

# S-matrices for Perturbed N=2 Superconformal Field Theory from Quantum Groups

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S-matrices for integrable perturbations of  $N = 2$  superconformal field theories are studied. The models we consider correspond to perturbations of the coset theory  $G_k \times H_{g-h}/H_{k+g-h}$ . The perturbed models are closely related to  $\hat{G}$ -affine Toda theories with a background charge tuned to  $H$ . Using the quantum group restriction of the affine Toda theories we derive the S-matrix.

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## 1. Introduction

Over the past few years many new integrable quantum field theories in two dimensions have been constructed as perturbations of conformal field theories. This program was initiated by A. Zamolodchikov [1]. Of special interest are the  $N = 2$  superconformal theories with perturbations that preserve the  $N = 2$  supersymmetry. For such models the Landau-Ginzburg structure and Bogomolnyi bounds of the superalgebra can be used to deduce much about the soliton spectrum [2][3]. Integrable perturbations of the minimal series of models (with central charge  $c = 3k/(k+2)$ ) were studied in [2], where it was shown that there are three different integrable perturbations, corresponding to perturbation by the least relevant, most relevant, and next-to-most relevant chiral primary fields. In [2] some features of the spectrum for the most relevant perturbations were proposed based on the Landau-Ginzburg formulation of the theories. More general  $N = 2$  supersymmetric integrable models were obtained in [4] and further results about the soliton spectrum were obtained using the Landau-Ginzburg picture in [3].

In [5] the S-matrices for the least relevant perturbations of the minimal series were proposed based on the quantum group symmetries that exist in the models. Indeed, the  $N = 2$  supersymmetry was understood as a special case of the quantum affine symmetry  $U_q(\widehat{SU}(2))$ , which occurs at  $q = -i$ . Further confirmation of these S-matrices was provided by a thermodynamic Bethe ansatz analysis [6]. Also in [6] the relation between the spectrum proposed in [5] and the Landau-Ginzburg formulation of the theories was clarified. The S-matrices for the most relevant perturbations of the minimal series of  $N = 2$  theories and other related models were proposed in [7], again based largely on the Landau-Ginzburg picture of the soliton spectrum. Here it was found that the solitons generally had fractional fermion number, and this fact was essential for obtaining the correct S-matrices. In the quantum group approach, these fractional fermion numbers are automatically incorporated.

In this paper we use the restricted quantum group approach [8] [9] to derive the S-matrices for a class of integrable perturbations of  $N = 2$  superconformal field theories. We will actually solve a larger class of models corresponding to perturbations of the cosets  $G_k \otimes H_l/H_{k+l}$ . The  $N = 2$  supersymmetric theories can essentially be obtained from these models by taking  $l = g - h$ , where  $g$  and  $h$  are the dual Coxeter numbers of  $G$  and  $H$  respectively[10]. The main features of this approach can be summarized as follows. Consider first the situation where  $G = H$ , which was studied in [5][11][12]. For  $k = 1$ ,

one begins with the  $\widehat{G}$ -affine Toda field theory, with zero background charge. (The affine extension of  $G$  will be denoted by  $\widehat{G}$ .) This field theory can be solved by using the affine quantum group symmetry,  $U_q(\widehat{G})$ , that exists in the model [12]. The coset model is obtained by turning on a background charge for the Toda fields. This background charge modifies the conformal dimension of the conserved charges that generate the finite quantum subalgebra,  $U_q(G)$ , of  $U_q(\widehat{G})$ . As a result the conserved charges become dimension zero screening charges. One then uses the  $U_q(G)$  symmetry algebra of screening charges to restrict the S-matrices of the  $\widehat{G}$ -affine Toda theory to obtain the S-matrices of the perturbed coset theory. In the conformal limit this restriction amounts to the usual projection of null states one encounters in the generalized Feigin-Fuchs construction [13]. For higher levels  $k > 1$ , one must begin with a generalization of the  $\widehat{G}$ -affine Toda theory so that it includes additional generalized para-fermions for the group  $G_k/[U(1)]^{\text{rank}(G)}$ . This generalization has been called the fractional super- $\widehat{G}$ -affine Toda or (affine) para-Toda theory [5][11][14]; here we will refer to the conformal, unperturbed combination of para-fermions and free bosons as the para-Toda theory, and the perturbed, integrable model as the  $k^{\text{th}}$   $\widehat{G}$ -affine Toda theory or the  $k^{\text{th}}$  affine para-Toda theory. For  $G = SU(2)$ , this latter model is the series of fractional super sine-Gordon models, which consists of an interacting system of a single boson and a  $Z_k$  para-fermion (for  $k = 2$  this is the usual supersymmetric sine-Gordon theory), and is described in detail in [5].

In order to study general perturbations of  $N = 2$  theories, one must understand how to extend the foregoing results to the situation where  $G \neq H$ . The way to begin to accomplish this is contained in the paper [4] for  $k = 1$  and for general  $k$  in [15] [14]. The main observation made in these papers that we will use here is that the perturbed coset theories are again related to  $\widehat{G}$ -affine Toda theory, but now with a background charge tuned to  $H$  rather than  $G$ . In the sequel we will explain how to implement this non-conventional background charge in the restricted quantum group approach, and thereby derive the S-matrices for the perturbed  $G_k \otimes H_l/H_{k+l}$  theories from the S-matrices of the  $k^{\text{th}}$   $\widehat{G}$ -affine Toda theories. We will show how the conserved  $N = 2$  charges are always a subset of the  $U_q(\widehat{G})$  quantum affine charges, with the  $N = 2$  supersymmetry occurring at a special value of the coupling.

The remainder of this paper is organized as follows. In section 2, we will review the relevant conformal field theories, and how their perturbations are related to affine Toda field theory. In section 3 we will describe in complete generality how one obtains the S-matrices from an appropriate restriction of the Toda theory. Finally in section 4 we will discuss specific examples in some detail.

## 2. The para-Toda models

In this section we review the para-Toda construction of the coset models:

$$\mathcal{M}_{k,\ell}(G; H) \equiv \frac{G_k \times H_\ell}{H_{k+\ell}}, \quad (2.1)$$

and their integrable perturbations. Here  $H$  is a subgroup of  $G$  with  $\text{rank}(H) = \text{rank}(G)^1$ . As we will describe, the  $N = 2$  superconformal coset models [10] are a subclass of these  $\mathcal{M}_{k,\ell}(G; H)$  models [15][14]. To fix our notation, let  $\alpha_1, \dots, \alpha_r$  be a system of simple roots for  $G$ , ordered in such a way that  $\alpha_1, \dots, \alpha_p$ , for  $p \leq r$ , is a system of simple roots for  $H$ . The highest root of  $G$  will be denoted by  $\psi$ . Let  $\rho_G$  and  $\rho_H$  denote the Weyl vectors, and  $g$  and  $h$  denote the dual Coxeter numbers of  $G$  and  $H$  respectively <sup>2</sup>. Finally, let  $\mathcal{U} = (U(1))^r$  be a torus for  $H$ , and hence a torus for  $G$ .

