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Connected 2-Domination in the Corona of Graphs

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Abstract

A connected 2-dominating set of a graph G is a set S of vertices of G such that every vertex not in S is dominated at least twice and the subgraph induced by S is connected. In this paper, we characterized the connected 2-dominating sets of the join $K_1 + G$ and the corona of graphs and, obtain their connected 2-domination numbers.

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1 Introduction and Preliminary Results

Let $G = (V(G), E(G))$ be an undirected graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* of X is $N(X) = \bigcup_{v \in X} N(v)$ and the *closed neighborhood* of X is $N[X] = \bigcup_{v \in X} N[v]$.

A set $S \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set $S \subseteq V(G)$ is a *connected dominating set* of G if the subgraph $\langle S \rangle$, induced by S, is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the smallest cardinality of a connected dominating set of G. A set $S \subseteq V(G)$ is a 2-*dominating set* of G if for every $v \in V(G) \backslash S$, $|S \cap N_G(v)| \geq 2$. The 2-*domination number* of G, denoted by $\gamma_2(G)$, is the smallest cardinality of a 2-dominating set of G. A 2-dominating set $S \subseteq V(G)$ is a *connected* 2-*dominating set* of G if the subgraph $\langle S \rangle$ is connected. The *connected* 2-*domination number* of G, denoted by $\gamma_{2c}(G)$, is the smallest cardinality of a connected 2-dominating set of G.

The concept of 2-domination was introduced by Fink and Jacobson and was studied in [1], where bounds on the 2-domination numbers of cactus graphs were obtained. In [2], the 2-domination numbers of trees were studied. A similar concept, the double domination in graphs, were studied in [3]. In this paper, the connected 2-dominating sets in the join $K_1 + G$ and the corona of graphs were characterized and their connected 2-domination numbers were obtained.

The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Let G and H be graphs of order m and n, respectively. The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H.

Remark 1.1 *Let G be a connected graph of order $n \geq 2$. Then $2 \leq \gamma_{2c}(G) \leq n$.*

2 Main Results

Theorem 2.1 *Let $K_1 = \langle \{v\} \rangle$ and let G be a graph of order at least 2. Then $\gamma_{2c}(K_1 + G) = 2$ if and only if one of the following holds:*
(i) $\gamma_{2c}(G) = 2$;
(ii) $\gamma(G) = 1$.

Proof: Suppose $\gamma_{2c}(K_1+G) = 2$. Let $S = \{u, w\}$ be a connected 2-dominating set of $K_1 + G$. If $v \notin S$, then $S \subseteq V(G)$. Clearly, $\langle S \rangle$ is a connected set of G. Let $z \in V(G) \backslash S$. Since S a 2-dominating set of $K_1 + G$, $|S \cap N_G(z)| \geq 2$. Thus, S a 2-dominating set of G. Consequently, S a connected 2-dominating set of G. Hence, $\gamma_{2c}(G) \leq |S| = 2$. By Remark 1.1, $2 \leq \gamma_{2c}(G)$. Therefore, $\gamma_{2c}(G) = 2$ and this proves (i). Suppose $v \in S$, say $u = v$. Then $S \backslash \{v\} = \{w\} \subseteq V(G)$. Let $y \in V(G) \backslash \{w\}$. Suppose $\{w\}$ is not a dominating set of G. Then $\{w\} \cap N_G(y) = \varnothing$, which implies that $\{w\} \cap N_{K_1+G}(y) = \varnothing$. Since

$v \in N_{K_1+G}(y)$, $S \cap N_{K_1+G}(y) < 2$. This contradicts the assumption that S a connected 2-dominating set of $K_1 + G$. Hence, $\{w\}$ is a dominating set of G and so $\gamma(G) = 1$. This proves (ii).

