Feedback Control for Oscillations with CPG Architecture

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Abstract—This paper presents a method for designing a nonlinear feedback controller for a linear plant to achieve an oscillation with a prescribed profile as a projection of a stable limit cycle of the closed-loop system. The controller architecture is based on the central pattern generator, a biological oscillator composed of interconnected neurons that control rhythmic body movements during animal locomotion. The nonlinear control design problem is reduced, approximately, to a linear eigenstructure assignment problem through describing functions and the multivariable harmonic balance method. We then provide a necessary and sufficient condition for existence of a feasible controller assigning a given eigenstructure, as well as a parametrization of all such controllers. A numerical example demonstrates the efficacy of the proposed design method.

I. INTRODUCTION

The problem of designing feedback controllers to achieve specified coordinated oscillations has not received as much attention as its utility would suggest. The ability to systematically design a controller to achieve coordinated oscillations would be particularly attractive in the field of robotics where repetitive motions are common and the ability to entrain to a specified profile autonomously would be a desirable property. Previous work in this field has primarily centered on the tracking of oscillatory inputs [1], [2], but reference tracking may be too strong as a requirement in certain applications because the trajectory of the system will attempt to converge to the reference value specified at each time rather than remain on a specified oscillation orbit (a closed curve in the state space). If the timing is not important for the design, it would be more preferable to design feedback controllers to achieve coordinated oscillations as the projection of a stable limit cycle of the closed-loop system.

A limit cycle is a self-sustaining oscillation that arises from nonlinear dynamics and it can be structurally stable, i.e., the trajectory starting from an initial point in the neighborhood of the periodic orbit approaches the orbit. Limit cycles are well-studied and there are established methods to analyze and predict their existence [3]. Presently, there have not been many general theories for the design of feedback controllers to achieve stable limit cycles with prescribed amplitudes, phases, and frequencies. Classical dynamical system theories have been used to design coupled oscillators to achieve specified phases [4], [5] and PD controllers have been used in conjunction with coupled nonlinear oscillators to induce limit cycle behavior in robots [6], [7]. However, a general theory has yet to be developed to enable design of feedback controllers for dynamical systems to achieve stable limit cycles with prescribed oscillation profiles. The central pattern generator (CPG) provides a solution to this design problem in the context of biological control systems, and has a potential to provide a basis for such general theory for engineering applications.

The CPG is a collection of interconnected neurons responsible for the repetitive movements of animal locomotion [8]. By itself, the CPG is a nonlinear oscillator and, when isolated from the body dynamics, the CPG will exhibit coordinated oscillation patterns similar to that of the observed body movements [9]. When placed in a closed-loop with animal body dynamics, the CPG creates various periodic body motions observed in different environments [10]. Because the CPG has been extensively studied and functions as a biological controller for rhythmic motions, it is one practical choice for the architecture of controllers to achieve coordinated oscillations.

Mathematical modeling and analyses of CPGs have concluded that both the stability of oscillation as well as the profile itself is intimately connected to the eigenvalue/eigenvectors of the neuronal interconnection matrix. More specifically, the frequency and phases of the CPG oscillation can be predicted by the maximal eigenvalue and corresponding eigenvector, respectively [11]. In a similar manner, when the CPG is used as a controller in feedback with a plant, the control design for closed-loop oscillation can be posed as an eigenstructure assignment problem [12]. Within this framework, much research has been done with a focus on CPG-based controller design for entrainment to resonance modes [13]–[16]. However, these results only cover a subset of possible oscillation profiles, and the full potential of CPG control has yet to be explored to achieve closed-loop oscillations with an arbitrary profile.