Consider first the conformal field theory. The para-Toda theory consists of the generalized para-fermions, constructed from the coset  $G/\mathcal{U}$ , tensored with a model consisting of  $r$  free bosons. The bosonic energy momentum tensor is

$$T_b(z) = -\frac{1}{2}(\partial\phi)^2 + i(\beta_+^{(H)} - \beta_-^{(H)})\rho_H \cdot \partial^2\phi, \quad (2.2)$$

where

$$\beta_\pm^{(H)} \equiv \frac{1}{\sqrt{k}} \left[ \sqrt{\frac{(k+\ell+h)}{(\ell+h)}} \right]^{\pm 1}. \quad (2.3)$$

As has been described in a number of places, such a bosonic free field description can be directly related to a Toda theory (see, for example: [16] [17] [4]).

The primary fields of the para-fermionic theory will be denoted by  $\mathcal{A}_\lambda^\Lambda(z)$ , where  $\Lambda$  is a highest weight of  $G_k$ , and  $\lambda$  is a vector of charges under the Cartan subalgebra (CSA),  $\mathcal{X}$ , of  $G$  that generates the torus,  $\mathcal{U}$ . The Toda, or free bosonic, field theory has a natural vertex operator representation for its highest weight states:

$$\mathcal{V}_{\lambda_+, \lambda_-}(z) = \exp[-i(\beta_+^{(H)}\lambda_+ - \beta_-^{(H)}\lambda_-) \cdot \phi(z)]. \quad (2.4)$$

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<sup>1</sup> All that is in fact required for the para-Toda construction is that  $H$  is a regularly embedded subgroup of  $G$ .

<sup>2</sup> Since  $H$  is, in general, a product of groups,  $h$  is to be thought of as a vector. The dual Coxeter number of a  $U(1)$  factor is defined to be zero.

The conformal weight of  $\mathcal{A}_\lambda^\Lambda$  is:

$$h_\lambda^\Lambda = \frac{\Lambda \cdot (\Lambda + 2\rho_G)}{2(k+g)} - \frac{\lambda^2}{2k} + \text{integer} ,$$

and that of  $\mathcal{V}_{\lambda_+, \lambda_-}$  is:

$$h_{\lambda_+, \lambda_-} = \frac{1}{2}(\beta_+^{(H)}\lambda_+ - \beta_-^{(H)}\lambda_-)^2 + (\beta_+^{(H)} - \beta_-^{(H)})\rho_H \cdot (\beta_+^{(H)}\lambda_+ - \beta_-^{(H)}\lambda_-) .$$

One finds that this can be rewritten as:

$$h_{\lambda_+, \lambda_-} = \frac{1}{2k} (\lambda_+ - \lambda_-)^2 + \frac{\lambda_+ \cdot (\lambda_+ + 2\rho_H)}{2(\ell + h)} - \frac{\lambda_- \cdot (\lambda_- + 2\rho_H)}{2(k + \ell + h)} . \quad (2.5)$$

From this, and a consideration of the CSA eigenvalues,  $\lambda$ , it is easy to identify representatives of the primary fields,  $\Phi_{\lambda_-}^{\Lambda, \lambda_+}$ , of the  $\mathcal{M}_{k, \ell}(G; H)$  coset model. ( The labels  $(\Lambda, \lambda_+; \lambda_-)$  are highest weight labels of affine  $G$  at level  $k$  and  $H$  at levels  $\ell$  and  $k + \ell$  respectively, and correspond to the numerator and denominator factors in  $\mathcal{M}_{k, \ell}(G; H)$ .) Indeed, we may take

$$\Phi_{\lambda_-}^{\Lambda, \lambda_+}(z) = \mathcal{A}_{(\lambda_- - \lambda_+)}^\Lambda(z) \mathcal{V}_{\lambda_+, \lambda_-}(z) . \quad (2.6)$$

One should also remember that there are field identifications induced by spectral flow in the CSA,  $\mathcal{X}$ , of  $H$  [18] [19] [20] [21]. Such field identifications map a coset state with weights  $(\Lambda, \lambda_+; \lambda_-)$  into another such state according to:

$$\Lambda \rightarrow \Lambda + k v , \quad \lambda_+ \rightarrow \lambda_+ + \ell v , \quad \lambda_- \rightarrow \lambda_- + (k + \ell)v , \quad (2.7)$$

where  $v$  is any vector. Spectral flow by an arbitrary vector,  $v$ , yields an automorphism of the coset theory provided we use appropriately twisted Kac-Moody currents (see, for example, [22]); that is, we replace the currents of  $G$  or  $H$  according to:

$$\begin{aligned} J_n^\alpha &\rightarrow J_{n+v \cdot \alpha}^\alpha \\ H_n^i &\rightarrow H_n^i + k v^i \delta_{n,0} \end{aligned}$$

To avoid using such twisted representations one usually restricts  $v$  to be a weight of  $G$ , and hence a weight of  $H$ .

Finally, to obtain the model  $\mathcal{M}_{k, \ell}(G; H)$  from the tensor product of the para-fermions and free bosons one needs the screening currents, and these are given by:

$$\begin{aligned} S_{+\alpha_i}(z) &= \Phi_0^{0, -\alpha_i}(z) \equiv \mathcal{A}_{\alpha_i}^0(z) \exp[+i\beta_+^{(H)} \alpha_i \cdot \phi(z)] \\ S_{-\alpha_i}(z) &= \Phi_{-\alpha_i}^{0, 0}(z) \equiv \mathcal{A}_{-\alpha_i}^0(z) \exp[-i\beta_- \alpha_i \cdot \phi(z)] , \end{aligned} \quad (2.8)$$

for  $i = 1, \dots, p$ .

If  $G/H$  is a symmetric space, one can conformally embed  $H_{g-h}$  into  $SO(\dim(\frac{G}{H}))$ . Therefore, for a special choice of modular invariant for  $H_{g-h}$ , the  $M_{k,\ell=g-h}(G; H)$  model is precisely the super-GKO coset model based on  $G/H$ . That is, one obtains the coset model:

$$\frac{G_k \times SO(\dim(G/H))}{H} . \quad (2.9)$$

Moreover, if  $G/H$  is a *hermitian* symmetric space then  $M_{k,\ell=g-h}(G; H)$  is an  $N = 2$  supersymmetric model [10]. For most of the remainder of this section we will restrict our attention to the models (2.1) in which  $G/H$  is hermitian symmetric. For such spaces the group  $H$  has the form  $H = H' \times U(1)$ , where  $H'$  is semi-simple. There is thus only one simple root,  $\gamma \equiv \alpha_r$ , of  $G$  that is not a simple root of  $H$ . The vector  $2(\rho_G - \rho_H)$  has the property that

$$2(\rho_G - \rho_H) \cdot \alpha = \begin{cases} 0 & \text{if } \alpha \text{ is a root of } H', \\ \pm g & \text{otherwise;} \end{cases} \quad (2.10)$$

where one has  $\pm g$  depending upon whether  $\alpha$  is a positive or negative root. Thus  $2(\rho_G - \rho_H)$  defines the free  $U(1)$  direction.

