Fundamental theorem of algebra

In mathematics, the fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. Equivalently, the field of complex numbers is algebraically closed.

Sometimes, this theorem is stated as: every non-zero single-variable polynomial, with complex coefficients, has exactly as many complex roots as its degree, if each root is counted up to its multiplicity. Although this at first appears to be a stronger statement, it is a direct consequence of the other form of the theorem, through the use of successive polynomial division by linear factors.

In spite of its name, there is no purely algebraic proof of the theorem, since any proof must use the completeness of the reals (or some other equivalent formulation of completeness), which is not an algebraic concept. Additionally, it is not fundamental for modern algebra; its name was given at a time in which algebra was mainly about solving polynomial equations with real or complex coefficients.

History

Peter Rothe (Petrus Roth), in his book *Arithmetica Philosophica* (published in 1608), wrote that a polynomial equation of degree \( n \) (with real coefficients) may have \( n \) solutions. Albert Girard, in his book *L'invention nouvelle en l'Algèbre* (published in 1629), asserted that a polynomial equation of degree \( n \) has \( n \) solutions, but he did not state that they had to be real numbers. Furthermore, he added that his assertion holds “unless the equation is incomplete”, by which he meant that no coefficient is equal to 0. However, when he explains in detail what he means, it is clear that he actually believes that his assertion is always true; for instance, he shows that the equation \( x^4 = 4x - 3 \), although incomplete, has four solutions (counting multiplicities): 1 (twice), \(-1 + i\sqrt{2}\), and \(-1 - i\sqrt{2}\).

As will be mentioned again below, it follows from the fundamental theorem of algebra that every non-constant polynomial with real coefficients can be written as a product of polynomials with real coefficients whose degree is either 1 or 2. However, in 1702 Leibniz said that no polynomial of the type \( x^4 + a^4 \) (with \( a \) real and distinct from 0) can be written in such a way. Later, Nikolaus Bernoulli made the same assertion concerning the polynomial \( x^4 - 4x^3 + 2x^2 + 4x + 4 \), but he got a letter from Euler in 1742 in which he was told that his polynomial happened to be equal to

\[
(x^2 - (2 + \alpha)x + 1 + \sqrt{7} + \alpha)(x^2 - (2 - \alpha)x + 1 + \sqrt{7} - \alpha),
\]

where \( \alpha \) is the square root of \( 4 + 2\sqrt{7} \). Also, Euler mentioned that

\[
x^4 + a^4 = (x^2 + a\sqrt{2} \cdot x + a^2)(x^2 - a\sqrt{2} \cdot x + a^2).
\]

A first attempt at proving the theorem was made by d'Alembert in 1746, but his proof was incomplete. Among other problems, it assumed implicitly a theorem (now known as Puiseux's theorem) which would not be proved until more than a century later, and furthermore the proof assumed the fundamental theorem of algebra. Other attempts were made by Euler (1749), de Foncenex (1759), Lagrange (1772), and Laplace (1795). These last four attempts assumed implicitly Girard's assertion; to be more precise, the existence of solutions was assumed and all that remained to be proved was that their form was \( a + bi \) for some real numbers \( a \) and \( b \). In modern terms, Euler, de Foncenex, Lagrange, and Laplace were assuming the existence of a splitting field of the polynomial \( p(z) \).

At the end of the 18th century, two new proofs were published which did not assume the existence of roots. One of them, due to James Wood and mainly algebraic, was published in 1798 and it was totally ignored. Wood's proof had an algebraic gap. The other one was published by Gauss in 1799 and it was mainly geometric, but it had a topological gap, filled by Alexander Ostrowski in 1920, as discussed in Smale 1981.
A rigorous proof was published by Argand in 1806; it was here that, for the first time, the fundamental theorem of algebra was stated for polynomials with complex coefficients, rather than just real coefficients. Gauss produced two other proofs in 1816 and another version of his original proof in 1849.

The first textbook containing a proof of the theorem was Cauchy's *Cours d'analyse de l'École Royale Polytechnique* (1821). It contained Argand's proof, although Argand is not credited for it.