The purpose of this paper is to present a solution to the problem of designing a CPG-based controller to achieve closed-loop oscillations with prescribed amplitudes, phases, and frequency. We consider the linear time-invariant plant and the nonlinear feedback controller based on the CPG architecture. The controller is represented as the interconnection of multiple neurons, each with identical dynamics described by a first-order low-pass filter followed by a static nonlinearity. In our approach, the nonlinear problem is first reduced to a tractable quasi-linear form by approximating the nonlinearity by a describing function. The method of harmonic balance [11] then predicts that the original nonlinear closed-loop system has a prescribed stable limit cycle if the
quasi-linear system is marginally stable with only one pair of conjugate eigenvalues on the imaginary axis corresponding to the frequency, and the respective eigenvectors specifying the phases and amplitudes. The aforementioned problem can thus be formulated as a search for the controller parameters satisfying both an eigenvalue/eigenvector equality constraint and a stability requirement on the rest of the eigenvalues. Our main result shows that, under a mild assumption on the assigned eigenvectors, the search reduces to two separate numerically tractable problems; one is the state feedback eigenstructure assignment described by a linear matrix equation, and the other is a standard dynamic output feedback stabilization. We demonstrate the efficacy of the proposed design method through an example of a three-link mechanical arm to achieve an arbitrarily specified oscillation. Due to space considerations, the proofs for the results in this paper have been omitted.

Notation: Let \( \mathbb{R} \), \( \mathbb{C} \), \( \mathbb{C}_- \), and \( \mathbb{I}_n \) denote the sets of real numbers, complex numbers, complex numbers with negative real part, and integers \( 1, \ldots, n \), respectively. Transfer function \( P(s) = C(sI - A)^{-1}B + D \) is denoted by its state space realization as \( P(s) \sim P := (A, B, C, D) \). The set of eigenvalues of a matrix \( M \) is denoted by \( \text{eig}(M) \).

II. PROBLEM FORMULATION AND APPROACH

A. Oscillation Control Problem

We consider a linear system described by

\[
\dot{x} = Ax + Bu, \quad y = Cx
\]

where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the control input, and \( y(t) \in \mathbb{R}^p \) is the measured output. Denote by \( P(s) \) the transfer function from \( u \) to \( y \). A general oscillation control problem can be stated as follows:

**Problem 1.** Let a linear plant (1) and a desired oscillation profile for \( x_i(t) \) be given, where the latter is specified in terms of the frequency \( \omega \), amplitude \( a_i \), phase \( b_i \), and shape \( \sigma_i \) (2\( \pi \)-periodic function with a normalized amplitude). Find a nonlinear dynamic output-feedback controller such that the closed-loop system has an orbitally stable limit cycle on which \( x_i(t) = a_i \sigma_i(\omega t + b_i) \) holds for all \( i \in \mathbb{I}_n \).

The orbital stability mentioned in Problem 1 is a property of a limit cycle which means that trajectories with initial conditions sufficiently close to the limit cycle orbit converge to the orbit. Since the plant is assumed linear, it would be necessary to have a nonlinear controller or else orbital stability could never be achieved due to the lack of structural stability for any periodic orbit of a linear system. To make the control design tractable, we will fix the nonlinear architecture of the controller and search for the design parameters. In particular, we choose the CPG control structure because the CPG has been extensively studied in biology as a nonlinear oscillator that is known to generate stable limit cycles when used in a feedback loop.

A CPG is a neuronal circuit responsible for controlling rhythmic body movements during animal locomotion. By itself, it is a nonlinear oscillator and its oscillation profile is similar to (but not quite the same as) observed body motion.

The CPG is placed in a feedback loop with the body so that the closed-loop system has a stable limit cycle whose projection onto the body variable space gives a gait. It is often represented mathematically as a set of interconnected neuron models, each of which is composed of a linear filter and static nonlinearity. More specifically, a CPG of \( n_c \) interconnected neurons can be represented by

\[
v_i = \psi(q_i), \quad q_i = f(s)w_i, \quad w_i = \sum_{j=1}^{n_c} \mu_{ij}v_j
\]