To determine exactly how to obtain the  $N = 2$  model from the  $\mathcal{M}_{k,\ell}(G; H)$  model, one needs to decompose the characters of  $SO(\dim(G/H))$  into characters of  $H_{g-h}$ . Let  $\chi_R^\pm(\tau; \nu)$  and  $\chi_{NS}^\pm(\tau; \nu)$  denote the Ramond and Neveu-Schwarz characters of  $SO(\dim(G/H))$ , where  $\nu$  is the character parameter of the embedded CSA of  $H$ . One finds that [14]

$$\chi_R^\pm(\tau; \nu) = \sum_{w \in \frac{W(G)}{W(H)}} \sum_{\alpha \in \frac{M(G)}{M(H)}} \epsilon^\pm(\lambda, w) \chi_{\lambda(\alpha, w)}^{H_{g-h}}(\tau; \nu) , \quad (2.11)$$

where

$$\lambda(\alpha, w) = w(\rho_G) - \rho_H + g\alpha , \quad (2.12)$$

and

$$\begin{aligned} \epsilon^-(\lambda, w) &= \epsilon(w) , \\ \epsilon^+(\lambda, w) &= \epsilon(w) e^{-\frac{2\pi i}{g}(\rho_G - \rho_H) \cdot \lambda(\alpha, w)} , \end{aligned} \quad (2.13)$$

and  $\chi_\lambda^{H_{g-h}}$  is the character of  $H$  at level  $g - h$  with highest weight  $\lambda$ . The Weyl element,  $w \in \frac{W(G)}{W(H)}$ , and the vector  $\alpha \in \frac{M(G)}{M(H)}$  in (2.11) are chosen so that  $\lambda(\alpha, w)$  is a highest weight of  $H_{g-h}$ . The corresponding result in the Neveu-Schwarz sector is almost identical, except that (2.12) is replaced by

$$\lambda(\alpha, w) = w(\rho_G) - \rho_G + g\alpha . \quad (2.14)$$

Note that the only difference between the sectors is a shift of  $\lambda(\alpha, w)$  by  $\rho_G - \rho_H$ , which is purely in the  $U(1)$  direction. One should also observe that in the Neveu-Schwarz sector  $\lambda(\alpha, w)$  is always a *root* of  $G$ .

In the bosonic sector of the para-Toda formulation there is a single free  $U(1)$  direction that is orthogonal to the charge at infinity and to the screening currents (2.8). This  $U(1)$  factor is, of course, the  $U(1)$  current of the  $N = 2$  superconformal theory, and is given by:

$$J_{U(1)}(z) = 2i (\beta'_+ - \beta'_-) (\rho_G - \rho_H) \cdot \partial\phi , . \quad (2.15)$$

The associated charge,  $q_{U(1)}$ , of a primary field  $\Phi_{\lambda_-}^{\Lambda, \lambda_+}$  is:

$$q_{U(1)} = -2(\rho_G - \rho_H) \cdot \left[ \frac{\lambda_+}{g} - \frac{\lambda_-}{k+g} \right] . \quad (2.16)$$

Recall that in the Neveu-Schwarz sector of the  $N = 2$  model, the vector  $\lambda_+$  is a root of  $G$  and hence  $\frac{2}{g}(\rho_G - \rho_H) \cdot \lambda_+$  is always an integer. The same conclusion is also true in the Ramond sector since  $2(\rho_G - \rho_H)^2$  is also always an integer.

The para-Toda formulation is not manifestly supersymmetric, but the supercharges have natural realizations in terms of vertex operators. Indeed, the operators

$$\begin{aligned} S_\gamma(z) &= \Phi_0^{0, -\gamma}(z) \equiv \mathcal{A}_\gamma^0(z) \exp[+i\beta_+^{(H)} \gamma \cdot \phi(z)] \\ S_{-\psi}(z) &= \Phi_0^{0, \psi}(z) \equiv \mathcal{A}_{-\psi}^0(z) \exp[-i\beta_+^{(H)} \psi \cdot \phi(z)] \end{aligned} \quad (2.17)$$

are representations of  $G^+(z)$  and  $G^-(z)$  respectively. The symmetric relationship between these operators and the screening currents has important implications for the structure of the super- $W$  algebra [23].

There are two other operators that are closely related to the screening currents:

$$\begin{aligned} S_{-\gamma}(z) &= \Phi_{-\gamma}^{0, 0}(z) \equiv \mathcal{A}_{-\gamma}^0(z) \exp[-i\beta_-^{(H)} \gamma \cdot \phi(z)] , \\ S_\psi(z) &= \Phi_\psi^{0, 0}(z) \equiv \mathcal{A}_\psi^0(z) \exp[+i\beta_-^{(H)} \psi \cdot \phi(z)] . \end{aligned} \quad (2.18)$$

These are representations of  $(G_{-\frac{1}{2}}^- X)(z)$  and  $(G_{-\frac{1}{2}}^+ \tilde{X})(z)$ , where  $X(z)$  is the chiral, primary field

$$X = \Phi_{-\gamma}^{0, -\gamma} , \quad (2.19)$$

with  $h = \frac{k}{2(k+g)}$  and  $q_{U(1)} = \frac{k}{(k+g)}$ . The field  $\tilde{X}(z)$  is the anti-chiral conjugate of  $X(z)$ .

We now turn to massive integrable perturbations. The set  $\{\alpha_0, \dots, \alpha_r\}$ , with  $\alpha_0 = -\psi$ , comprise the simple roots of the affine Lie algebra  $\widehat{G}$ . Define the  $k^{\text{th}}$   $\widehat{G}$ -affine Toda theory at coupling  $\beta$  by the action:

$$S = \frac{1}{4\pi} \int d^2z \partial_z \Phi \cdot \partial_{\bar{z}} \Phi + S_{G_k/\mathcal{U}} + \frac{\lambda}{2\pi} \int d^2z \sum_{i=0}^r \mathcal{A}_{\alpha_i}^{(0)} \overline{\mathcal{A}}_{\alpha_i}^{(0)} \exp(-i\beta\alpha_i \cdot \Phi), \quad (2.20)$$

where  $S_{G_k/\mathcal{U}}$  is the formal action of the para-fermions and the operator  $\overline{\mathcal{A}}_{\alpha_i}^{(0)}$  is the anti-holomorphic counterpart of  $\mathcal{A}_{\alpha_i}^{(0)}$ . The terms with  $i = 1, \dots, p$  in the sum in (2.20) characterize the  $\mathcal{M}_{k,l}(G; H)$  conformal field theory, when  $\beta = \beta_-^{(H)}$ . The additional terms ( $i = 0$  and  $i = p + 1, \dots, r$ ) are to be thought of as perturbations of the conformal field theory. The fact that (2.20) is a classically integrable model suggests very strongly that the model will be quantum integrable. From the perspective of perturbed conformal field theory, the perturbing operators in (2.20) are related to the screening operators by an automorphism of the Lie algebra of  $G$ . This provides very strong evidence that the perturbed conformal model is indeed a massive, quantum integrable field theory [24] [4]. Note that for the  $N = 2$  supersymmetric models, the perturbations in (2.20) are simply the operators (2.18) (paired, of course, with the anti-holomorphic counterparts). When  $G = H$  there is only one perturbing operator (the  $i = 0$  term in (2.20)) and it corresponds to the field  $\Phi_{\text{Adj}}^{0,0}$  in  $G \times G/G$ , where Adj refers to the adjoint representation of  $G$ . In the situation where  $G$  has level one (*i.e.*  $k = 1$ ), this way of relating integrable perturbations of conformal theories to affine Toda theory was described in [24][17][4][25].

