Int. Journal of Math. Analysis, Vol. 2, 2008, no. 3, 133 - 142

Approximating Solution for Systems of Strongly Accretive Operator Equations on Weakly Continuous Duality Maps

Issara Inchan

Department of Mathematics and Computer
Uttaradit Rajabhat University
Uttaradit 53000, Thailand
peissara@uru.ac.th

Somyot Plubtieng

Department of Mathematics
Naresuan University
Phitsanulok 65000, Thailand
somyotp@nu.ac.th

Abstract

This paper introduce a iteration scheme for approximation of strongly accretive operator equations in reflexive Banach space with weakly continuous duality mapping. Our result improve the results of Y. Xu [Ishikawa and Mann Iterative Processes with Errors for Nonlinear Strongly Accretive Operator Equations, J. Math. Anal Appl. 224(1998) 91-101] and I. Inchan and S. Plubtieng [Approximating solutions for the systems of strongly accretive operator equations, Com. Math. Appl. 53(2007) 1317-1324] and many author.

Mathematics Subject Classification: 47H10, 47H09, 46B20

Keywords: fixed point, strongly accretive operator equations, strongly pseudocontractive mapping, weakly continuous duality maps

1 Introduction.

Mann and Ishikawa iteration processes have been studied extensively by various authors for approximating either fixed points of nonlinear mappings or solutions of nonlinear operator equations in Banach (see, e.g., [3, 5, 7, 8, 10, 11, 15, 19, 22, 25]. Recently, Chidume [4], Liu [14], Osilike [20] and Xu [23], introduced the concepts of Isikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in uniformly smooth Banach spaces. On the other hand Noor [17], has suggested and analyzed three-step iterative method for finding the approximate solutions of the variational indusions (inequalities) in a Hilbert space. Glowinski and Le Tallee [9] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal Theory, and eigenvalue problem. It has been shown in [9] that the three-step iterative scheme gives better numerical results than the two-step and one-step. In 2002, Noor, Rassias and Huang [18] has suggested the three-step iteration process for solving the nonlinear strongly accretive operators equations in a real uniformly smooth Banach spaces. Moreover, Inchan and Plubtieng [12] introduced and studied a multi-step scheme with errors to approximate fixed points of asymptotically nonexpansive mappings in a real uniformly smooth Banach spaces. These facts motivated us to introduce and analyze a class of multi-step iterative scheme with errors for solving nonlinear strongly accretive operator equation in a Banach space with weakly continuous duality map with gauge φ.

In 1998, Xu [23] introduce the Ishikawa iteration with errors defined by

$$\begin{cases} x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \\ y_n = \alpha_n' x_n + \beta_n' T x_n + \gamma_n' v_n, n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}, \{\alpha_n'\}, \{\beta_n\}, \{\beta_n'\}, \{\gamma_n\}, \{\gamma_n'\}$ are six sequences in $[0, 1]$ satisfying the conditions $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n'$ for all $n \geq 1$ and $\{u_n\}, \{v_n\}$ are bounded sequence in K. Then $\{x_n\}$ converge strongly to the unique fixed point of T.

Recently, Noor, Rassias and Huang [18] introduce three-step iterations defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1, \\ y_n = x_n^2 = (1 - \beta_n)x_n + \beta_n T z_n, \quad n \geq 1, \\ z_n = (1 - \gamma_n)x_n + \gamma_n T x_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three real sequences in $[0, 1]$. Then $\{x_n\}$ converge strongly to the unique fixed point of T.

Moreover, Inchan and Plubtieng [12], extended and improve (1.2) the following iteration scheme with errors to finite mappings $\{T_1, T_2, ..., T_N\}$ of bounded closed convex C defined by

$$\begin{cases} x_1 = x \in C, \\ x_n^{(1)} = \alpha_n^{(1)} x_n + \beta_n^{(1)} T_1 x_n + \gamma_n^{(1)} u_n^{(1)}, \\ x_n^{(2)} = \alpha_n^{(2)} x_n + \beta_n^{(2)} T_2 x_n^{(1)} + \gamma_n^{(2)} u_n^{(2)}, \\ \vdots \\ x_{n+1} = x_n^{(N)} = \alpha_n^{(N)} x_n + \beta_n^{(N)} T_N x_n^{(N-1)} + \gamma_n^{(N)} u_n^{(N)}, n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n^1\}, ..., \{\alpha_n^N\}, \{\beta_n^1\}, ..., \{\beta_n^N\}, \{\gamma_n^1\}, ..., \{\gamma_n^N\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, 3, ..., N$ and $\{u_n^1\}, \{u_n^2\}, ..., \{u_n^N\}$ are bounded sequence in C. Then $\{x_n\}$ converge strongly to the unique common fixed point of $T_1, ..., T_N$.