for \( i \in \mathbb{I}_{n_c} \), where \( w_i \) is the input into the cell, \( \mu_{ij} \) represents the strength and type (either inhibitory or excitatory) of connection from neuron \( j \) to neuron \( i \), \( q_i \) is the internal variable, \( v_i \) is the output, \( f(s) \) is a transfer function that captures time lag or adaptation properties observed in neuronal dynamics, and \( \psi \) is the static nonlinearity that captures the threshold property also observed in biology [17]. Although alternative choices for \( \psi \) and \( f(s) \) exist, the work in this paper will use

\[
\psi(x) = \tanh(x), \quad f(s) = \frac{1}{1 + \tau s}
\]

where \( \tau \) is the time constant for neuronal information processing and we assume that each neuron has identical dynamics, thereby sharing the same \( f(s) \).

For compactness, the CPG can also be represented in vector form as

\[
q = F(s)M\Psi(q), \quad v = \Psi(q), \quad w = Mv
\]

where \( q(t), v(t), \) and \( w(t) \) are \( n_c \)-dimensional vectors, \( M \) is the interconnectivity matrix which has \( \mu_{ij} \) as its \((i, j)\)th entry, \( \Psi(q) \) is a vector that has \( \psi(q_i) \) as its \( i \)th entry, and \( F(s) = f(s)I \).

In order to utilize the CPG as a feedback controller, it is necessary to insert an input-output interface to the CPG model. Let such a form be represented by

\[
q = F(s)(M\Psi(q) + Hy), \quad u = G\Psi(q) + Ly,
\]

where \( G, H, \) and \( L \) are constant matrices [10]. This can be visualized through the block diagram in Fig. 1.

![Fig. 1. Closed-Loop System of CPG and Plant](image-url)
The closed-loop system can be represented by
\[
\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} = \begin{bmatrix} A & 0 \\
0 & A_f \end{bmatrix} \begin{bmatrix} x \\
\xi \end{bmatrix} + \begin{bmatrix} B & 0 \\
0 & B_f \end{bmatrix} \begin{bmatrix} L \\
G \\
H \\
M \end{bmatrix} \begin{bmatrix} 0 \\
I \\
\Psi(q) \end{bmatrix}
\]
where \( q = C_f \xi \), and \((A_f, B_f, C_f)\) is a minimal state space realization of \( F(s) \) with state vector \( \xi \). Within this framework, the control design parameters are \( L, G, H, M, \) and \( \tau \), and we search for these parameters to satisfy the design requirements described in Problem 1. Since \( \tau \) is just a scalar and can be determined by a line search, we consider \( \tau \) to be fixed in the theoretical development and discuss its effect later in a numerical example. In the rest of the paper, we assume that the targeted oscillations for \( x_i \) are identical and sinusoidal, i.e., \( \hat{\alpha}(\theta) := \sin \theta \). Because Problem 1 is difficult to solve exactly, we will use the multivariable harmonic balance (MHB) method [11] to reduce it to an approximate but tractable problem.

**B. The Multivariate Harmonic Balance**

Consider the class of systems described as a feedback connection of a time-invariant system and static nonlinearities:
\[
\dot{x} = Ax + Bw, \quad z = Cx, \quad w = \Psi(z)
\]
where \( w \) and \( z \) are vectors and the \( i \)th entry of \( \Psi(z) \) is assumed to depend only on the \( i \)th entry of \( z \). Note that the closed-loop system in Fig. 1 is just a special case of this type of system with \( z := q \).

In order to simplify the analysis or design of oscillations for the closed-loop system, it would be useful to eliminate the nonlinearity. To this end, we place the system in a quasi-linear form by approximating the static nonlinearity by its describing function,
\[
\Psi(z) \cong K(\alpha)z \quad \text{for} \quad z_i = \alpha_i \sin(\omega t),
\]
where \( \alpha \) is a vector with \( i \)th entry \( \alpha_i \), and \( K(\alpha) \) is a diagonal matrix such that \( K(\alpha)z \) coincides with the first harmonic of \( \Psi(z) \). The resulting quasi-linearized system becomes
\[
\dot{x} = Ax + Bw, \quad z = Cx, \quad w = K(\alpha)z.
\]

