

Research Article

Robust Exponential Converge Controller Design for a Unified Chaotic System with Structured Uncertainties via LMI

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This paper focuses on the chaos control problem of the unified chaotic systems with structured uncertainties. Applying Schur-complement and some matrix manipulation techniques, the controlled uncertain unified chaotic system is then transformed into the linear matrix inequality (LMI) form. Based on Lyapunov stability theory and linear matrix inequality (LMI) formulation, a simple linear feedback control law is obtained to enforce the prespecified exponential decay dynamics of the uncertain unified chaotic system. Numerical results validate the effectiveness of the proposed robust control scheme.

1. Introduction

Chaotic behavior of physical systems has been studied since 1963 when the Lorenz system was first introduced [1]. Since the pioneering work by Ott et al. [2], chaos control and synchronization has been extensively investigated in recent years. Although the chaotic systems are deterministic systems, their behavior is very sensitive to initial conditions and system parameters and it is unpredictable. The Lorenz system is often taken as a paradigm for demonstrating the effectiveness of control design techniques, since it captures many of the features of chaotic dynamics. The Chen and Ueta [3] and Lü and Chen [4] systems, which are derived from Lorenz system, are other popular paradigms. The unified chaotic system was introduced in [5], which unified the Lorenz, Chen, and Lü systems.

Recently, controlling and synchronizing this kind of complex dynamical systems has attracted a great deal of attention within the engineering society. Chaos control, in

a broader sense, can be divided into two categories: one is to suppress the chaotic dynamical behavior and the other is to generate or enhance chaos in nonlinear systems. Nowadays, many methods and techniques have been developed, such as OGY method [2], bang-bang control [6], optimal control [7], intelligent control base on neural network [8], feedback linearization [9], differential geometric method [10], adaptive control [11, 12], H^∞ control method [13], and linear matrix inequality (LMI) technique [14], among many others [1].

An increasing number of studies have formulated the problem of obtaining the Lyapunov-based quadratic stability of uncertain systems as a linear matrix inequality (LMI) optimization problem [15, 16] lately. However, the use of a fixed quadratic function as an optimization criterion generates conservative solutions in the case of parametric variations. Accordingly, Gahinet et al. [17] proposed an improved (i.e., less conservative) test for the robust stability of linear systems with uncertain real parameters in which the fixed Lyapunov function was replaced by a Lyapunov function with an affine dependence on the uncertain parameters. Similarly, Yang and Lum [18] showed that a Lyapunov function formulated as a linear combination of the uncertain system parameters yielded less conservative results than those obtained using a fixed Lyapunov function. However, the use of a parameter-dependent Lyapunov function rather than a fixed function increases the number of decision variables to be solved in the LMI optimization process and therefore increased the time of the computational procedure. In 2009, Lien et al. [19] studied global exponential stability for a class of uncertain delayed neural networks of neutral type. In the basis of Lien et al., we brought up the robust exponential stability and controller design of systems with structured uncertainties. Based on the Lyapunov stability criterion, the problem of achieving the robust exponential stability of the uncertain system has first been formulated as an LMI feasible problem. The problem of determining the exponential decay rate of the structured uncertain system has then been formulated as an LMI-generalized eigenvalue problem. Finally, the controller required to achieve a robust exponential stability of the uncertain system has been formulated as a second LMI feasible problem. Therefore, the control gain could be found from the LMI formulation in the MATLAB environment.

In 2009, Kuntanapreeda [14] presents an alternative synchronization design for the unified chaotic systems based on Lyapunov stability theory and LMI formulation, when the system uncertainty is not considered. However, the system uncertainties do exist in real physical system. Therefore, we will take the time-varying perturbation of parameters into account for the robust chaos suppression control problem of uncertain unified chaotic systems in this paper, just by a simple structure linear feedback controller. Besides, as is known, if physical systems operate in a state of aperiodic motion, the subsequently large broadband variation may increase the likelihood of fatigue failure and shorten the system lifetime. Therefore, designing a controller to suppress the chaotic behavior is more important than designing a controller to synchronize the chaotic systems. Motivated by the above observations, in this paper, we revisit the chaos suppressing control problem of uncertain unified chaotic system. Based on the concept by LMI, a robust controller design for chaos control can be easily obtained. There is guaranteed the exponential stability of the controlled uncertain unified chaotic system. The rest of this paper is organized as follows. Section 2 addresses the system description for the uncertain unified chaotic system. In Section 3, an LMI based robust controller is derived. For illustration, a numerical example is shown in Section 4. Finally, some conclusions are drawn in Section 5.