None of the proofs mentioned so far is constructive. It was Weierstrass who raised for the first time, in the middle of the 19th century, the problem of finding a constructive proof of the fundamental theorem of algebra. He presented his solution, that amounts in modern terms to a combination of the Durand–Kerner method with the homotopy continuation principle, in 1891. Another proof of this kind was obtained by Hellmuth Kneser in 1940 and simplified by his son Martin Kneser in 1981.

Without using countable choice, it is not possible to constructively prove the fundamental theorem of algebra for complex numbers based on the Dedekind real numbers (which are not constructively equivalent to the Cauchy real numbers without countable choice\[^{[3]}\]). However, Fred Richman proved a reformulated version of the theorem that does work\[^{[4]}\].

### Proofs

All proofs below involve some analysis, at the very least the concept of continuity of real or complex functions. Some also use differentiable or even analytic functions. This fact has led some to remark that the Fundamental Theorem of Algebra is neither fundamental, nor a theorem of algebra.

Some proofs of the theorem only prove that any non-constant polynomial with real coefficients has some complex root. This is enough to establish the theorem in the general case because, given a non-constant polynomial \(p(z)\) with complex coefficients, the polynomial

\[
q(z) = p(z)\overline{p(\overline{z})}
\]

has only real coefficients and, if \(z\) is a zero of \(q(z)\), then either \(z\) or its conjugate is a root of \(p(z)\).

A large number of non-algebraic proofs of the theorem use the fact (sometimes called “growth lemma”) that an \(n\)-th degree polynomial function \(p(z)\) whose dominant coefficient is 1 behaves like \(z^n\) when \(|z|\) is large enough. A more precise statement is: there is some positive real number \(R\) such that:

\[
\frac{1}{2}\ |z^n| < |p(z)| < \frac{3}{2}\ |z^n|
\]

when \(|z| > R\).

### Complex-analytic proofs

Find a closed disk \(D\) of radius \(r\) centered at the origin such that \(|p(z)| > |p(0)|\) whenever \(|z| \geq r\). The minimum of \(|p(z)|\) on \(D\), which must exist since \(D\) is compact, is therefore achieved at some point \(z_0\) in the interior of \(D\), but not at any point of its boundary. The minimum modulus principle implies then that \(p(z_0) = 0\). In other words, \(z_0\) is a zero of \(p(z)\).

A variation of this proof that does not require the use of the minimum modulus principle (most of whose proofs in turn require the use of Cauchy's integral theorem or some of its consequences) is based on the observation that for the special case of a polynomial function, the minimum modulus principle can be proved directly using elementary arguments. More precisely, if we assume by contradiction that \(a := p(z_0) \neq 0\), then, expanding \(p(z)\) in powers of \(z - z_0\) we can write...
Here, the \( c_j \)'s are simply the coefficients of the polynomial \( z \rightarrow p(z + z_0) \), and we let \( k \) be the index of the first coefficient following the constant term that is non-zero. But now we see that for \( z \) sufficiently close to \( z_0 \) this has behavior asymptotically similar to the simpler polynomial \( q(z) = a + c_k(z - z_0)^k \), in the sense that (as is easy to check) the function \( \frac{|p(z) - q(z)|}{(z - z_0)^{k+1}} \) is bounded by some positive constant \( M \) in some neighborhood of \( z_0 \). Therefore if we define \( \theta_0 = (\arg(a) + \pi - \arg(c_k)) / k \) and let \( z = z_0 + re^{i\theta_0} \), then for any sufficiently small positive number \( r \) (so that the bound \( M \) mentioned above holds), using the triangle inequality we see that

\[
|p(z)| < |q(z)| + r^{k+1} \left| \frac{p(z) - q(z)}{r^{k+1}} \right| \leq |a + (-1)c_k r^k e^{i\arg(a) - \arg(c_k)}| + Mr^{k+1} = |a - c_k r^k|.
\]

When \( r \) is sufficiently close to 0 this upper bound for \( |p(z)| \) is strictly smaller than \( |a| \), in contradiction to the definition of \( z_0 \). (Geometrically, we have found an explicit direction \( \theta_0 \) such that if one approaches \( z_0 \) from that direction one can obtain values \( p(z) \) smaller in absolute value than \( |p(z_0)| \).)