Previous MHB analysis [11] has concluded that the nonlinear system in (5) is expected to have a stable limit cycle when the quasi-linear system in (7) is marginally stable for some vector \( \alpha \), with a pair of eigenvalues of
\[
\mathcal{A} := A + BK(\alpha)C
\]
on the imaginary axis and all the others in the open left half plane. Furthermore, the oscillation profile for the limit cycle is predicted as
\[
x_i(t) \cong \gamma_i \sin(\omega t + \delta_i),
\]
where \( \omega \), \( \gamma_i \), and \( \delta_i \) satisfy the MHB equation
\[
(j\omega I - \mathcal{A})\hat{x} = 0, \quad \hat{x}_i = \gamma_i e^{j\delta_i},
\]
and \( |\hat{z}_i| = \alpha_i \) with \( \hat{z} := Cx \). Here, \( \hat{x} \) is a phasor representation of the sinusoid in (8), which can also be written as \( 3|\hat{x}e^{j\omega t}| \).

Note that the eigenvalue \( j\omega \) on the imaginary axis specifies the oscillation frequency, and the eigenvector \( \hat{x} \) specifies the amplitudes and phases. Solving the MHB equation for \( (\omega, x) \) is nontrivial due to the coupling of \( \mathcal{A} \) and \( x \) through \( \alpha \).

**C. Reduction to Eigenstructure Assignment**

We now apply the MHB method to the closed-loop system in (4) and approximately reformulate Problem 1 in a more tractable form. Suppose \( q(t) \) oscillates as
\[
q_i(t) \cong \alpha_i \sin(\omega t + \beta_i)
\]
when the desired limit cycle is achieved for the closed-loop system. Let \( \hat{q} \) be the corresponding phasor, i.e., \( \hat{q}_i = \alpha_i e^{j\beta_i} \).

Approximating the nonlinearity \( \Psi(q) \) by its describing function \( K(\alpha)q \), the closed-loop system (4) can be simplified to
\[
\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} = \begin{bmatrix} A & 0 \\
0 & A_f \end{bmatrix} \begin{bmatrix} x \\
\xi \end{bmatrix} + \begin{bmatrix} B & 0 \\
0 & B_f \end{bmatrix} \begin{bmatrix} L \\
G \\
H \\
M \end{bmatrix} \begin{bmatrix} 0 \\
I \\
\Psi(q) \end{bmatrix}.
\]

Let us introduce the change of variables \((L, G, H, M) \leftrightarrow (A_c, B_c, C_c, D_c)\) defined by
\[
\begin{bmatrix} D_c & C_c \\
B_c & A_c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\
0 & A_f \end{bmatrix} + \begin{bmatrix} I & 0 \\
0 & B_f \end{bmatrix} \begin{bmatrix} L \\
G K(\alpha) \\
H \\
M K(\alpha) \end{bmatrix} \begin{bmatrix} 0 \\
I \\
0 \\
C_f \end{bmatrix},
\]
where the mapping is invertible because \( B_f \) and \( C_f \) are square invertible for the first order low pass filter \( f(s) \). The system (10) can then be expressed as
\[
\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} = \mathcal{A}_{cl} \begin{bmatrix} x \\
\xi \end{bmatrix},
\]
where
\[
\mathcal{A}_{cl} = A + BK C
\]
on the standard output feedback problem with plant \( C(sI - A)^{-1}B \) and controller \( C_c(sI - A_c)^{-1}B_c + D_c \).