If we start from an  $N = 2$  superconformal field theory then the massive perturbed model is also  $N = 2$  supersymmetric. In addition, if the degeneracy of the Ramond ground state of the conformal model is  $\mu$ , then the perturbed theory has  $\mu$  distinct ground states that are fully resolved by the different expectation values of the chiral primary fields in those ground states [14]. Moreover, if  $G$  has level one, then there is a Landau-Ginzburg formulation and there are  $\mu$  distinct minima of the potential. All the classical small oscillations about these minima correspond to massive excitations [20][3]. This sort of vacuum structure leads one to consider the soliton sector of the effective field theory for the chiral primary fields. A great deal can be said about the soliton structure[2][3], but for the present we simply wish to note that if one introduces the soliton sectors into the perturbed conformal model and then returns to a conformal model by taking the ultra-violet limit then one has gone beyond the usual (modular invariant) conformal model. This is because, in the ultra-violet limit, the soliton creation operators become purely

holomorphic or purely anti-holomorphic fields with possibly fractional fermion number. A super-selection rule was advanced in [13] to determine which operators in the conformal field theory correspond to the soliton operators of the perturbed model. This selection rule states that one should include all the holomorphic and anti-holomorphic operators that are local with respect to the perturbing operators.

Consider once again the completely general conformal  $\mathcal{M}_{k,\ell}(G; H)$  model. It is elementary to identify the operators that are local with respect to the perturbing operators (2.18). Since these perturbing operators have the form  $\Phi_\alpha^{0,0}(z)$ , it is clear that any operator of the form  $\Phi_0^{\Lambda,\lambda_+}(z)$  is local with respect to the perturbation. The fact that one has  $\lambda_- = 0$  means that  $\Lambda + \lambda_+$  must be on the root lattice<sup>3</sup> of  $H$ . If  $G$  has level one then  $\Lambda$  is uniquely determined by the choice of  $\lambda_+$ . The problem now is to determine what subset of the operators  $\Phi_0^{\Lambda,\lambda_+}(z)$  correspond to soliton creation operators, and what are the set of possible choices for  $\lambda_+$ . For the  $N = 2$  superconformal models it is tempting to restrict  $\lambda_+$  to those of (2.11) and (2.14), however the resulting operators will only have integer fermion number,  $q_{U(1)}$ , while the work of [6] shows that solitons can have fractional fermion number. In the next section we obtain a consistent scattering matrix for the solitons of the abovementioned integrable models, and our results lead to the conclusion that the vectors,  $\lambda_+$ , in the soliton creation operators can be any of the fundamental weights of  $G$ . In this way we will also recover the results of [6].

Before concluding this section, we wish to remark upon the generalizations beyond the  $N = 2$  supersymmetric integrable models. For general  $G$  and  $H$ , with  $\text{rank}(G) = \text{rank}(H)$ , one can certainly construct the conformal model (2.1) using the para-Toda formalism, however if one wants to construct a unitary integrable model by using perturbations similar to (2.18) in an action of the form (2.20), then  $G/H$  must be a symmetric space [25]<sup>4</sup>. If  $G/H$  is a real symmetric space then there is only one operator that is analogous to (2.18) and which leads to an integrable model. This operator is real and so perturbation hamiltonian is hermitian. If  $G/H$  is a hermitian symmetric space then the two perturbations (2.18) are hermitian conjugates of each other. If  $G/H$  is not symmetric then there is no way to use operators of the form (2.18) to obtain a hermitian perturbation hamiltonian: The extra terms,  $i = 0$  and  $i = p + 1, \dots, r$  in (2.20) do not possess any form of hermiticity in the

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<sup>3</sup> This lattice is of dimension  $r$  with appropriate basis vectors chosen in the  $U(1)$  directions. When  $G/H$  is a hermitian symmetric space this lattice has basis vectors  $\alpha_1, \dots, \alpha_{r-1}, 2(\rho_G - \rho_H)$ .

<sup>4</sup> Note that we define the trivial quotient  $G/G$  to be symmetric.

quantum group truncated model. Thus the restriction to symmetric spaces, though not necessarily to hermitian symmetric spaces, is required. The restriction to supersymmetric models is optional and would require  $\ell = g - h$  and a special choice of modular invariant. Therefore the freedom to generalize the quantum integrable models arising out of perturbations of the  $\mathcal{M}_{k,\ell}(G; H)$  models is not so much in the choice of  $H$  as it is in the choice of  $\ell$  and the modular invariant for  $\mathcal{M}_{k,\ell}(G; H)$ .

### 3. The $U_q(H)$ restriction of the $\widehat{G}$ -affine Toda theories

Let us summarize the results of the previous section. The perturbed  $G_k \otimes H_l/H_{k+l}$  models were shown to be related to the  $k^{\text{th}}$   $\widehat{G}$ -affine Toda field theory at the specific coupling  $\beta = \beta_-^{(H)}$ . The energy-momentum tensor in the conformal limit has the background charge appropriate to  $H$ . In this section we describe how to obtain the spectrum and S-matrices of the perturbed coset theories from the S-matrices of the  $k^{\text{th}}$   $\widehat{G}$ -affine Toda theory. We first review the S-matrices of the  $\widehat{G}$ -affine Toda theory with zero background charge, and then describe how to restrict the model in a way appropriate to the value of the background charge (2.2).

For simplicity we begin with the case  $k = 1$ . The S-matrices for the  $\widehat{G}$ -affine Toda theory (at ‘imaginary coupling’, which is the only case of relevance here) can be determined by requiring that they commute with the  $U_q(\widehat{G})$  quantum affine symmetry that exists in the model [12]. Let us review this construction. Let  $\vec{\alpha}_i, i = 0, \dots, r$  be the simple roots for the affine Lie algebra  $\widehat{G}$ . For simplicity we assume that  $\widehat{G}$  is simply laced<sup>5</sup> so that  $\vec{\alpha}_i^2 = 2$ . The  $\widehat{G}$ -affine Toda theory with the coupling  $\beta$  is defined by the action

$$S = \frac{1}{4\pi} \int d^2z \partial_z \Phi \cdot \partial_{\bar{z}} \Phi + \frac{\lambda}{2\pi} \int d^2z \sum_{i=0}^r \exp(-i\beta\alpha_i \cdot \Phi). \quad (3.1)$$

Let  $\phi$  and  $\bar{\phi}$  denote the non-local quasi-chiral components of the field  $\Phi = \phi + \bar{\phi}$ :

$$\begin{aligned} \phi(x, t) &= \frac{1}{2} \left( \Phi(x, t) + \int_{-\infty}^x dy \partial_t \Phi(y, t) \right) \\ \bar{\phi}(x, t) &= \frac{1}{2} \left( \Phi(x, t) - \int_{-\infty}^x dy \partial_t \Phi(y, t) \right). \end{aligned}$$

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<sup>5</sup> The following results can be generalized to the non-simply laced case [13].