Theorem 1.1 *[12] Let X be a real uniformly smooth Banach sapce and let K be a nonempty bounded closed convex subset of X, and let $T_1, T_2, ..., T_N : K \longrightarrow K$ be strongly pseudo-contractive mappings. Suppose $F = \cap_{i=1}^N F(T_i) \neq \phi$ and the sequence $\{x_n\}$ be defined by (1.3), where $\{\alpha_n^1\}, ..., \{\alpha_n^N\}, \{\beta_n^1\}, ..., \{\beta_n^N\}$, $\{\gamma_n^1\}, ..., \{\gamma_n^N\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, ..., N$ and satisfying the conditions:*

(i) $\lim_{n\to\infty} \beta_n^i = 0 \ \forall i = 1, 2, ..., N$ *and* $\Sigma_{n=0}^{+\infty} \beta_n^N = +\infty$ *;*

(ii) $\lim_{n\to\infty} \gamma_n^i = 0, \ \forall i = 1, 2, ..., N-1$ *and* $\Sigma_{n=0}^{+\infty} \gamma_n^N < +\infty$.

Then $\{x_n\}$ converges strongly to the unique common fixed point of $T_1, ..., T_N$.

Theorem 1.2 *[12] Let X be a real uniformly smooth Banach space and let $T_1, ..., T_N : X \longrightarrow X$ be a strongly accretive mappings. For a fixed $f \in X$, define $S_1, ..., S_N : X \longrightarrow X$ by $S_i x = x - T_i x + f$ for all $i = 1, ..., N$ and suppose that each range of S_i are bounded. For arbitary $x_1 \in X$ the sequence $\{x_n\}$ with errors is defined by*

$$\begin{cases} x_1 \in X, \\ x_n^1 = \alpha_n^1 x_n + \beta_n^1 S_1 x_n + \gamma_n^1 u_n^1, \\ x_n^2 = \alpha_n^2 x_n + \beta_n^2 S_2 x_n^1 + \gamma_n^2 u_n^2, \\ \vdots \\ x_{n+1} = x_n^N = \alpha_n^N x_n + \beta_n^N S_N x_n^{N-1} + \gamma_n^N u_n^N, n \geq 1, \end{cases} \tag{1.4}$$

where $\{u_n^1\}, \{u_n^2\}, ..., \{u_n^N\}$ are bounded sequences in X and $\{\alpha_n^1\}, ..., \{\alpha_n^N\}, \{\beta_n^1\}$, ..., $\{\beta_n^N\}, \{\gamma_n^1\}, ..., \{\gamma_n^N\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, ..., N$ and satisfying the following conditions:

(i) $\lim_{n\to\infty} \beta_n^i = 0 \ \forall i = 1, 2, ..., N$ *and* $\Sigma_{n=1}^{+\infty} \beta_n^N = +\infty$ *;*

(ii) $\lim_{n\to\infty} \gamma_n^i = 0, \forall i = 1, 2, ..., N-1$*; and* $\Sigma_{n=1}^{+\infty} \gamma_n^N < +\infty$,

If the systems of operator equations $T_1 x = f, ..., T_N x = f$ has solution in X, then the sequence $\{x_n\}$ converges strongly to the unique solution of operator equations $T_1 x = f, ..., T_N x = f$.

It is our purpose in this paper to establish several strong convergence theorems of multi-step iterative scheme with errors for approximating either common fixed points of nonlinear strongly pseudocontractive mappings or solutions of nonlinear strongly accretive operators equations. The results presented in this paper improve the corresponding Inchan and Plubtieng [12] for reflexive Banach space with weakly continuous duality map with gauge φ, and many others.

2 Preliminalies.

Now, we recall the well-known concept and results.