Based on the MHB analysis in the previous section, the nonlinear closed-loop system (4) is expected to have a stable limit cycle on which \( x_i(t) \cong \alpha_i \sin(\omega t + \beta_i) \) if
\[
(j\omega I - \mathcal{A}_{cl}) \begin{bmatrix} \hat{x} \\
\hat{\xi} \end{bmatrix} = 0
\]
holds for \( \hat{x}_i = a_ie^{j\beta_i} \) and for some complex vector \( \hat{\xi} \), and all the eigenvalues of \( \mathcal{A}_{cl} \) other than \( \pm j\omega \) are in the open left-half plane. Here, \( \hat{\xi} \) is the phasor of \( \xi(t) \), and is constrained by \( |\hat{q}_i| = \alpha_i \) for \( \hat{\xi} := C_f \hat{\xi} \). The design problem has now reduced to the search for real matrices \((A_c, B_c, C_c, D_c)\), complex vector \( \hat{\xi} \), and real scalar \( \alpha_i \), satisfying \( |C_f \hat{\xi}| = \)
α and (13), the eigenvalue (marginal stability) condition. Note that α_i appears only in the latter constraint since the design freedom associated with α_i in (13) is absorbed into the new parameters (A_c, B_c, C_c, D_c) during the change of variables in (11). Consequently, the essential problem is to find \( K := (A_c, B_c, C_c, D_c) \) and \( \xi \) satisfying (13) and the marginal stability requirement since the parameter α_i can always be chosen as α_i := |\( \bar{q}_i \)| after the design. This is an eigenstructure assignment problem.

D. Further Reformulation

The eigenstructure assignment problem has two specifications: one is the MHB condition (13) on the eigenvalue/eigenvector pair specifying the desired oscillation, and the other is on the location of the rest of the eigenvalues aiming at orbital stability of the oscillation. It would be beneficial for the design to isolate the eigenspaces associated with those on the imaginary axis and the rest.

To this end, let

\[
V := \begin{bmatrix} X \end{bmatrix}, \quad \Omega := \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}
\]

\[
X := \begin{bmatrix} \Re(\dot{x}) \\ \Im(\dot{x}) \end{bmatrix}, \quad \Xi := \begin{bmatrix} \Re(\dot{\xi}) \\ \Im(\dot{\xi}) \end{bmatrix}.
\]

Then (13) becomes

\[
A_{cl} V = V \Omega. \tag{14}
\]

Note that, for the oscillation control problem, V is necessarily full column-rank because if V were rank-deficient, then (14) would imply \( \omega = 0 \) or \( \dot{x} = 0 \), leading to a trivial solution to the MHB equation. Since V is full column-rank, there exists a matrix \( N \) such that \([V \quad N]\) is square nonsingular. Furthermore, matrices \( W \) and \( U \) can be uniquely defined by

\[
\begin{bmatrix} U^T \\ W^T \end{bmatrix} \begin{bmatrix} V \quad N \end{bmatrix} = I. \tag{15}
\]

Below, \((N, U, W)\) denotes any one of such matrix triples determined from V as described above. Then, when (14) is satisfied, the similarity transformation

\[
\begin{bmatrix} U^T \\ W^T \end{bmatrix} A_{cl} \begin{bmatrix} V \quad N \end{bmatrix} = \begin{bmatrix} \Omega & U^T A_{cl} N \\ 0 & W^T A_{cl} N \end{bmatrix} \tag{16}
\]

shows that the eigenvalues of \( A_{cl} \) are those of \( \Omega \) and \( W^T A_{cl} N \). Hence, the eigenstructure assignment problem can be restated as follows:

**Problem 2.** Let a linear time-invariant plant of order \( n \) be given in terms of the state space realization \((A, B, C)\) and let a desired oscillation profile be specified by \( X \in \mathbb{R}^{n \times 2} \) and \( \Omega \in \mathbb{R}^{2 \times 2} \). Find a controller \( K \) of order \( n_c \) and \( \Xi \in \mathbb{R}^{n_c \times 2} \) such that a state-space realization of \( K \) exists to satisfy

(a) \( A_{cl} V = V \Omega \),

(b) \( \text{eig}(W^T A_{cl} N) \subset \mathbb{C}_- \),

where matrix \( A_{cl} \) is defined below (12) and matrices \( U, V, \) and \( W \) are defined in the preceding paragraph.

