimes \mathcal{E}_2$, normed as a subspace of $\mathcal{X} \otimes_\omega \mathcal{Y}$. Thus

$$\|w_1 \odot w_2\| < 1.$$

By definition of $\|\cdot\|_\wedge$, we now have

$$\|u\|_{\mathcal{E}_1 \otimes \mathcal{E}_2} < 1. \tag{2.9}$$

Then (2.8), combined with (2.9) and Lemma 2.1, shows that $\|R\|_{cb} > \lambda\mu$, contradicting (2.7) and completing the proof.

We have already noted that the minimal tensor product norm, the Haagerup norm and the operator space projective tensor product norm are admissible, so the following is a special case of the theorem.

Corollary 2.3. *If $\mathcal{X}$ and $\mathcal{Y}$ are operator spaces then*

$$\Lambda_1(\mathcal{X} \otimes_{\min} \mathcal{Y}) = \Lambda_1(\mathcal{X})\Lambda_1(\mathcal{Y}), \tag{2.10}$$

$$\Lambda_1(\mathcal{X} \otimes_h \mathcal{Y}) = \Lambda_1(\mathcal{X})\Lambda_1(\mathcal{Y}), \tag{2.11}$$

$$\Lambda_1(\mathcal{X} \widehat{\otimes} \mathcal{Y}) = \Lambda_1(\mathcal{X})\Lambda_1(\mathcal{Y}). \tag{2.12}$$

Remark 2.4. (i) Equation (2.10) for C^*-algebras is the analogue of the von Neumann algebra result for $\Lambda(\cdot)$ proved in [16]. The definitions of $\Lambda(\cdot)$ and $\Lambda_1(\cdot)$ are formally very similar, so it is natural to ask whether results for one can be deduced from results for the other. Specifically, a relationship between $\Lambda_1(\mathcal{A})$ and $\Lambda(\mathcal{A}_1^{**})$ or between $\Lambda_1(\mathcal{A})$ and $\Lambda(\mathcal{A}'')$ would be required. The following examples suggest that this is not a fruitful approach.

In [4], the authors found reduced group C^*-algebras $\mathcal{A}_n$ for which

$$\Lambda_1(\mathcal{A}_n) = 2n+1, \quad n \geq 1,$$

and subsequently it was shown in [1] that these algebras are simple. Consequently they have faithful irreducible representations as algebras $\mathcal{B}_n$ on a separable Hilbert space H. Then the ultraweak closures are all equal to $B(H)$, so we have the situation

$$\Lambda_1(\mathcal{B}_n) = 2n+1, \ \ \Lambda(\mathcal{B}_n'') = 1, \quad n \geq 1.$$

In [11] an example is given of a C^*-algebra $\mathcal{A}$ containing the algebra of compact operators $\mathcal{K}$ so that $\mathcal{A}$ is not exact, but $\Lambda_1(\mathcal{A}/\mathcal{K}) = 1$. Exactness is implied by finiteness of $\Lambda_1(\cdot)$, so we have

$$\Lambda_1(\mathcal{A}) = \infty, \quad \Lambda_1(\mathcal{A}/\mathcal{K}) = 1,$$

but

$$\Lambda(\mathcal{A}^{**}) = \Lambda((\mathcal{A}/\mathcal{K})^{**})$$

since

$$A^{**} = (\mathcal{A}/\mathcal{K})^{**} \oplus B(H).$$

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(ii) We do not know if a similar result holds for the maximal tensor product of C^*-algebras. Huruya [10] has shown that (P3) fails for this tensor product. This does not rule out the possibility of such a result, but seems to make it unlikely. A weaker question is: if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $\Lambda_1(\mathcal{A})$ and $\Lambda_1(\mathcal{B})$ are finite, is $\Lambda_1(\mathcal{A} \otimes_{\max} \mathcal{B})$ finite?

References

[1] M. Bekka, M. Cowling and P. de la Harpe, Some groups whose reduced C^*-algebra is simple, I.H.E.S. Publ. Math., to appear.

[2] D.P. Blecher and V.I. Paulsen, Tensor products of operator spaces, J. Funct. Anal., 99 (1991), 262-292.

[3] M.-D. Choi and E.G. Effros, Injectivity and operator spaces, J. Funct. Anal., 24 (1977), 156-209.

[4] M. Cowling and U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, Invent. Math., 96 (1989), 507-549.

[5] J. de Cannière and U. Haagerup, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups, Amer. J. Math., 107 (1984), 455-500.

[6] E.G. Effros and Z.J. Ruan, On approximation properties for operator spaces, Internat. J. Math., 1 (1990), 163-187.

[7] U. Haagerup, An example of a non-nuclear C^*-algebra which has the metric approximation property, Invent. Math., 50 (1979), 279-293.

[8] U. Haagerup, Group C^*-algebras without the completely bounded approximation property, preprint.

[9] U. Haagerup and J. Kraus, Approximation properties for group C^*-algebras and group von Neumann algebras, Trans. Amer. Math. Soc., to appear.

[10] T. Huruya, On compact completely bounded maps of C^*-algebras, Michigan Math. J., 30 (1983), 213-220.

[11] E. Kirchberg, On non-semisplit extensions, tensor products and exactness of group C^*-algebras, Invent. Math., 112 (1993), 449-489.

[12] V.I. Paulsen, Completely bounded maps and dilations, Notes in Mathematics Series No. 146, Pitman, New York, 1986.

[13] V.I. Paulsen and R.R. Smith, Multilinear maps and tensor norms on operator systems, J. Funct. Anal., 73 (1987), 258-276.

[14] G. Pisier, Remarks on complemented subspaces of von Neumann algebras, Proc. Roy. Soc. Edinburgh, 121A (1992), 1-4.

[15] Z.J. Ruan, Subspaces of C^*-algebras, J. Funct. Anal., 76 (1988), 217-230.

[16] A.M. Sinclair and R.R. Smith, The Haagerup invariant for von Neumann algebras, Amer. J. Math., to appear.

[17] G. Wittstock, Ein operatorwertiger Hahn-Banach Satz, J. Funct. Anal., 40 (1981), 127-150.