**Another** analytic proof can be obtained along this line of thought observing that, since \( |p(z)| > |p(0)| \) outside \( D \), the minimum of \( |p(z)| \) on the whole complex plane is achieved at \( z_0 \). If \( |p(z_0)| > 0 \), then \( 1/p \) is a bounded holomorphic function in the entire complex plane since, for each complex number \( z \), \( |1/p(z)| \leq |1/p(z_0)| \). Applying Liouville's theorem, which states that a bounded entire function must be constant, this would imply that \( 1/p \) is constant and therefore that \( p \) is constant. This gives a contradiction, and hence \( p(z_0) = 0 \).

**Yet another** analytic proof uses the argument principle. Let \( R \) be a positive real number large enough so that every root of \( p(z) \) has absolute value smaller than \( R \); such a number must exist because every non-constant polynomial function of degree \( n \) has at most \( n \) zeros. For each \( r > R \), consider the number

\[
\frac{1}{2\pi i} \int_{c(r)} \frac{p'(z)}{p(z)} \, dz,
\]

where \( c(r) \) is the circle centered at 0 with radius \( r \) oriented counterclockwise; then the argument principle says that this number is the number \( N \) of zeros of \( p(z) \) in the open ball centered at 0 with radius \( r \), which, since \( r > R \), is the total number of zeros of \( p(z) \). On the other hand, the integral of \( n/z \) along \( c(r) \) divided by \( 2\pi i \) is equal to \( n \). But the difference between the two numbers is

\[
\frac{1}{2\pi i} \int_{c(r)} \left( \frac{p'(z)}{p(z)} - \frac{n}{z} \right) \, dz = \frac{1}{2\pi i} \int_{c(r)} \frac{zp'(z) - np(z)}{zp(z)} \, dz.
\]

The numerator of the rational expression being integrated has degree at most \( n - 1 \) and the degree of the denominator is \( n + 1 \). Therefore, the number above tends to 0 as \( r \) tends to +\( \infty \). But the number is also equal to \( N - n \) and so \( N = n \).

**Still another** complex-analytic proof can be given by combining linear algebra with the Cauchy theorem. To establish that every complex polynomial of degree \( n > 0 \) has a zero, it suffices to show that every complex square matrix of size \( n > 0 \) has a (complex) eigenvalue\(^5\). The proof of the latter statement is by contradiction.
Let $A$ be a complex square matrix of size $n > 0$ and let $I_n$ be the unit matrix of the same size. Assume $A$ has no eigenvalues. Consider the resolvent function

$$R(z) = (zI_n - A)^{-1},$$

which is a meromorphic function on the complex plane with values in the vector space of matrices. The eigenvalues of $A$ are precisely the poles of $R(z)$. Since, by assumption, $A$ has no eigenvalues, the function $R(z)$ is an entire function and Cauchy's theorem implies that

$$\int_{c(r)} R(z) dz = 0.$$

On the other hand, $R(z)$ expanded as a geometric series gives:

$$R(z) = z^{-1}(I_n - z^{-1}A)^{-1} = z^{-1}\sum_{k=0}^{\infty} \frac{1}{z^k}A^k.$$

This formula is valid outside the closed disc of radius $||A||$ (the operator norm of $A$). Let $r > ||A||$. Then

$$\int_{c(r)} R(z) dz = \sum_{k=0}^{\infty} \int_{c(r)} \frac{dz}{z^{k+1}}A^k = 2\pi i I_n$$

(in which only the summand $k = 0$ has a nonzero integral). This is a contradiction, and so $A$ has an eigenvalue.

**Topological proofs**

Let $z_0 \in \mathbb{C}$ be such that the minimum of $|p(z)|$ on the whole complex plane is achieved at $z_0$; it was seen at the proof which uses Liouville's theorem that such a number must exist. We can write $p(z)$ as a polynomial in $z - z_0$: there is some natural number $k$ and there are some complex numbers $c_k, c_{k+1}, \ldots, c_n$ such that $c_k \neq 0$ and that

$$p(z) = p(z_0) + c_k(z - z_0)^k + c_{k+1}(z - z_0)^{k+1} + \cdots + c_n(z - z_0)^n.$$

It follows that if $a$ is a $k$th root of $-p(z_0)/c_k$ and if $t$ is positive and sufficiently small, then $|p(z_0 + ta)| < |p(z_0)|$, which is impossible, since $|p(z_0)|$ is the minimum of $|p|$ on $D$.