Throughout we assume that X is a real Banach space and X^* is the dual space of X. Let J denote the normalized duality from X to 2^{X^*} defined by

$$J(x) = \{j \in X^* : (x, j) = \|x\|\|j\|, \|j\| = \|x\|\}$$

where $\langle \cdot, \cdot \rangle$ denote the generalized duality pairing.

Definition 2.1 *A mapping* $T : X \longrightarrow X$ *is called strongly accretive if there exists a constant* $0 < k < 1$ *such that for each* $x, y \in X$ *there is a* $j \in J(x - y)$ *satisfying*

$$\langle Tx - Ty, j(x - y)\rangle \geq k\|x - y\|^2.$$

Definition 2.2 *An operator* T *with domain* $D(T)$ *and range* $R(T)$ *in* X *is called strongly pseudo-contractive if for all* $x, y \in D(T)$*, there exists* $j(x-y) \in J(x - y)$ *and a constant* $0 < k < 1$ *such that*

$$\langle Tx - Ty, j(x - y)\rangle \leq (1 - k)\|x - y\|^2.$$

It is well known (see, for example, Theorem 13.1 of Deimling [6]) that for any given $f \in X$ the equation

$$Tx = f$$

has a unique solution if $T : X \to X$ is strongly accretive and continuous, on X is uniformly smooth and $T : X \to X$ is strongly accretive and demicontinuous (i.e., $x_n \to x \Rightarrow Tx_n \rightharpoonup Tx$). Martin [16] has also proved that if $T : X \to X$ is continuous and accretive then for any given $f \in X$ the equation

$$x + Tx = f$$

has a unique solutions.

The concept of accretive mapping was introduced independently by Browder [1] and Kato [13] in 1967. An early fundamental result in the theory of accretive mapping, due to Browder, states that the initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0, \quad u(0) = u_0$$

is solvable if T is locally Lipschitzian and accretive on X. The reader is refered to Browder [2] for more details of the theory of accretive operators.

Recall that a Banach space X is said to satisfy *Opial's condition* [21] if for any sequence $\{x_n\}$ in X, the condition that $\{x_n\}$ converges weakly to $x \in X$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for $y \in X, y \neq x$.

By a gauge we mean a continuous strictly increasing function φ defined $R^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. We associate with a gauge φ a (generally multivalued) duality map $J_\varphi : X \to X^*$ defined by

$$J_\varphi = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|) \text{ and } \|x^*\| = \varphi(\|x\|)\}.$$

clearly the (normalized) duality map J corresponds to the gauge $\varphi(t) = t$. Browder [1] initiated the study of certain classes of nonlinear operators by means of a duality map J_φ. Set for $t \geq 0$,

$$\Phi(t) = \int_0^t \varphi(r)dr.$$

Then it is known that $J_\varphi(x)$ is the convex function $\Phi(\|\cdot\|)$ at x. Browder [1] say that a Banach space X has a weakly continuous duality map if there exists a gauge φ such that the duality map J_φ is single valued and $weak-to-weak^*$ sequentially continuous (i.e. if $\{x_n\}$ is sequence in X weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ convergent weak*ly to $J_\varphi(x)$). A space with a weakly continuous duality map is easily seen to satisfy *Opial's condition* (cf.[1]). Every $l^p (1 < p < \infty)$ space has a weakly continuous duality map with the gauge $\varphi(t) = t^{p-1}$. In the sequel, we shall use the following lemmas.

Lemma 2.3 *[24] Assume that X has a weakly continuous duality map J_φ with gauge φ.*
(i) For all $x, y \in X$, there holds the inequality

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y) \rangle.$$

(ii) Assume a sequence $\{x_n\}$ in X is weakly convergent to a point x. Then there holds the identity

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|),\ x, y \in X.$$

Notation $' \rightharpoonup '$ stands for weak convergence and $' \to '$ for strong convergence.

Lemma 2.4 *[13] Let $a_n, b_n,$ and c_n be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

with $t_n \in [0, 1), \Sigma t_n = +\infty, b_n = o(t_n),$ and $\Sigma c_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

3 Main Results.

In this section, we study the strong convergence of multi-step iterative process with errors define by (1.3) for the strongly pseudo-contractive mappings on weakly continuous duality maps with gauge φ.