The matrix \( \Xi \) dictates how the controller states \( \xi \) oscillate when the plant states \( x \) oscillate as desired in the closed-loop system. Since we do not know a priori how \( \xi \) should oscillate, \( \Xi \) is an unknown design parameter in Problem 2 to be found during the design. It turns out, however, that \( \Xi \) can be chosen to be an arbitrary full column-rank matrix under a mild assumption. This is shown through the following two lemmas.

**Lemma 1:** Suppose \( A_{cl} V = V \Omega \) holds for a controller \( K \) and the matrix \( CX \) is full column rank. Then there exist matrices \( \bar{K} \) and \( \hat{\Xi} \) of arbitrarily small norms such that the matrix \( \Xi + \hat{\Xi} \) is full column rank and

\[
\bar{A}_{cl} (V + \bar{V}) = (V + \bar{V}) \Omega,
\]

\[
\bar{A}_{cl} = A + B(K + \bar{K})C, \quad \bar{V} = \begin{bmatrix} 0 \\ \hat{\Xi} \end{bmatrix}.
\]

When \( X \) is rank-deficient, \( \Xi \) is necessarily full column-rank to ensure satisfaction of the MHB equation by a nontrivial oscillation profile \((\omega, \dot{\omega})\). When \( X \) is full column-rank, it may be possible to achieve the plant state oscillation specified by \( X \) with a rank-deficient \( \Xi \). However, Lemma 1 shows that a small perturbation of the controller can always achieve the same plant state oscillation with full column-rank \( \Xi \), provided \( CX \) is full column-rank. Hence, we can restrict our search for \( \Xi \) in Problem 2 to full column-rank matrices.

**Lemma 2:** Consider Problem 2 and let \( K \) and \( \Xi \) be a solution, i.e.,

\[
A_{cl} V = V \Omega, \quad \text{eig}(W^T A_{cl} N) \subset \mathbb{C}_-.
\]

Suppose \( \Xi \) is full column rank. Then, for an arbitrary full column rank matrix \( \Xi \in \mathbb{R}^{n_c \times 2} \), there exists \( \bar{K} \) such that

\[
\bar{A}_{cl} \bar{V} = \bar{V} \Omega, \quad \text{eig}(\bar{W}^T \bar{A}_{cl} \bar{N}) \subset \mathbb{C}_-
\]

where \( \bar{W} \) and \( \bar{N} \) are defined for \( \bar{V} \) in the same way as \( W \) and \( N \) for \( V \), and

\[
\bar{A}_{cl} = A + B\bar{K}C, \quad \bar{V} = \begin{bmatrix} X \\ \hat{\Xi} \end{bmatrix}.
\]

Furthermore, this \( \bar{K} \) can be represented as

\[
\bar{K} = T^{-1} CT, \quad T = \text{diag}(I, T_c)
\]

where \( T_c \) is a square invertible matrix such that \( T_c \hat{\Xi} = \Xi \).

When a controller \( K \) solves Problem 2 with a full column-rank \( \Xi \), Lemma 2 implies the existence of another controller \( \bar{K} \) that solves Problem 2 with an arbitrarily fixed, full column-rank \( \bar{\Xi} \). In fact, the two controllers are related by state coordinate transformation \( T_c \). Thus, the choice of \( \Xi \) does not affect the controller transfer function as long as \( \Xi \) is full column-rank. Consequently, the design parameter \( \Xi \) in Problem 2 can be fixed to an arbitrary full column-rank matrix to simplify the problem.