For another topological proof by contradiction, suppose that $p(z)$ has no zeros. Choose a large positive number $R$ such that, for $|z| = R$, the leading term $z^n$ of $p(z)$ dominates all other terms combined; in other words, such that $|z|^n > |a_{n-1}z^{n-1} + \cdots + a_0|$. As $z$ traverses the circle given by the equation $|z| = R$ once counter-clockwise, $p(z)$, like $z^n$, winds $n$ times counter-clockwise around 0. At the other extreme, with $|z| = 0$, the “curve” $p(z)$ is simply the single (nonzero) point $p(0)$, whose winding number is clearly 0. If the loop followed by $z$ is continuously deformed between these extremes, the path of $p(z)$ also deforms continuously. We can explicitly write such a deformation as $H(Re^{i\theta}, t) = p((1 - t)Re^{i\theta})$ where $t$ is greater than or equal to 0 and less than or equal to 1. If one views the variable $t$ as time, then at time zero the curve is $p(z)$ and at time one the curve is $p(0)$. Clearly at every point $t$, $p(z)$ cannot be zero by the original assumption, therefore during the deformation, the curve never crosses zero. Therefore the winding number of the curve around zero should never change. However, given that the winding number started as $n$ and ended as 0, this is absurd. Therefore, $p(z)$ has at least one zero.
Algebraic proofs

These proofs use two facts about real numbers that require only a small amount of analysis (more precisely, the intermediate value theorem):

- every polynomial with odd degree and real coefficients has some real root;
- every non-negative real number has a square root.

The second fact, together with the quadratic formula, implies the theorem for real quadratic polynomials. In other words, algebraic proofs of the fundamental theorem actually show that if \( R \) is any real-closed field, then its extension \( C = R(\sqrt{-1}) \) is algebraically closed.

As mentioned above, it suffices to check the statement “every non-constant polynomial \( p(z) \) with real coefficients has a complex root”. This statement can be proved by induction on the greatest non-negative integer \( k \) such that \( 2^k \) divides the degree \( n \) of \( p(z) \). Let \( a \) be the coefficient of \( z^n \) in \( p(z) \) and let \( F \) be a splitting field of \( p(z) \) over \( C \); in other words, the field \( F \) contains \( C \) and there are elements \( z_1, z_2, \ldots, z_n \) in \( F \) such that

\[
p(z) = a(z - z_1)(z - z_2) \cdots (z - z_n).
\]

If \( k = 0 \), then \( n \) is odd, and therefore \( p(z) \) has a real root. Now, suppose that \( n = 2^km \) (with \( m \) odd and \( k > 0 \)) and that the theorem is already proved when the degree of the polynomial has the form \( 2^{k-1}m' \) with \( m' \) odd. For a real number \( t \), define:

\[
q_t(z) = \prod_{1 \leq i < j \leq n} (z - z_i - z_j - tz_i z_j).
\]

Then the coefficients of \( q_t(z) \) are symmetric polynomials in the \( z_i \)'s with real coefficients. Therefore, they can be expressed as polynomials with real coefficients in the elementary symmetric polynomials, that is, in \(-a_1, a_2, \ldots, (-1)^n a_n\). So \( q_t(z) \) has in fact real coefficients. Furthermore, the degree of \( q_t(z) \) is \( n(n-1)/2 = 2^{k-1}m(n-1) \), and \( m(n-1) \) is an odd number. So, using the induction hypothesis, \( q_t \) has at least one complex root; in other words, \( z_i + z_j + tz_i z_j \) is complex for two distinct elements \( i \) and \( j \) from \( \{1, \ldots, n\} \). Since there are more real numbers than pairs \((i,j)\), one can find distinct real numbers \( t \) and \( s \) such that \( z_i + z_j + tz_i z_j \) and \( z_i + z_j + sz_i z_j \) are complex (for the same \( i \) and \( j \)). So, both \( z_i + z_j \) and \( z_i z_j \) are complex numbers. It is easy to check that every complex number has a complex square root, thus every complex polynomial of degree 2 has a complex root by the quadratic formula. It follows that \( z_i \) and \( z_j \) are complex numbers, since they are roots of the quadratic polynomial \( z^2 - (z_i + z_j)z + z_i z_j \).