Conversely, suppose first that $\gamma_{2c}(G) = 2$. Let $S = \{a, b\}$ be a connected 2-dominating set of G. Clearly, $\langle S\rangle$ is a connected set of $K_1 + G$. Let $c \in V(K_1 + G)\backslash S$. Suppose $c \in V(G)\backslash S$. Since S is a connected 2-dominating set of G, $|S \cap N_G(c)| \geq 2$. Thus, $|S \cap N_{K_1+G}(c)| \geq 2$. Suppose $c = v$. Then $S \subseteq N_{K_1+G}(v)$, which means that $|S \cap N_{K_1+G}(v)| = |S| \geq 2$. Hence, S is a connected 2-dominating set of K_1+G and so, $\gamma_{2c}(K_1+G) = 2$. Next, suppose $\gamma(G) = 1$. Let $\{d\}$ be a dominating set of G. Then $S = \{v, d\}$. Then $\langle S\rangle$ is a connected set of $K_1 + G$. Let $y \in V(K_1 + G)\backslash S$. Since $\{d\}$ is a dominating set of G, $\{d\} \cap N_G(y) \neq \varnothing$, that is, $\{d\} \cap N_{K_1+G}(y) \neq \varnothing$. But $v \in N_{K_1+G}(y)$. Hence, $|S \cap N_{K_1+G}(y)| \geq 2$. This implies that S is a connected 2-dominating set of $K_1 + G$. Thus, $\gamma_{2c}(K_1 + G) = 2$. □

The next result characterizes the connected 2-dominating sets of the join $K_1 + G$.

Theorem 2.2 *Let $K_1 = \langle\{v\}\rangle$ and let G be a graph of order at least 2. Then $S \subseteq V(K_1 + G)$ is a connected 2-dominating set of $K_1 + G$ if and only if one of the following holds:*

(i) $S \subseteq V(G)$ and S is a connected 2-dominating set of G;

(ii) $v \in S$ and $S\backslash\{v\}$ is a dominating set of G.

Proof: Suppose $S \subseteq V(K_1 + G)$ is a connected 2-dominating set of $K_1 + G$. Consider the following cases:

Case 1. $v \notin S$.

Then $S \subseteq V(G)$. Clearly, S is a connected 2-dominating set of G.

Case 2. $v \in S$.

Then $S\backslash\{v\} \subseteq V(G)$. Let $w \in V(G)\backslash(S\backslash\{v\})$. Suppose $S\backslash\{v\}$ is not a dominating set of G. Then $(S\backslash\{v\}) \cap N_G(w) = \varnothing$, which implies that $(S\backslash\{v\}) \cap N_{K_1+G}(w) = \varnothing$. Hence, $S \cap N_{K_1+G}(w) = \{v\}$, which implies that $|S \cap N_{K_1+G}(w) = \{v\}| < 2$. this contradicts the assumption that S is a connected 2-dominating set of $K_1 + G$. Thus $S\backslash\{v\}$ is a dominating set of G.

Conversely, suppose first that $S \subseteq V(G)$ and S is a connected 2-dominating set of G. Clearly, $\langle S\rangle$ is a connected set of $K_1 + G$. Let $x \in V(K_1 + G)\backslash S$. Suppose $x \in V(G)\backslash S$. Since S is a connected 2-dominating set of G, $|S \cap N_G(x)| \geq 2$. Thus, $|S \cap N_{K_1+G}(x)| \geq 2$. Suppose $x = v$. Then $S \subseteq N_{K_1+G}(v)$, which means that $|S \cap N_{K_1+G}(v)| = |S| \geq 2$. Hence, S is a connected 2-dominating set of $K_1 + G$. Next, suppose $v \in S$ and $S\backslash\{v\}$ is a dominating set of G. Then $\langle S\rangle$ is a connected set of $K_1 + G$. Let $y \in V(K_1 + G)\backslash S$. Then $v \in N_{K_1+G}(y)$. Since $y \in V(G)\backslash(S\backslash\{v\})$, there exists $z \in (S\backslash\{v\}) \cap N_G(y)$ since $S\backslash\{v\}$ is a dominating set of G. This implies that $z \in (S\backslash\{v\}) \cap N_{K_1+G}(y)$. Hence, $v, z \in S \cap N_{K_1+G}(y)$, that is, $|S \cap N_{K_1+G}(y)| \geq 2$. Therefore, S is a connected 2-dominating set of K_1+G. $\square$

Corollary 2.3 *Let $K_1 = \langle\{v\}\rangle$ and let G be a graph of order at least 2. Then $\gamma_{2c}(K_1 + G) = \min\{1 + \gamma(G), \gamma_{2c}(G)\}$.*