One can show that the model (3.1) possesses non-local conserved charges  $Q_{\alpha_i}, \bar{Q}_{-\alpha_i}, i = 0, \dots, r$  which together with the topological charges

$$h_i = \frac{\beta}{2\pi} \int dx \alpha_i \cdot \partial_x \Phi, \quad (3.2)$$

generate the  $U_q(\widehat{G})$  quantum affine algebra. The conserved currents  $J_{\alpha_i}^\mu, \bar{J}_{-\alpha_i}^\mu$  for the charges  $Q_{\alpha_i}, \bar{Q}_{-\alpha_i}$  respectively, which satisfy  $\partial_\mu J_{\alpha_i}^\mu = \partial_\mu \bar{J}_{-\alpha_i}^\mu = 0$ , are the following:

$$\begin{aligned} J_{\alpha_i, z}(z, \bar{z}) &= \exp\left(\frac{i}{\beta} \alpha_i \cdot \phi\right) \\ J_{\alpha_i, \bar{z}}(z, \bar{z}) &= \lambda \frac{\beta^2}{1 - \beta^2} \exp\left(-i\left(\beta - \frac{1}{\beta}\right) \alpha_i \cdot \phi - i\beta \alpha_i \cdot \bar{\phi}\right) \\ \bar{J}_{-\alpha_i, \bar{z}}(z, \bar{z}) &= \exp\left(\frac{i}{\beta} \alpha_i \cdot \bar{\phi}\right) \\ \bar{J}_{-\alpha_i, z}(z, \bar{z}) &= \lambda \frac{\beta^2}{1 - \beta^2} \exp\left(-i\left(\beta - \frac{1}{\beta}\right) \alpha_i \cdot \bar{\phi} - i\beta \alpha_i \cdot \phi\right) \end{aligned} \quad (3.3)$$

These charges can be shown to satisfy the  $U_q(\widehat{G})$  algebra:<sup>6</sup>

$$\begin{aligned} [h_i, Q_{\alpha_j}] &= a_{ij} Q_{\alpha_j}, & [h_i, \bar{Q}_{-\alpha_j}] &= -a_{ij} \bar{Q}_{-\alpha_j} \\ Q_{\alpha_i} \bar{Q}_{-\alpha_j} - q^{-a_{ij}} \bar{Q}_{-\alpha_j} Q_{\alpha_i} &= a \delta_{ij} (1 - q^{2h_i}), \end{aligned} \quad (3.4)$$

where  $a_{ij} = \alpha_i \cdot \alpha_j$  is the Cartan matrix,  $a$  is a constant, and  $q$  is given by

$$q = \exp\left(-\frac{i\pi}{\beta^2}\right) \quad (k = 1). \quad (3.5)$$

The deformed Serre relations were proven in [13]. The Lorentz spin of the quantum affine charges is given by

$$\frac{1}{\gamma} \equiv \text{spin}(Q_{\alpha_i}) = -\text{spin}(\bar{Q}_{-\alpha_i}) = \frac{1 - \beta^2}{\beta^2}. \quad (3.6)$$

We now describe the spectrum of massive particles of the  $\widehat{G}$ -affine Toda field theory. As usual we parameterize the momentum of asymptotic one-particle states in terms of rapidity  $\theta$ ,

$$E = m \cosh(\theta), \quad P = m \sinh(\theta). \quad (3.7)$$

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<sup>6</sup> The following relations are isomorphic to the standard relations in [26] [27].

The spectrum consists of  $r$  separate finite dimensional multiplets of solitons  $K^{(n)}(\theta)$ ,  $n = 1, \dots, r$ . The solitons in each separate multiplet  $K^{(n)}$  are all of the same mass  $M_n$ , and are characterized as transforming under a representation  $\widehat{\rho}_n$  of  $U_q(\widehat{G})$ . We denote by  $W_n$  the finite dimensional representation vector space of  $\widehat{\rho}_n$ , such that the states  $K^{(n)}$  are vectors in  $W_n$ .

The representation vector spaces  $W_n$  are described more precisely as follows. Let  $V_n$ ,  $n = 1, \dots, r$  denote the vector spaces of the fundamental representations of  $G$ . For each of the fundamental representations  $V_n$  that correspond to integrable representations in the  $G$ -Wess-Zumino-Witten (WZW) model at level one (the number of these is always less than or equal to  $r$ ), there exists a multiplet of solitons where  $W_n = V_n$ . The fields that create these solitons can be taken as

$$K_{\mu_n}^{(n)}(z, \bar{z}) = \exp\left(-\frac{i}{\beta}\mu_n \cdot \phi(z, \bar{z})\right), \quad \overline{K}_{\mu_n}^{(n)}(z, \bar{z}) = \exp\left(\frac{i}{\beta}\mu_n \cdot \bar{\phi}(z, \bar{z})\right), \quad (3.8)$$

where  $\mu_n$  is any weight in  $V_n$ . The fields (3.8) are local with respect to the perturbing field, and thus define meaningful superselection sectors[13]. For example for  $G = SU(N)$ ,  $W_n = V_n$  for all  $n$ . For other groups the remaining  $W_n$  are direct sums of a finite number of  $V_n$ , and as vector spaces are not reducible with respect to  $G$ . The precise decomposition of the spaces  $W_n$ , as well as the masses  $M_n$  are known for all  $G$ [28][29][30], and can be traced to the representation theory of the algebra  $U_q(\widehat{G})$ <sup>7</sup>.

The representations  $\widehat{\rho}_n$  of  $U_q(\widehat{G})$  are rapidity dependent due to the non-zero Lorentz spin of the charges. This can be expressed as follows:

$$\widehat{\rho}_n(Q_{\alpha_i}) = x \widehat{\rho}'_n(Q_{\alpha_i}), \quad \widehat{\rho}_n(\overline{Q}_{-\alpha_i}) = x^{-1} \widehat{\rho}'_n(\overline{Q}_{-\alpha_i}), \quad (3.9)$$

where

$$x \equiv \exp(\theta/\gamma) \quad (3.10)$$

is a ‘spectral’ parameter, and  $\widehat{\rho}'_n$  are rapidity independent representations of  $U_q(\widehat{G})$ . The  $x$  dependence of the representation (3.10) defines it to be in the so-called principal gradation.

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<sup>7</sup> The papers [29][30] are not concerned with  $\widehat{G}$ -affine Toda theory, but rather with the generalized Gross-Neveu models, or non-abelian Thirring models, which actually have a  $G$ -Yangian invariance. One can appeal to these results in the present context since the  $\widehat{G}$ -affine Toda theories are equivalent to the Yangian invariant ones as  $\beta \rightarrow 1$ . We refer the reader to [12] for details of this argument.