Theorem 3.1 *Suppose X is a reflexive Banach space with weakly continuous duality map J_φ with gauge φ. Let C be nonempty bounded closed convex subset of X and $T_1, ..., T_N : C \to C$ be strongly pseudo-contractive mappings with duality map J_φ and $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be sequence defined by (1.3) satisfying*

(i) $\lim_{n\to\infty} \beta_n^{(i)} = 0$, for all $i = 1, 2, ..., N$ and $\Sigma_{n=1}^\infty \beta_n^{(1)} = \infty$,

(ii) $\Sigma_{n=1}^\infty \gamma_n^{(i)} < \infty$ for all $i = 1, 2, ..., N$.

Then $\{x_n\}$ converge strongly to unique common fixed point in F.

Proof For each $i \in \{1, 2, ..., N\}$ and C is bounded, we note that

$$\begin{aligned}\|x_n - x_n^{(i)}\| &\leq \alpha_n^{(i)}\|x_n - x_n\| + \beta_n^{(i)}\|x_n - T_i x_n^{(i-1)}\| + \gamma_n^{(i)}\|x_n - u_n^{(i)}\| \\ &= \beta_n^{(i)}\|x_n - T_i x_n^{(i-1)}\| + \gamma_n^{(i)}\|x_n - u_n^{(i)}\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Let $p \in F(T)$ and for all $x, y \in C$. Since $T_1, ..., T_N$ are strongly pseudo-contractive mappings with duality map J_φ there exists $j_\varphi(x-y) \in J_\varphi(x-y)$ and $k_i \in (0, 1)$ such that

$$\langle T_i x - T_i y, j_\varphi(x-y)\rangle \leq (1-k_i)\|x-y\|^2, \quad \forall\, x, y \in X.$$

Let $k = \min\{k_i : i = 1, ..., N\}$ and $p \in F$. Since $\{u_n^1\}, ..., \{u_n^N\}$ are bounded and C is bounded, we can set

$$d = \max\left\{\max_{1\leq i\leq N} \sup_{n\geq 0}\{\|u_n^{(i)} - p\|\}, \max_{1\leq i\leq N} \sup_{x\in K}\{\|T_i x - p\|\}, \|x_1 - p\|\right\}.$$

Obviously $d < \infty$. Next, we will prove that $\|x_n - p\| \leq d$ for all $n \in \mathbf{N}$.
In fact, for $n = 1$ is clearly right. Assume the inequality is true for $n = k$. Then for $n = k + 1$, we note that

$$\begin{aligned}\|x_{k+1} - p\| &= \|\alpha_k^{(N)} x_k + \beta_k^{(N)} T_N x_k^{(N-1)} + \gamma_k^{(N)} u_k^{(N)} - p\| \\ &\leq \alpha_k^{(N)}\|x_k - p\| + \beta_k^{(N)}\|T_N x_k^{(N-1)} - p\| + \gamma_k^{(N)}\|u_k^{(N)} - p\| \\ &\leq (\alpha_k^{(N)} + \beta_k^{(N)} + \gamma_k^{(N)})d = d.\end{aligned}$$

So, from the above discussion, we can conclude that

$$\|x_n - p\| \leq d, \ \forall n \geq 1.$$

For any $i = 1, ..., N$, we see that

$$\begin{aligned}\|x_n^{(i)} - p\| &= \|\alpha_n^{(i)} x_n - \beta_n^{(i)} T_i x_n^{(i-1)} + \gamma_n^{(i)} u_n^{(i)} - p\|) \\ &= \|\alpha_n^{(i)}(x_n - p) + \beta_n^{(i)}(T_i x_n^{(i-1)} - p) + \gamma_n^{(i)}(u_n^{(i)} - p)\|)\end{aligned}$$

$$\leq \|\alpha_n^{(i)}(x_n - p)\| + \|\beta_n^{(i)}(T_i x_n^{(i-1)} - p)\| + \|\gamma_n^{(i)}(u_n^{(i)} - p)\|)$$
$$\leq (\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)})d = d.$$