2. System Formulation

Consider the following unified chaotic system described by [5]

$$\begin{aligned}\dot{x}_1 &= (25\alpha + 10)(x_2 - x_1), \\ \dot{x}_2 &= (28 - 35\alpha)x_1 + (29\alpha - 1)x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - \frac{8 + \alpha}{3}x_3,\end{aligned}\tag{2.1}$$

where x_1 , x_2 , and x_3 are state variables and the parameter α belongs to $[0, 1]$. When $\alpha \in [0, 0.8)$, system (2.1) is a Lorenz chaotic system, while the system becomes Lü's chaotic system for $\alpha = 0.8$ and then becomes Chen's chaotic system for $\alpha \in (0.8, 1]$. Obviously, system (2.1) always displays chaotic motion in the whole interval $\alpha \in [0, 1]$ [1]. Taking into consideration the system's uncertainties, system (2.1) can be expressed as

$$\begin{aligned}\dot{x}_1 &= (25(\alpha_0 + p_1) + 10)(x_2 - x_1), \\ \dot{x}_2 &= (28 - 35(\alpha_0 + p_1))x_1 + (29(\alpha_0 + p_1) - 1)x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - \frac{8 + (\alpha_0 + p_1)}{3}x_3,\end{aligned}\tag{2.2}$$

where α_0 is the nominal value of α and p_1 is a system structured uncertainty of parameter α . In general, it is assumed that $\Delta\alpha$ is bounded, that is,

$$|p_1| \leq \tilde{p}_1.\tag{2.3}$$

It is assumed that the uncertain unified chaotic system (2.2) still demonstrated a chaotic behavior. To suppress the undesired chaotic dynamics of the uncontrolled system (2.2), the proposed method adds a control-input $u(t)$ to the differential equation of state x_2 . The controlled system thus becomes

$$\begin{aligned}\dot{x}_1 &= (25(\alpha_0 + p_1) + 10)(x_2 - x_1), \\ \dot{x}_2 &= (28 - 35(\alpha_0 + p_1))x_1 + (29(\alpha_0 + p_1) - 1)x_2 - x_1x_3 + u(t), \\ \dot{x}_3 &= x_1x_2 - \frac{8 + (\alpha_0 + p_1)}{3}x_3.\end{aligned}\tag{2.4}$$

The addition of $u(t)$ will be used to provide a robust controller to derive the system state to the zero point in the state space, even when the system is experiencing match and mismatch uncertainties. From (2.4), if x_1 and x_2 are both controlled to zeros and the coefficient $(8 + (\alpha_0 + p_1))/3$ is positive, then x_3 converges to zero when $t \rightarrow \infty$. As the result, the closed loop system state is $(x_1(t), x_2(t), x_3(t)) \rightarrow (0, 0, 0)$ as $t \rightarrow \infty$. Consequentially, the chaos behavior of system is suppressed.

3. Robust Controller Design

Let the control input be $u(t) = u_1(t) + u_2(t)$ and $u_1(t) = x_1 \cdot x_3$ such that the nonlinear term is cancelled, and then the controlled system (2.4) becomes

$$\dot{x}_1 = (25(\alpha_0 + p_1) + 10)(x_2 - x_1), \quad (3.1a)$$

$$\dot{x}_2 = (28 - 35(\alpha_0 + p_1))x_1 + (29(\alpha_0 + p_1) - 1)x_2 + u_2(t),$$

$$\dot{x}_3 = x_1x_2 - \frac{8 + (\alpha_0 + p_1)}{3}x_3. \quad (3.1b)$$

From (3.1a) and (3.1b), it can be seen that the dynamics of x_1 and x_2 are decoupled from x_3 . Equations (3.1a) and (3.1b) show that x_3 represents the internal dynamics of the control system, which will converge and stabilize when x_1 and x_2 converge to zero.