J. Shipman showed in 2007 that the assumption that odd degree polynomials have roots is stronger than necessary; any field in which polynomials of prime degree have roots is algebraically closed (so "odd" can be replaced by "odd prime" and furthermore this holds for fields of all characteristics). This is the best possible, as there are counterexamples if a single prime is excluded.

Another algebraic proof of the fundamental theorem can be given using Galois theory. It suffices to show that \( C \) has no proper finite field extension. Let \( K/C \) be a finite extension. Since the normal closure of \( K \) over \( R \) still has a finite degree over \( C \) (or \( R \)), we may assume without loss of generality that \( K \) is a normal extension of \( R \) (hence it is a Galois extension, as every algebraic extension of a field of characteristic 0 is separable). Let \( G \) be the Galois group of this extension, and let \( H \) be a Sylow 2-group of \( G \), so that the order of \( H \) is a power of 2, and the index of \( H \) in \( G \) is odd. By the fundamental theorem of Galois theory, there exists a subextension \( L \) of \( K/R \) such that \( Gal(K/L) = H \). As \([L:R] = [G:H] \) is odd, and there are no nonlinear irreducible real polynomials of odd degree, we must have \( L = R \), thus \([K:R]\) and \([K:C]\) are powers of 2. Assuming for contradiction \([K:C] > 1\), the 2-group \( Gal(K/C) \) contains a subgroup of index 2, thus there exists a subextension \( M \) of \( C \) of degree 2. However, \( C \) has no extension of degree 2, because every quadratic complex polynomial has a complex root, as mentioned above.
Corollaries

Since the fundamental theorem of algebra can be seen as the statement that the field of complex numbers is algebraically closed, it follows that any theorem concerning algebraically closed fields applies to the field of complex numbers. Here are a few more consequences of the theorem, which are either about the field of real numbers or about the relationship between the field of real numbers and the field of complex numbers:

- The field of complex numbers is the algebraic closure of the field of real numbers.
- Every polynomial in one variable $x$ with real coefficients is the product of a constant, polynomials of the form $x + a$ with $a$ real, and polynomials of the form $x^2 + ax + b$ with $a$ and $b$ real and $a^2 - 4b < 0$ (which is the same thing as saying that the polynomial $x^2 + ax + b$ has no real roots).
- Every rational function in one variable $x$, with real coefficients, can be written as the sum of a polynomial function with rational functions of the form $a/(x - b)^n$ (where $n$ is a natural number, and $a$ and $b$ are real numbers), and rational functions of the form $(ax + b)/(x^2 + cx + d)^n$ (where $n$ is a natural number, and $a$, $b$, $c$, and $d$ are real numbers such that $c^2 - 4d < 0$). A corollary of this is that every rational function in one variable and real coefficients has an elementary primitive.
- Every algebraic extension of the real field is isomorphic either to the real field or to the complex field.

Bounds on the zeroes of a polynomial

While the fundamental theorem of algebra states a general existence result, it is of some interest, both from the theoretical and from the practical point of view, to have information on the location of the zeroes of a given polynomial. The simpler result in this direction is a bound on the modulus: all zeroes $\zeta$ of a monic polynomial $a_n \zeta^n + a_{n-1} \zeta^{n-1} + \cdots + a_1 \zeta + a_0$ satisfy an inequality $|\zeta| \le R_\infty$, where

$$ R_\infty := 1 + \max \{|a_0|, \ldots, |a_{n-1}|\}. $$

Notice that, as stated, this is not yet an existence result but rather an example of what is called an a priori bound: it says that if there are solutions then they lay inside the closed disk of center the origin and radius $R_\infty$. However, once coupled with the fundamental theorem of algebra it says that the disk contains in fact at least one solution. More generally, a bound can be given directly in terms of any $p$-norm of the $n$-vector of coefficients $a := (a_0, a_1, \ldots, a_{n-1})$, that is $|a| \le R_p$, where $R_p$ is precisely the $q$-norm of the 2-vector $(1, ||a||_p)$, $q$ being the conjugate exponent of $p$, $1/p + 1/q = 1$, for any $1 \le p \le \infty$. Thus, the modulus of any solution is also bounded by