Define  $S_{nm}^{(\widehat{G})}(\theta, q)$  to be the two-body S-matrix for the scattering of particles in  $W_n$  with those in  $W_m$ :

$$S_{nm}^{(\widehat{G})}(\theta, q) : \quad W_n \otimes W_m \rightarrow W_m \otimes W_n \quad (3.11)$$

( $\theta = \theta_1 - \theta_2$ ). Requiring the S-matrices to commute with the  $U_q(\widehat{G})$  symmetry leads to the following result:

$$S_{nm}^{(\widehat{G})}(\theta, q) = X_{nm}(\theta) v_{nm}(\theta, q) R_{nm}^{(\widehat{G})}(\theta, q). \quad (3.12)$$

In (3.12) the various factors have the following meaning. The last term,  $R_{nm}^{(\widehat{G})}(\theta, q)$ , is the standard  $R$ -matrix for the quantum group  $U_q(\widehat{G})$  in the principal gradation, and its structure is completely fixed by the  $U_q(\widehat{G})$  symmetry. It also automatically satisfies the Yang-Baxter equation. The rapidity dependence enters through the spectral parameter  $x$ , whereas the coupling  $\beta$  dependence enters through both  $x$  and  $q$ . Many of the  $R_{nm}^{(\widehat{G})}$  are explicitly known[31][32]; they can be computed in principle from the fusion procedure[33]. See [34] for a review<sup>8</sup>. The scalar factor  $v_{nm}(\theta, q)$  is the minimal factor that makes the product  $v_{nm}R_{nm}$  crossing symmetric and unitary. For  $G = SU(N)$  they were computed in [35]. For more general groups they are straightforward, though tedious, to compute. The additional factor  $X_{nm}(\theta)$  is independent of the coupling  $\beta$ , and is a CDD factor. This factor contains all of the necessary poles for closure of the bootstrap with the spectrum of masses  $M_n$ . The  $X_{nm}$  are also known from the relation with the Gross-Neveu type models[29][30].

We now turn on the background charge (2.2), and restrict the model to obtain the S-matrices for the perturbed cosets. This proceeds in two stages. One must first modify the S-matrices (3.12) such that they reflect the presence of the background charge. One then uses the screening quantum group sub-algebra to truncate the Hilbert space.

The non-zero background charge modifies the conformal dimension of the quantum affine currents (3.3). A simple computation shows that the charges  $Q_{\alpha_i}, \overline{Q}_{-\alpha_i}$ , for  $\alpha_i$  a root of  $H$  become dimension zero operators, and thus have Lorentz spin zero; these generate a screening quantum group sub-algebra  $U_q(H)$ . The remaining quantum affine charges have modified non-zero Lorentz spin (which is easily computed from in the conformal limit from

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<sup>8</sup> One needs to be careful about the rapidity dependence of the spectral parameter  $x$  when borrowing  $R$ -matrices from the mathematics literature. For example, in [31] the  $R$ -matrices by construction commute with the affine charges in the homogeneous gradation, and are related to the  $R$ -matrices in (3.12) by an automorphism (see below).

(3.3) and (2.2)) and will generate residual quantum symmetries of the S-matrices of the kind described in [36][37][5][13]. The effect of the background charges on the representations  $\hat{\rho}_n$  can be expressed as follows. Let  $\hat{\rho}_n^{(H)}(Q_{\alpha_i})$ ,  $\hat{\rho}_n^{(H)}(\overline{Q}_{-\alpha_i})$  denote the representations of  $U_q(\widehat{G})$  when the background charge is present. Then  $\hat{\rho}_n$  and  $\hat{\rho}_n^{(H)}$  are related by an automorphism:

$$\hat{\rho}_n^{(H)}(Q_{\alpha_i}) = \sigma_H^{-1} \hat{\rho}_n(Q_{\alpha_i}) \sigma_H, \quad \hat{\rho}_n^{(H)}(\overline{Q}_{-\alpha_i}) = \sigma_H^{-1} \hat{\rho}_n(\overline{Q}_{-\alpha_i}) \sigma_H, \quad (3.13)$$

where

$$\sigma_H = x^{\rho_H \cdot h} \quad (3.14)$$

and  $h$  in the foregoing exponent represents a vector of Cartan elements. When  $G = H$ , the representations  $\hat{\rho}_n^{(G)}$  are sometimes referred to as being in the homogeneous gradation.

Since the S-matrices are completely characterized by their  $U_q(\widehat{G})$  symmetry, one can deduce the effect of the background charge on the S-matrices from (3.13). Let  $S_{nm}^{(\widehat{G}/H)}$  denote such an S-matrix. It is given by the formula

$$S_{nm}^{(\widehat{G}/H)}(\theta, q) = (\sigma_H^{-1} \otimes \sigma_H^{-1}) S_{nm}^{(\widehat{G})}(\theta, q) (\sigma_H \otimes \sigma_H). \quad (3.15)$$

By construction, the S-matrices  $S_{nm}^{(\widehat{G}/H)}$  commute with the action of the finite quantum group  $U_q(H)$ . These charges act in a rapidity independent fashion on the states, since they have Lorentz spin zero. Therefore one can use the  $U_q(H)$  symmetry to restrict the model. The restriction may be described as follows. For each weight  $\alpha$  of  $W_n$  we introduce formal operators  $K_\alpha^{(n)}(\theta)$ . These operators satisfy an S-matrix exchange relation:

$$K_\alpha^{(n)}(\theta_1) K_\beta^{(m)}(\theta_2) = \sum_{\alpha', \beta'} \left( S_{nm}^{(\widehat{G}/H)}(\theta, q) \right)_{\alpha\beta}^{\alpha'\beta'} K_{\beta'}^{(m)}(\theta_2) K_{\alpha'}^{(n)}(\theta_1). \quad (3.16)$$

Let  $\mathcal{F}$  denote the multiparticle fock space generated by the formal action of the operators  $K_\alpha^{(n)}(\theta)$  on the vacuum. The space  $\mathcal{F}$  is an  $U_q(H)$  module, and reducible (for  $q$  not a root of unity):

$$\mathcal{F} = \bigoplus_i V^{(\lambda_i^{(H)})}, \quad (3.17)$$

where  $V^{(\lambda^{(H)})}$  is an  $U_q(H)$  module of highest weight  $\lambda^{(H)}$ . Since the  $K_\alpha^{(n)}(\theta)$  act on  $\mathcal{F}$ , one can consider their reduction

$$K_{\lambda_j^{(H)} \lambda_i^{(H)}}^{(n)}(\theta) : \quad V^{(\lambda_i^{(H)})} \longrightarrow V^{(\lambda_j^{(H)})}. \quad (3.18)$$

These operators satisfy the exchange relation:

$$\begin{aligned}
& K_{\lambda_j^{(H)} \lambda_i^{(H)}}^{(n)}(\theta_1) K_{\lambda_i^{(H)} \lambda_k^{(H)}}^{(m)}(\theta_2) = \\
& \sum_{\lambda_l^{(H)}} \left( S_{nm}^{(\widehat{G}/H)}(\theta, q) \right)_{\lambda_i^{(H)} \lambda_k^{(H)}}^{\lambda_j^{(H)} \lambda_l^{(H)}} K_{\lambda_j^{(H)} \lambda_l^{(H)}}^{(m)}(\theta_2) K_{\lambda_l^{(H)} \lambda_k^{(H)}}^{(n)}(\theta_1). \tag{3.19}
\end{aligned}$$

The S-matrix for the kinks in (3.19) is in the so-called SOS (solid-on-solid) form. The above construction is the usual vertex/SOS correspondence which is commonplace in lattice statistical mechanics[38].