Let the uncertain control system (3.1a) be expressed as follows:

$$\dot{x} = (A + E)x + Bu, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m, \quad (3.2a)$$

where $A \in \mathbf{R}^{n \times n}$ and $B \in \mathbf{R}^{n \times m}$ are the nominal system matrices, and $E \in \mathbf{R}^{n \times n}$ is the system uncertainties. Furthermore, assume that the uncertainties E are linear combinations of constant matrices E_i , respectively, and can be described as

$$E = \sum_{i=1}^r p_i E_i, \quad |p_i| \leq \tilde{p}_i \quad \text{for } i = 1, \dots, r. \quad (3.2b)$$

In synthesizing the controller, the following state feedback control law is defined:

$$u = Kx. \quad (3.3)$$

Thus, the controlled system becomes

$$\dot{x} = (A + BK + E)x. \quad (3.4)$$

The controller proposed in this study is designed to enforce certain prespecified exponential decay dynamics. It is well known that the controlled system in (3.4) is exponentially stable if and only if there exist a positive-definite matrix $P = P^T \in \mathbf{R}^n$ and a positive scalar $\eta > 0$ such that the following Lyapunov inequality holds:

$$(A + BK + E + \eta I)^T P + P(A + BK + E + \eta I) < 0, \quad (3.5)$$

where η is the corresponding exponential decay rate. The condition required to achieve the robust exponential stabilizing control of the uncertain system is defined in the following theorem.

Theorem 3.1. For the uncertain control system described in (3.2a) and (3.2b), if there exist symmetric positive-definite matrices $X \in \mathbf{R}^{n \times n}$, $W_i \in \mathbf{R}^{n \times n}$, $W_i = W_i^T > 0$, $i = 1, \dots, r$, and $Z \in \mathbf{R}^{m \times n}$ such that the following condition holds:

$$\begin{bmatrix} \Phi(X, Z, W_i) & X^T & \cdots & \cdots & X^T \\ X & -W_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ X & 0 & \cdots & 0 & -W_r \end{bmatrix} < 0, \quad (3.6a)$$

where

$$\Phi(X, Z, W_i) = XA^T + AX + Z^T B^T + BZ + 2\eta X + \sum_{i=1}^r \tilde{p}_i^2 E_i^T W_i E_i, \quad (3.6b)$$

then the uncertain system can be robustly exponentially stabilized by the state feedback gain:

$$K = ZX^{-1}, \quad (3.7)$$

Proof. Let P be a symmetric positive-definite matrix and multiply each term in (3.5) by the matrix $X = P^{-1}$ such that (3.5) becomes

$$X(A + BK + E + \eta I)^T + (A + BK + E + \eta I)X < 0. \quad (3.8)$$

Furthermore, introduce the matrix $Z = KX$ and substitute the structured uncertainties given in (3.2b) into (3.8) such that the left-hand side of (3.8) becomes

$$\begin{aligned} & XA^T + AX + Z^T B^T + BZ + 2\eta X + XE^T + EX \\ &= XA^T + AX + Z^T B^T + BZ + 2\eta X + \sum_{i=1}^r p_i (XE_i^T + E_i X). \end{aligned} \quad (3.9)$$

Let $W_i \in \mathbf{R}^{n \times n}$, $W_i = W_i^T > 0$, $i = 1, \dots, r$. From [15], it can then be shown that

$$\begin{aligned} & XA^T + AX + Z^T B^T + BZ + 2\eta X + \sum_{i=1}^r p_i (XE_i^T + E_i X) \\ &\leq XA^T + AX + Z^T B^T + BZ + 2\eta X + \sum_{i=1}^r (p_i^2 E_i^T W_i E_i + XW_i^{-1} X) \\ &\leq XA^T + AX + Z^T B^T + BZ + 2\eta X + \sum_{i=1}^r (\tilde{p}_i^2 E_i^T W_i E_i + XW_i^{-1} X). \end{aligned} \quad (3.10)$$

Thus, the dynamics of the uncertain system in (3.2a) and (3.2b) can be exponentially stabilized if the following inequality holds:

$$\Phi(X, Z, W_i) + \sum_{i=1}^r XW_i^{-1}X \leq 0, \quad (3.11)$$

where $\Phi(X, Z, W_i) = XA^T + AX + Z^T B^T + BZ + 2\eta X + \sum_{i=1}^r \tilde{p}_i^2 E_i^T W_i E_i$.