$$ R_1 := \max \left\{ 1, \sum_{0 < k < n} |a_k| \right\}, $$

$$ R_p := \left[ 1 + \left( \sum_{0 < k < n} |a_k|^p \right)^{q/p} \right]^{1/q}, $$

for $1 < p \le \infty$, and in particular

$$ R_2 := \sqrt{\sum_{0 \le k \le n} |a_k|^2}. $$

The case of a generic polynomial of degree $n$, $P(\zeta) := a_n \zeta^n + a_{n-1} \zeta^{n-1} + \cdots + a_1 \zeta + a_0$, is of course reduced to the case of a monic, dividing all coefficients by $a_n \neq 0$. Also, in case that 0 is not a root, i.e. $a_0 \neq 0$, bounds from below on the roots $\zeta$ follow immediately as bounds from above on $1/\zeta$, that is, the roots of $a_0 \zeta^n + a_1 \zeta^{n-1} + \cdots + a_{n-1} \zeta + a_n$. Finally, the distance $|\zeta - \zeta_0|$ from the roots $\zeta$ to any point $\zeta_0$ can be
estimated from below and above, seeing $\zeta - \zeta_0$ as zeroes of the polynomial $P(z + \zeta_0)$, whose coefficients are the Taylor expansion of $P(z)$ at $z = \zeta_0$.

We report here the proof of the above bounds, which is short and elementary. Let $\zeta$ be a root of the polynomial $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$; in order to prove the inequality $|\zeta| \leq R_0$ we can assume, of course, $|\zeta| > 1$. Writing the equation as $-\zeta^n = a_{n-1} \zeta^{n-1} + \cdots + a_1 \zeta + a_0$, and using the Hölder’s inequality we find $|\zeta|^n \leq \|a\|_p \|(\zeta^{n-1}, \cdots, \zeta, 1)|_q$. Now, if $p = 1$, this is $|\zeta|^n \leq \|a\|_1 \max\{|z|^{n-1}, \cdots, |z|, 1\} = \|a\|_1 |\zeta|^{n-1}$, thus $|\zeta| \leq \max\{1, \|a\|_\infty\}$. In the case $1 < p \leq \infty$, taking into account the summation formula for a geometric progression, we have

$$|\zeta|^n \leq \|a\|_p \left( |\zeta|^{(n-1)} + \cdots + |\zeta|^{q+1} \right)^{1/q} = \|a\|_p \left( |\zeta|^{qn - 1} \right)^{1/q} \leq \|a\|_p \left( \frac{|\zeta|^{qn}}{|\zeta|^{q} - 1} \right)^{1/q}$$

thus $|\zeta|^n \leq \|a\|_p^{q} \frac{|\zeta|^{qn}}{|\zeta|^{q} - 1}$ and simplifying, $|\zeta|^q \leq 1 + \|a\|_p^{q}$. Therefore $|\zeta| \leq \|1, \|a\|_p \|_q = R_0$ holds, for all $1 < p \leq \infty$.

See also Properties of polynomial roots, for further results about the location of zeroes.

Notes

2. ^ Concerning Wood’s proof, see the article A forgotten paper on the fundamental theorem of algebra, by Frank Smithies.
3. ^ For the minimum necessary to prove their equivalence, see Bridges, Schuster, and Richman; 1998; A weak countable choice principle; available from [1].
4. ^ See Fred Richman; 1998; The fundamental theorem of algebra: a constructive development without choice; available from [2].
5. ^ A proof of the fact that this suffices can be seen here.
6. ^ A proof of the fact that this suffices can be seen here.

References

Historic sources

- Gauss, Carl Friedrich (1799), *Demonstratio nova theorematis omnem functionem algebraicam rationalem integrum unius variabilis in factores reales primi vel secundi gradus resolvi posse*, Helmstedt: C. G. Fleckeisen (tr. New proof of the theorem that every integral rational algebraic function of one variable can be resolved into real factors of the first or second degree).
- C. F. Gauss, “Another new proof of the theorem that every integral rational algebraic function of one variable can be resolved into real factors of the first or second degree”, 1815
Recent literature

- Conrad, Keith, *The fundamental Theorem of Algebra via Linear Algebra*

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