Finally, the restriction amounts to taking  $q$  to be a root of unity and imposing a limitation on the allowed highest weight labels  $\lambda^{(H)}$ , due to the fact that  $q$  is a root of unity. From (2.3) and (3.5) one finds that  $q$  has the value

$$q = -\exp(-i\pi/(l+h)). \tag{3.20}$$

The limitations on the labels  $\lambda^{(H)}$  can be deduced from the representation theory of  $U_q(H)$  [39]; the result is equivalent to the statement that  $\lambda^{(H)}$  must correspond to an integrable representation of the  $H$ - $WZW$  model at level  $l$ . Additionally, the pair  $\lambda_j^{(H)}, \lambda_i^{(H)}$  must be admissible, which means that  $V^{(\lambda_j^{(H)})}$  must be contained in the tensor product  $W_n \otimes V^{(\lambda_i^{(H)})}$  when the spaces are considered as  $H$  modules.

To summarize, the spectrum of the perturbed  $G_1 \otimes H_l/H_{l+1}$  coset theory consists of RSOS kinks  $K_{\lambda_j^{(H)} \lambda_i^{(H)}}^{(n)}(\theta)$ , with the above limitations on the labels  $\lambda^{(H)}$ , characterized by an integer  $l$ . The S-matrix for these kinks is the RSOS form of  $S_{nm}^{(\widehat{G}/H)}$  which we denote as  $S_{nm}^{(\widehat{G}/H);l}$ . For  $G = H$  the above result was described in [11][40] [41].

The action of the residual symmetries on the kink states is described as follows. The residual charges are  $Q_\alpha, \overline{Q}_{-\alpha}$  for  $\alpha$  not a root of  $H$ . Each of these charges can be associated to a representation  $\lambda_\alpha^{(H)}$  of  $H$  by considering  $\alpha$  a weight of  $H$ . The charges must be decomposed into the components  $(Q_\alpha)_{\lambda_2^{(H)} \lambda_1^{(H)}}$  that intertwine the sectors  $\lambda_2^{(H)}$  and  $\lambda_1^{(H)}$ , since these are what have a well-defined action on the states. The currents for these intertwiners have well-defined, generally non-abelian, braiding relations. One finds

$$(Q_\alpha)_{\lambda_3^{(H)} \lambda_2^{(H)}} |K_{\lambda_2^{(H)} \lambda_1^{(H)}}^{(n)}(\theta)\rangle = e^{s\theta} \left\{ \begin{array}{ccc} \lambda_3^{(H)} & \lambda_\alpha^{(H)} & \lambda_2^{(H)} \\ \lambda_n^{(H)} & \lambda_1^{(H)} & \lambda_n^{(H)} \end{array} \right\}_q |K_{\lambda_3^{(H)} \lambda_1^{(H)}}^{(n)}(\theta)\rangle, \tag{3.21}$$

where  $s$  is the Lorentz spin of  $Q_\alpha$ ,  $\{*\}_q$  represents a generalized  $6j$  symbol for  $U_q(H)$ , and  $\lambda_n^{(H)}$  refers to the space  $W_n$  viewed as a  $U_q(H)$  module. Similar formulas give the action of  $\overline{Q}_\alpha$  with  $s \rightarrow -s$ . One has the qualitative rules

$$(Q_\alpha)_{\lambda_3^{(H)}\lambda_2^{(H)}} |K_{\lambda_2^{(H)}\lambda_1^{(H)}}^{(n)}(\theta)\rangle \neq 0 \quad \text{if } V^{\lambda_3^{(H)}} \subset V^{\lambda_\alpha^{(H)}} \otimes V^{\lambda_2^{(H)}}. \quad (3.22)$$

General quantum group theoretic arguments show that this action is necessarily a symmetry of the above S-matrix. For  $G = H = SU(2)$  this construction was described in detail in [37][13].

For  $G = H$  the above result was described in [11][42] [41]. The remaining quantum affine charges in this situation are  $Q_{\alpha_0}$  and  $\overline{Q}_{-\alpha_0}$ , with fractional spin  $\pm g/(g+l)$ , and they generate residual quantum symmetries of the RSOS S-matrices. In order to describe properly the action of these charges on the RSOS kinks and verify that they commute with the S-matrices  $S^{(\widehat{G}/G);l}$  one must screen them in a manner described for  $SU(2)$  in [13][37]. In a coset description these symmetries can be identified with the fractional supersymmetries generated by the conserved currents

$$J^{(l)} = \Phi_0^{0,\text{Adj}} \quad (3.23)$$

(for  $k = 1$ ). We will use this fact later.

Let us specialize now to the perturbations of the  $N = 2$  theories, again when  $k = 1$ . In this situation, the level  $l = g-h$ , and  $q = \exp(-i\pi/g)$ . Furthermore  $H = H' \otimes U(1)$ , so that  $\rho_H = \rho_{H'}$ . Therefore the restriction described above is performed with the quantum group  $U_q(H')$ . The spectrum consists of RSOS kinks  $K_{\lambda_j^{(H)}\lambda_i^{(H)}}^{(n)}(\theta)$ , whose scattering is given by the S-matrices  $S_{nm}^{(\widehat{G}/H');g-h}$ . When the background charge is as in (2.2), the quantum affine charges  $Q_{\alpha_0}, \overline{Q}_{-\alpha_0}$  and  $Q_{\alpha_r}, \overline{Q}_{-\alpha_r}$  have Lorentz spin  $\pm 1/2$ , and with proper screening are identified with the  $N = 2$  supercharges. Due to the automorphisms  $\sigma_H$  that define  $S^{(\widehat{G}/H');g-h}$ , by design this S-matrix commutes with the action of these supercharges.

The  $U(1)$  current of the  $N = 2$  superconformal algebra is given by

$$J_{U(1)}(z) = \frac{2i}{\sqrt{g(g+1)}}(\rho_G - \rho_H) \cdot \partial\phi, \quad (3.24)$$

and similarly for  $\overline{J}_{U(1)}$ . These currents are normalized such that the  $U(1)$  charge of the  $N = 2$  supercharges is  $\pm 1$ . The perturbed theory is invariant under a diagonal  $U(1)$ .

The  $U(1)$  charges of the kink states can be computed from the vertex operators (3.8) that create them, at least when  $W_n = V_n$ . For the kinks  $K_{\mu_n}^{(n)}$  one obtains

$$q_{U(1)} = -\frac{1}{g} 2(\rho_G - \rho_H) \cdot \mu_n. \quad (3.25)$$

For example, for  $G = SU(N+1)$ ,  $H' = SU(N)$ , these  $U(1)$  charges are multiples of  $1/(N+1)$ , in accordance with the fractional fermion numbers in [6][7].