Using the Schur-complement lemma [15], (3.11) can be reformulated in the form of (3.6a) and (3.6b). Thus, the uncertain control system in (3.2a) and (3.2b) can be stabilized by the state feedback gain $K = ZX^{-1}$ and the resulting exponential decay rate of the system dynamics is given by η . \square

Corollary 3.2 (Obtain a robust exponential stabilizing controller). *Consider the following LMI:*

$$\begin{bmatrix} \Phi(X, Z, W_i) & X^T & \cdots & \cdots & X^T \\ X & -W_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ X & 0 & \cdots & 0 & -W_r \end{bmatrix} < \beta I, \quad (3.12)$$

where $\Phi(X, Z, W_i) = XA^T + AX + Z^T B^T + BZ + 2\eta X + \sum_{i=1}^r \tilde{p}_i^2 E_i^T W_i E_i$. The uncertain control system described in (3.2a) and (3.2b) can be robustly stabilized if the following LMI feasible problem has an optimal solution of $\beta_{\min} < 0$:

$$\beta_{\min} = \min_{X, Z, W_1, \dots, W_r, V_1, \dots, V_s} \beta \quad (3.13)$$

subject to (3.12),

$$\begin{aligned} X &= X^T > 0, \quad X \in \mathbf{R}^{n \times n}, \\ W_i &= W_i^T > 0, \quad i = 1, \dots, r, \end{aligned} \quad (3.14)$$

and $\beta < 0$.

The corresponding state feedback control law is

$$u = ZX^{-1}x. \quad (3.15)$$

Remark 3.3. In solving an LMI optimization problem utilizing a numerical technique, the number of decision variables is an important concern. A matrix variable $W \in \mathbf{R}^{n \times n}$, $W = W^T > 0$, has a total of $n(n+1)/2$ decision variables. However, if matrix W is replaced by a diagonal matrix $W = \text{diag}(w_{ii}) > 0$ such that the condition $W = W^T > 0$ is satisfied, then the total number of decision variables is reduced to n . In other words, for systems with a large order, the use of a diagonal matrix yields an effective reduction in the computational burden.

Remark 3.4. If the multiscroll chaotic system [20–22] with system uncertainties can be written as the form of (3.2a), then it also can be controlled by the same scheme in this paper. Therefore, the main Theorem 3.1 can be extended to chaos suppression control of multiscroll chaotic systems or other typical chaotic system which can be described as in (3.2a).

4. Numerical Simulations

In this section, computer simulations are used to illustrate the effectiveness of the proposed scheme. The initial condition of uncertain unified chaotic system is $(0.5, 2, -2)$. The fourth-order Runge-Kutta technique is applied to the simulation results with a step size 0.0001. For the controlled system (3.1a) and (3.1b), the matrices in (3.2a) are

$$A = \begin{bmatrix} -(25\alpha_0 + 10) & 25\alpha_0 + 10 \\ 28 - 35\alpha_0 & 29\alpha_0 - 1 \end{bmatrix}, \quad E = p_1 \cdot \begin{bmatrix} -25 & 25 \\ -35 & 29 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (4.1)$$

Furthermore, the matrix in (3.2b) is

$$E_1 = \begin{bmatrix} -25 & 25 \\ -35 & 29 \end{bmatrix}. \quad (4.2)$$

The parameter and uncertain term of uncontrolled system (2.2) are $\alpha_0 = 0.5$ and $p_1 = 0.49 \times \sin(t)$. Therefore, the parameter $\alpha = \alpha_0 + p_1$ varied in the range of $[0.01, 0.99]$. It guarantees that the uncontrolled uncertain unified system is in the state of chaotic motion, and its behavior changes among Chen, Lorenz, and Lü systems. The complex trajectory in phase plane is shown in Figure 1. The perturbed parameter $p_1 = 0.49 \times \sin(t)$ is bounded as $|p_1| \leq 0.49 = \tilde{p}_1$. Assume that the objective here is to design a robust exponential stabilizing controller which ensures that the dynamics of the controlled system have an exponential decay rate of $\eta = 0.5$. By using Corollary 3.2, the decision variables can be obtained as

$$X = \begin{bmatrix} 0.4631 & -0.1608 \\ -0.1608 & 0.3026 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.6612 & -0.4226 \\ -0.4226 & 0.3555 \end{bmatrix}, \quad Z = [10.6817 \quad -15.0971]. \quad (4.3)$$

Therefore, the resulting robust state feedback gain is calculated as

$$K = ZX^{-1} = [7.0413 \quad -46.1504]. \quad (4.4)$$

The simulation results are shown in Figures 2 and 3 under the proposed control law (3.15) which is active at time $t = 10$ seconds. Figure 2 shows the time responses of controlled system. The continuous control signal is shown in Figure 3. These results show that the proposed method works successfully to suppress chaos in the unified chaotic systems even though the uncertainty is present.