It follows that $\{x_n^{(i)} - p\}$ are bounded sequence for all $i = 1, ..., N$. Consider, for $n \geq 1$,

$$\begin{aligned}
\Phi(\|x_n^{(1)} - p\|) &= \Phi(\|\alpha_n^{(1)} x_n - \beta_n^{(1)} T_1 x_n + \gamma_n^{(1)} u_n^{(1)} - p\|) \\
&= \Phi(\|\alpha_n^{(1)}(x_n - p) + \beta_n^{(1)}(T_1 x_n - p) + \gamma_n^{(1)}(u_n^{(1)} - p)\|) \\
&\leq \Phi(\|\alpha_n^{(1)}(x_n - p)\|) + \langle \beta_n^{(1)}(T_1 x_n - p) + \gamma_n^{(1)}(u_n^{(1)} - p), j_\varphi(x_n^{(1)} - p)\rangle \\
&\leq \Phi(\|\alpha_n^{(1)}(x_n - p)\|) + \beta_n^{(1)}\langle (T_1 x_n - p), j_\varphi(x_n - p)\rangle \\
&\quad + \beta_n^{(1)}\langle (T_1 x_n - p), j_\varphi(x_n^{(1)} - x_n)\rangle + \langle \gamma_n^{(1)}(u_n^{(1)} - p), j_\varphi(x_n^{(1)} - p)\rangle \\
&\leq \Phi(\|\alpha_n^{(1)}(x_n - p)\|) + \beta_n^{(1)}(1-k)\|x_n - p\|^2 \\
&\quad + \beta_n^{(1)}\|T_1 x_n - p\|\|j_\varphi(x_n^{(1)} - p)\| + \gamma_n^{(1)}\|u_n^{(1)} - p\|\|j_\varphi(x_n^{(1)} - p)\| \\
&\leq \alpha_n^{(1)}\Phi(\|(x_n - p)\|) + b_n^{(1)} + c_n
\end{aligned}$$

where $b_n^{(1)} = \beta_n^{(1)}(1-k)\|x_n - p\|^2 + \beta_n^{(1)}\|T_1 x_n - p\|\|j_\varphi(x_n^{(1)} - p)\|$ and $c_n = \gamma_n^{(1)}\|u_n^{(1)} - p\|\|j_\varphi(x_n^{(1)} - p)\|$. From bounded of $\{x_n^{(i)} - p\}$, it follows by (i) and (ii) that $\lim_{n\to\infty} b_n^{(1)} = 0$ and $\Sigma_{n=1}^{\infty} c_n < \infty$. Next, we note that

$$\begin{aligned}
\Phi(\|x_n^{(2)} - p\|) &= \Phi(\|x_n^{(2)} - x_n^{(1)} + x_n^{(1)} - p\|) \\
&\leq \Phi(\|x_n^{(1)} - p\|) + \langle x_n^{(2)} - x_n^{(1)}, j_\varphi(x_n^{(2)} - p)\rangle \\
&\leq \Phi(\|x_n^{(1)} - p\|) + \|x_n^{(2)} - x_n^{(1)}\|\|j_\varphi(x_n^{(2)} - p)\| \\
&\leq \alpha_n^{(1)}\Phi(\|(x_n - p)\|) + b_n^{(2)} + c_n
\end{aligned}$$

where $b_n^{(2)} = b_n^{(1)} + \|x_n^{(2)} - x_n^{(1)}\|\|j_\varphi(x_n^{(2)} - p)\|$. By above of prove we have $\lim_{n\to\infty}\|x_n^{(2)} - x_n^{(1)}\| = 0$ and so it follows that $\lim_{n\to\infty} b_n^{(2)} = 0$. Next, we note that

$$\begin{aligned}
\Phi(\|x_n^{(3)} - p\|) &= \Phi(\|x_n^{(3)} - x_n^{(2)} + x_n^{(2)} - p\|) \\
&\leq \Phi(\|x_n^{(2)} - p\|) + \langle x_n^{(3)} - x_n^{(2)}, j_\varphi(x_n^{(3)} - p)\rangle \\
&\leq \Phi(\|x_n^{(2)} - p\|) + \|x_n^{(3)} - x_n^{(2)}\|\|j_\varphi(x_n^{(3)} - p)\| \\
&\leq \alpha_n^{(1)}\Phi(\|(x_n - p)\|) + b_n^{(3)} + c_n
\end{aligned}$$

where $b_n^{(3)} = b_n^{(2)} + \|x_n^{(3)} - x_n^{(2)}\|\|j_\varphi(x_n^{(3)} - p)\|$. Since $\lim_{n\to\infty}\|x_n^{(3)} - x_n^{(2)}\| = 0$ and so it follows that $\lim_{n\to\infty} b_n^{(3)} = 0$. By continuity of above method there exists nonnegative real sequence $\{b_n^{(N)}\}$ with $\lim_{n\to\infty} b_n^{(N)} = 0$ and $\Sigma_{n=1}^{\infty} c_n < \infty$ such that

$$\Phi(\|x_{n+1} - p\|) \leq (1 - \beta_n^{(1)})\Phi(\|x_n - p\|) + b_n^{(N)} + c_n.$$