Proof: Suppose $\gamma_{2c}(G) \leq 1+\gamma(G)$. Let S be a minimum connected 2-dominating set of G. Then $|S| = \gamma_{2c}(G)$. By Theorem 2.2(ii), S is a connected 2-dominating set of $K_1 + G$. Thus, $\gamma_{2c}(K_1 + G) \leq |S| = \gamma_{2c}(G)$. Next, suppose that S' is a minimum connected 2-dominating set of $K_1 + G$. Then $\gamma_{2c}(K_1 + G) = |S'|$. By Theorem 2.2($ii$), S' is a connected 2-dominating set of G. Hence, $\gamma_{2c}(K_1+G) = |S'| \geq \gamma_{2c}(G)$. Therefore, $\gamma_{2c}(K_1+G) = \gamma_{2c}(G)$. Similarly, if we asssume that $1 + \gamma(G) \leq \gamma_{2c}(G)$, then $\gamma_{2c}(K_1 + G) = 1 + \gamma(G)$. Consequently, $\gamma_{2c}(K_1 + G) = \min\{1 + \gamma(G), \gamma_{2c}(G)\}$. $\square$

The next result characterizes the connected 2-dominating set of $G \circ H$.

Theorem 2.4 *Let G be a connected graph and H any graph, each of order at least 2. Then $C \subseteq V(G \circ H)$ is a connected 2-dominating set of $G \circ H$ if and only if $C = V(G) \cup \left(\bigcup_{v \in V(G)} S^v\right)$, where S^v is a dominating set of H^v for all $v \in V(G)$.*

Proof: Suppose C is a connected 2-dominating set of $G \circ H$. Let $v \in V(G)$. By Theorem 2,2(ii), $C \cap V(H^v)$ is a dominating set of H^v. Let $S^v = C \cap V(H^v)$. Therefore, $C = V(G) \cup \left(\bigcup_{v \in V(G)} S^v\right)$.

Conversely, let $C = V(G) \cup \left(\bigcup_{v \in V(G)} S^v\right)$, where S^v is a dominating set of H^v for all $v \in V(G)$. Clearly, $\langle C\rangle$ is a connected set of $G \circ H$. Let $v \in V(G)$ and let $x \in V(G \circ H)\backslash C$. Then $x \in V(H^v)\backslash S^v$. Since S^v is a dominating set of H^v, there exist $y \in S^v \cap N_{H^v}(x)$. Thus, $y \in C \cap N_{G\circ H}(x)$. But $v \in C \cap N_{G\circ H}(x)$. Hence, $|C \cap N_{G\circ H}(x)| \geq 2$. This shows that C is a connected 2-dominating set of $G \circ H$. $\square$

The next result is an immediate consequence of Theorem 2.4.

Corollary 2.5 *Let G be a connected graph of order $m \geq 2$ and H any graph of order at least 2. Then $\gamma_{2c}(G \circ H) = m(1 + \gamma(H))$.*

Proof: Let D be a minimum dominating set of H. Define $S^v \subseteq H^v$ be such that $|S^v| = |D|$ for all $v \in V(G)$. By Theorem 2.4, $C = V(G) \cup \left(\bigcup_{v \in V(G)} S^v \right)$ is a connected 2-dominating set of $G \circ H$. Thus,

$$\gamma_{2c}(G \circ H) \leq |C| = |V(G)| + \sum_{v \in V(G)} |S^v| = m + m \cdot \gamma(H) = m(1 + \gamma(H)).$$

Next, suppose C' is a minimum connected 2-dominating set of $G \circ H$. By Theorem 2.4, $C' = V(G) \cup \left(\bigcup_{v \in V(G)} S^v \right)$, where S^v is a dominating set of H^v for all $v \in V(G)$. Hence,

$$\gamma_{2c}(G \circ H) = |C'| = |V(G)| + \sum_{v \in V(G)} |S^v| \geq m + m \cdot \gamma(H) = m(1 + \gamma(H).$$

Therefore, $\gamma_{2c}(G \circ H) = m(1 + \gamma(H).$ □

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