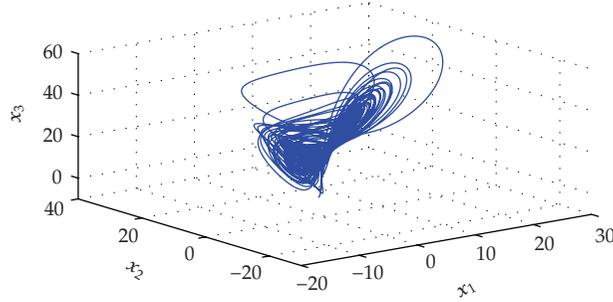


Figure 1: The phase plane trajectory of uncontrolled uncertain unified chaotic system when $\alpha_0 = 0.5$ and $p_1 = 0.49 \times \sin(t)$.

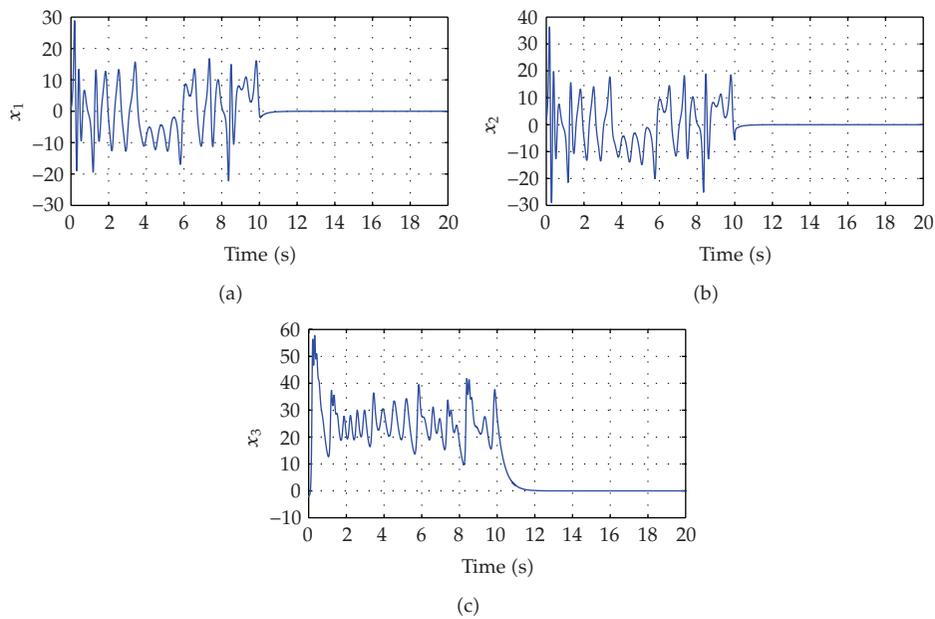


Figure 2: The time responses of controlled uncertain unified chaotic system when $\alpha_0 = 0.5$ and $p_1 = 0.49 \times \sin(t)$. The control is activated at time $t = 10$.

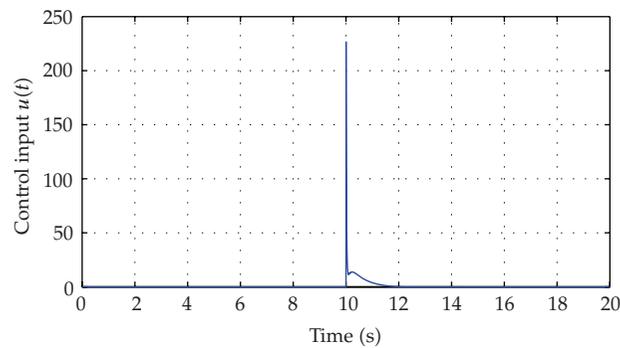


Figure 3: The time response of control input when $\alpha_0 = 0.5$ and $p_1 = 0.49 \times \sin(t)$. The control is activated at time $t = 10$.

5. Conclusion

In this paper, the robust exponential stability and controller design of systems with structured uncertainties is studied. Based on the Lyapunov stability criterion and linear matrix inequality (LMI) technique, the controller of achieving the robust exponential stability of the uncertain unified chaotic system can be obtained. Computer simulations show that the proposed method is effective. For all simulation cases, the obtained control law is able to chaos suppression control of the uncertain unified chaotic system as desired.

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