Since, $1 - \beta_n^{(1)} < 1$ and $\beta_n^{(1)} \to 0$ as $n \to \infty$, there exists $N \in \mathbf{N}$ and r is real number such that

$$1 - \beta_n^{(1)} < r < 1, \quad \text{for all } n \geq N.$$

Put $b_n = \frac{\beta_n^{(1)} b_n^{(N)}}{1-r}$. Therefore, $b_n = \circ(\beta_n^{(1)})$. Then we have

$$\Phi(\|x_{n+1} - p\|) \leq (1 - \beta_n^{(1)})\Phi(\|x_n - p\|) + b_n + c_n.$$

It follows form Lemma 2.4, $\lim_{n\to\infty}\Phi(\|x_n - p\|) = 0$ and hence $\lim_{n\to\infty}\|x_n - p\| = 0$. Finally, we will show that q is a unique common fixed point in F.

Suppose $q^* \in F$ is another common fixed point in F, we have

$$\|q - q^*\|^2 = (T_i q - T_i q^*, j(q - q^*)) \leq (1 - k_i)\|q - q^*\|^2.$$

Since $k_i \in (0, 1)$, we obtain the equality that $q = q^*$. This completes the proof. ◇

Theorem 3.2 *Suppose X is a reflexive Banach space with weakly continuous duality map J_φ with gauge φ. Let $T_1, ..., T_N : X \to X$ be strongly accretive mappings with duality map J_φ. For a fixed $f \in X$, defined $S_1, ..., S_N : X \to X$ by $S_i x = x - T_i x + f$ for all $i = 1, ..., N$ and suppose that each range of S_i are bounded. For arbitary $x_1 \in X$, the sequence $\{x_n\}$ with errors is defined by (1.4) satisfying the following conditions:*

(i) $\lim_{n\to\infty} \beta_n^{(i)} = 0$, for all $i = 1, 2, ..., N$ and $\Sigma_{n=1}^{\infty} \beta_n^{(1)} = \infty$,

(ii) $\Sigma_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i = 1, 2, ..., N$.

If the systems of operator equations $T_1 x = f, ..., T_N x = f$ has a solution in X, then the sequence $\{x_n\}$ converges strongly to the unique solution of operator equations $T_1 x = f, ..., T_N x = f$.

Proof. Since $T_1, ..., T_N$ are strongly accretive mappings, for all $x, y \in X$ there exists $j(x - y) \in J(x - y)$ and $k_i \in (0, 1)$ such that

$$\langle T_i x - T_i y, j(x - y)\rangle \geq k_i \|x - y\|^2 \ \ \forall x, y \in X$$

for all $i = 1, ..., N$. Consider;

$$\begin{aligned} \langle S_i x - S_i y, j(x - y)\rangle &= \langle (x - T_i x + f) - (y - T_i y + f), j(x - y)\rangle \\ &= \langle x - y, j(x - y)\rangle - \langle T_i x - T_i y, j(x - y)\rangle \\ &\leq \|x - y\|^2 - k_i\|x - y\|^2 = (1 - k_i)\|x - y\|^2. \end{aligned}$$

Hence $S_1, ..., S_N$ are strongly pseudo-contractive mappings. Let $p \in \cap_{i=1}^{N} F(S_i)$ and $k = \min\{k_1, ..., k_N\}$. Since $\{u_n^{(1)}\}, ..., \{u_n^{(N)}\}$ are bounded sequences and each range of $S_1, ..., S_N$ are bounded, we can set

$$d = \max\{\max_{1\leq i\leq N} \sup_{n\geq 1}\{\|u_n^{(i)} - p\|\}, \max_{1\leq i\leq N} \sup_{x\in X}\{\|S_i x - p\|\}, \|x_1 - p\|\}.$$

By using the same argument in the proof of Theorem 3.1, we have that the sequences $\{x_n\}$ converges strongly to a unique common solution of the equations $T_1 x = T_2 x = ... = T_N x = f$. ◇

Acknowledgement. The authors would like to thanks The Uttaradit Rajabhat University for financial support.