**III. Control Design**

A. Main Result

The following is the main result of this paper that provides a complete solution to Problem 2, restricting \( \Xi \) to an arbitrarily chosen full column-rank matrix without loss of generality.
Theorem 1: Consider a linear time-invariant plant of order \( n \) which is given in terms of a minimal state space realization \((A,B,C)\). Let \( X \in \mathbb{R}^{n \times 2} \) and \( \Omega \in \mathbb{R}^{2 \times 2} \) be given. There exists a controller that solves Problem 2 if and only if there exist \( F \) and \( K \) such that
\[
\begin{align*}
(\hat{\alpha}) & \quad AX + BF = X \Omega, \\
(\hat{\beta}) & \quad K \text{ stabilizes } \mathcal{P}(s) := C(sI - A)^{-1} [B \ -X].
\end{align*}
\]
In this case, all such controllers with \( \Xi = [I_2 \ 0]^T \) are parameterized by
\[
\begin{align*}
\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} &= \begin{bmatrix} F & D_c & \hat{C}_c \\ 0 & B_c & A_c \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ I & -CX & 0 \\ 0 & 0 & I \end{bmatrix}, \\
F &= \begin{bmatrix} \hat{F} \end{bmatrix}, \quad \hat{K} := (\hat{A}_c, \hat{B}_c, \hat{C}_c, \hat{D}_c).
\end{align*}
\]

Theorem 1 separates the control design into two independent problems: (a) eigenstructure assignment by state feedback, and (b) output feedback stabilization.

The former problem is reduced to a linear matrix equation with the unknown parameter \( F \), which is easily solved analytically or numerically. In particular, a solution exists if and only if \( (I - BB^T)(X \Omega - AX) = 0 \) holds, where \( B^T \) denotes the Moore-Penrose inverse of \( B \). This condition characterizes the assignability of the eigenstructure \( X \) by state feedback. Thus, an eigenstructure for the plant states is assignable by dynamic output feedback if and only if it is assignable by static state feedback provided that \( X \) has full column-rank.

The latter stabilization problem (b) is standard in the literature and is solvable if and only if the pair \((C,A)\) is detectable and \((A,[B \ -X])\) is stabilizable, in which case, the order of the controller \( K \) can always be chosen to be less than or equal to the plant order \( n \). It is interesting to note from the formula (18) that the order of the original controller \( K \) is less than or equal to \( n + 2 \) Thus, the eigenstructure assignment problem may not be solvable by a controller of the same order as the plant, but is solvable by a controller of order \( n+2 \) when \( X \) is assignable and the detectability/stabilizability conditions are satisfied.

Once a controller \( K \) is found through Theorem 1, the CPG control in Fig. 1 can be computed as follows. First, choose an arbitrary positive value of \( \tau \) and find a minimal realization \((A_f,B_f,C_f)\) of \( F(s) \). The state space formula in (18) is derived for the particular eigenstructure \( \Xi = [I_2 \ 0]^T \), as permitted by Lemma 2. Determine \( \hat{\xi} \) so that its real and imaginary parts are the first and second columns of this \( \Xi \). Find \( \hat{q} := C_f \hat{\xi} \) and set \( \alpha_i := |\hat{q}_i| \). Compute the value of the describing function \( K(\alpha) \), and solve (11) for \( L, G, H, \) and \( M \). We now have a CPG controller that satisfies the harmonic balance condition with the desired oscillation profile \((\omega, \hat{x})\) for the plant states.

B. Design Example

In order to illustrate the utility of the CPG control theory developed here, we apply the results to the design of a controller to achieve coordinated oscillations for a three link mechanical arm [14].

The plant is described by
\[
J \ddot{\theta} + D \dot{\theta} + K \theta = Bu, \quad y = B^T \theta,
\]
where \( \theta(t) \in \mathbb{R}^3 \) is the link angles with respect to the inertial frame, \( y(t) \in \mathbb{R}^3 \) is the angular displacements of the three joints connecting the links in series, and \( u(t) \in \mathbb{R}^3 \) is the joint torque inputs. Assuming identical links, the coefficient matrices are given by
\[
J = (mL^2)(I/3 + L^T L), \quad K = lBB^T, \quad D = \rho K,
\]
with parameter values
\[
l = 0.5, \quad k = 1.0, \quad m = 1.0, \quad \rho = 0.1.
\]
We set the target oscillation profile as follows:
\[
\theta_i(t) \cong a_i \sin(\omega t + b_i),
\]
where
\[
a = \begin{bmatrix} 20 \\ 35 \\ 60 \end{bmatrix} \text{ deg}, \quad b = \begin{bmatrix} 0 \\ 120 \\ 240 \end{bmatrix} \text{ deg}, \quad \omega = 3 \text{ rad/s}.
\]