References

[1] F. E. Browder, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, Bull. Amer. Math. Soc. **73**, (1967), 875-882.

[2] F. E. Browder, Nonlinear operations and nonlinear equations of evolution in Banach spaces, Proc. Sympos. Pure. Math. **18(2)**, (1976).

[3] C. E. Chidume, Approximation of fixed point of strongly pseudocontractive mappings, Proc. Amer. Math. Soc. **120**, (1994), 545-551.

[4] C. E. Chidume, Convergence theorems for strongly pseudo- contractive and strongly accretive maps, J. Math. Anal. Appl. **228**, (1998), 254-264.

[5] C. E. Chidume, and M. O. Osilike, Ishikawa iteration process for nonlinear Lipschitz strongly accretive mappings, J. Math. Anal. Appl. **192**, (1995), 727-741.

[6] K. Deimling, "Nonlinear Functional Analysis." Spinger-verlag. New York/ berlin, 1985.

[7] L. Deng, An Iterative process for nonlinear Lipschizian and strongly accretive mappings in uniformly convex and uniformly smooth Banach spaces, Acta. Appl. Math. **32**, (1993), 183-196.

[8] L. Deng and X. P. Ding, Iterative approximation of Lipschiz stricly pseudocontractive mappings in uniformly smooth Banach spaces, Nonlinear Anal. **24**, (1995), 981-987.

[9] R. Glowinski and P. Le. Tellec, Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics, SIAM, Philadephia, 1989.

[10] Z. Haiyun and J. Yuting, Approximation of fixed points of strongly pseudocontractive maps without Lipschitz assumption, Proc. Amer. Math. Soc. **125**, (1995), 1705-1709.

[11] S. Ishikawa, Fixed point and iteration of a nonexpansive mappings in a Banach spaces, Proc. Amer. Math. Soc. **73**, (1976), 65-71.

[12] I. Inchan and S. Plubtieng, Approximating solutions for the systems of strongly accretive operator equations, Com. Math. Appl. **53**, (2007), 1317 - 1324.

[13] T. Kato, Nonlinear semigroups and evolution equation, J. Math. Soc. Japan, **19**, (1967), 502-508.

[14] L. S. Liu, Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl. **194**, (1995), 114-125.

[15] W. R. Mann, Mean value methods in itertion, Proc. Amer. Math. Soc. **4**, (1953), 506-510.

[16] R. H. Martin, Jr., Differential equation on closed subsets of Banach space, Trans. Amer. Math. Soc. **179**, (1973), 399-414.

[17] M. A. Noor, Three-step iterative algorithms for multivalued quasi variational inclusions, J. Math. Anal. Appl. **255**, (2001), 589-604.

[18] M. A. Noor, T. M. Rassias and Z. Huang, Three-step iterations for nonlinear accretive operator equations, J. Math. Anal. Appl. **274**, (2002), 59-68.

[19] M. O. Osilike, Iterative solution of nonlinear equations of the ϕ-strongly accretive type, J. Math. Anal. Appl. **200**, (1996), 259-271.

[20] M. O. Osilike, Ishikawa and Mann iteration methods with errors for nonlinear equations of the accretive type, J. Math. Anal. Appl. **213**, (1997), 91-105.

[21] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73**, (1967), 591-597.

[22] K. K. Tan and H. K. Xu, Iterative solutions to nonlinear equations of strongly accretive operators in Banach spaces, J. Math. Anal. Appl. **178**, (1993), 9-21.

[23] Y. Xu, Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations, J. Math. Anal. Appl. **224**, (1998), 91-101.

[24] Y. Yu, Y. Liu and R. Chen, Viscosity approximation of zero of accretive operator in Banach space, Int. Journal of Math. Analysis, Vol. 1, 2007, no. **13**, 603 - 612.

[25] H. Zhou, Some convergence theorems for the Ishikawa iterative sequences of certain nonlinear operators in uniformly smooth Banach spaces, Acta. Math. Sinica **40**, (1997), 751-758.

Received: September 5, 2007