Theorem 1 reduces the oscillation control problem from a search for a single set of design parameters \( K \) satisfying the MHB equation and stability condition to a search for a parameter, \( F \), to satisfy the MHB equation and an output-feedback controller, \( \hat{K} \), to satisfy the stability condition. We utilize this by first solving the problem in terms of \((F, \hat{K})\) and then converting it into the original controller parameters, \( K \). Note that since Theorem 1 has fixed the value of \( \Xi \), the choice of \( \alpha \) and \( \beta \) in the oscillation profile of \( q(t) \) has also been fixed.

The only remaining parameter is \( \tau \) in \( f(s) \), which is an important design parameter that affects both the convergence rate and accuracy of \( x(t) \). Specifically, a smaller \( \tau \) decreases the rate of convergence, but improves the accuracy in matching up to the desired profile. We set \( \tau = 10 \) because it gave us a fairly quick convergence rate without significantly affecting how well the resulting oscillation profile of the plant matched the desired specifications.

Recall that the problem centers around enforcing marginal stability of the quasi-linear system with one pair of conjugate eigenvalues on the imaginary axis at \( \pm j\omega \), where \( \omega \) is the oscillation frequency of the desired profile. Thus we would anticipate seeing this property satisfied for the closed-loop, quasi-linear system (12).

According to the eigenvalues of the closed-loop system displayed in Fig. 2, the marginal stability condition is satisfied and the eigenvalues on the imaginary axis have the correct values \( \pm j3 \). This marginal stability of the quasi-linear closed-loop system is also expected to result in the orbital stability of the nonlinear closed-loop system. As a test of this, we let the closed-loop system start with initial conditions away from the designed limit cycle and see that it converges to the desired profile.
IV. Conclusion

Our work considered the problem of designing a controller such that the closed-loop system will have a specified oscillatory trajectory. To this end, we based our controller on the central pattern generator, a well-studied nonlinear oscillator in biology that can be represented by an interconnection of neurons, each of which contains dynamics described by a transfer function followed by a static nonlinearity. After simplifying the nonlinear problem via a describing function and the method of harmonic balance, we reduced the problem to an eigenstructure assignment problem characterized by a single controller satisfying an output-feedback problem and an eigenvalue/eigenvector equality relationship. We provided a solution to this general eigenstructure assignment problem, in which the control problem was reduced to a search for an output-feedback controller and another independent parameter to satisfy two independent conditions equivalent to those posed in the original problem. We presented a numerical example demonstrating the utility of eigenstructure assignment for the design of CPG-based controllers to achieve orbitally stable closed-loop oscillation.

REFERENCES


TABLE I

<table>
<thead>
<tr>
<th>Target Frequency [rad/s], Simulated Frequency= 2.99 rad/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ₁</td>
</tr>
<tr>
<td>----------------------------------------------</td>
</tr>
<tr>
<td>Target Amplitude [deg]</td>
</tr>
<tr>
<td>Closed-Loop Amplitude [deg]</td>
</tr>
<tr>
<td>Target Phase [deg]</td>
</tr>
<tr>
<td>Closed-Loop Phase [deg]</td>
</tr>
</tbody>
</table>

Although the numerically simulated oscillations in Fig. 3 are not sinusoidal due to the nonlinearities in the CPG control, Fourier analysis on the results show that the amplitudes, phases, and frequency of the first harmonic component are very close to those of the target oscillations. Thus, the approximation through the use of the describing function did not significantly alter the resulting trajectory of the nonlinear closed-loop system with respect to the original specifications.