Consider now the case of level  $k > 1$ . The arguments leading to the spectrum and S-matrices given in [5][11] for the case  $G = H$  can be repeated with little modification. We briefly outline the argument and the result. The  $k^{\text{th}}$   $\widehat{G}$ -affine Toda theory is characterized by a  $U_q(\widehat{G})$  symmetry for all  $k$ . The  $U_q(\widehat{G})$  currents are simple modifications of the currents in (3.3), where now the quasi-chiral components  $J_{\alpha_i, z}$  and  $\overline{J}_{-\alpha_i, \bar{z}}$  of the currents are multiplied by  $G_k/[U(1)]^r$  para-fermions  $\mathcal{A}_{\alpha_i}^{(0)}$ ,  $\overline{\mathcal{A}}_{-\alpha_i}^{(0)}$  respectively:

$$\begin{aligned} J_{\alpha_i, z}(z, \bar{z}) &= \mathcal{A}_{\alpha_i}^{(0)}(z, \bar{z}) \exp\left(\frac{i}{\beta k} \alpha_i \cdot \phi\right) \\ \overline{J}_{-\alpha_i, \bar{z}}(z, \bar{z}) &= \overline{\mathcal{A}}_{-\alpha_i}^{(0)}(z, \bar{z}) \exp\left(\frac{i}{\beta k} \alpha_i \cdot \overline{\phi}\right). \end{aligned} \quad (3.26)$$

The other components of the conserved currents can be deduced from conformal perturbation theory. The spin of the para-fermions  $\mathcal{A}_{\alpha_i}^{(0)}$ ,  $\overline{\mathcal{A}}_{-\alpha_i}^{(0)}$  is  $\pm(k-1)/k$ . Therefore the spin of the  $U_q(\widehat{G})$  charges now becomes

$$\frac{1}{\gamma_k} \equiv \text{spin}(Q_{\alpha_i}) = -\text{spin}(\overline{Q}_{-\alpha_i}) = \frac{1 - k\beta^2}{\beta^2 k^2}. \quad (3.27)$$

The braiding of the para-fermions, in addition to the braiding of the vertex operators in the currents (3.26), now imply that  $q$  is changed to  $-\exp(-i\pi/\gamma_k)$ . Therefore the representations (3.9) are valid with

$$x = \exp(\theta/\gamma_k) \quad , \quad q = -\exp(-i\pi/\gamma_k). \quad (3.28)$$

For  $k > 1$ , the  $k^{\text{th}}$   $\widehat{G}$ -affine Toda theory has some additional symmetries generated by charges  $Q^{(k)}$ ,  $\overline{Q}^{(k)}$ , not present at  $k = 1$ . These symmetries have fractional spin  $\pm g/(g+k)$ , and are independent of the  $U_q(\widehat{G})$  symmetries. In the conformal limit, the currents for these symmetries are of the form  $\epsilon \partial\phi$ , where  $\epsilon$  is an ‘energy’ operator in the para-fermion theory. When the background charges are turned on, in the coset theory this current corresponds to the field

$$J^{(k)} = \Phi_0^{\text{Adj}, 0}. \quad (3.29)$$

In the conformal theory, these currents play the role of generating a non-local chiral algebra for the coset models [43]. In the massive theory these symmetries are unbroken and must be symmetries of the S-matrix. The fact that the S-matrix must be invariant under two independent sets of symmetries  $Q^{(k)}, \overline{Q}^{(k)}$  and  $U_q(\widehat{G})$  implies that it must be the tensor product of two factors. The  $Q^{(k)}, \overline{Q}^{(k)}$  invariant factor can be deduced as follows. When  $G = H$ , under the  $k \Leftrightarrow l$  duality of the coset models, the  $Q^{(k)}, \overline{Q}^{(k)}$  symmetries are dual to the residual  $U_q(\widehat{G})$  symmetries coming from the current (3.23); thus one concludes that the S-matrix contains a factor  $S^{(\widehat{G}/G);k}$ . The spectrum of the  $k^{\text{th}}$   $\widehat{G}$ -affine Toda theory thus consists of kinks with a RSOS  $\otimes$  vertex structure  $K_{\lambda_j^{(G)} \lambda_i^{(G)}; \alpha}^{(n)}$  of mass  $M_n$ , where  $\alpha$  is a weight of  $W_n$ . The S-matrix is given by

$$S_{nm}^{k^{\text{th}} \widehat{G}\text{-Toda}}(\theta, q) = X_{nm}(\theta) \widetilde{S}_{nm}^{(\widehat{G}/G);k}(\theta) \otimes \widetilde{S}_{nm}^{(\widehat{G})}(x, q), \quad (3.30)$$

where  $\widetilde{S}_{nm} \equiv S_{nm}/X_{nm}$ . The factor  $\widetilde{S}^{(\widehat{G})}$  acts on the  $W_n$  indices of the kinks and is  $U_q(\widehat{G})$  symmetric, whereas  $\widetilde{S}^{(\widehat{G}/G);k}$  acts on the  $(\lambda_j^{(G)} \lambda_i^{(G)})$  RSOS labels.

When the background charges are turned on, only the  $U_q(\widehat{G})$  symmetries are affected, which implies the  $\widetilde{S}^{(\widehat{G})}$  factor must be restricted as before using the  $U_q(H)$  invariance. The  $Q^{(k)}, \overline{Q}^{(k)}$  symmetries are unaffected. For  $\beta = \beta_-^{(H)}$ ,  $1/\gamma_k = 1/(l+h)$  is independent of  $k$ , and  $q$  is still given by (3.20). This leads to the following result for the perturbed coset theories. The spectrum consists of kinks with a RSOS  $\otimes$  RSOS structure

$$K_{\lambda_j^{(G)} \lambda_i^{(G)}; \lambda_{j'}^{(H)} \lambda_{i'}^{(H)}}^{(n)}(\theta) \quad (3.31)$$

of mass  $M_n$ . The S-matrix for these kinks is given by

$$S_{nm}^{(G/H);(k,l)}(\theta) = X_{nm}(\theta) \widetilde{S}_{nm}^{(\widehat{G}/G);k}(\theta) \otimes \widetilde{S}_{nm}^{(\widehat{G}/H);l}(\theta). \quad (3.32)$$

#### 4. Examples

The simplest applications of our techniques is to the  $\mathbb{C}\mathbb{P}^N$  models:

$$\mathcal{M}_{N,k} \equiv \frac{SU_k(N+1) \times SO(2N)}{SU_{k+1}(N) \times U(1)}.$$

The S-matrices were obtained in [7] for the solitons in the affine Toda perturbations of these models. The results for the  $\mathcal{M}_{N,1}$  models were derived by a considerable amount of hard work using the Landau-Ginzburg structure and bootstrap methods. The S-matrices

for the general  $\mathcal{M}_{N,k}$  models were then conjectured and a compelling body of evidence was presented. Here we will derive, in a rather straightforward way, the general S-matrix for the  $\mathcal{M}_{N,k}$  models. Our techniques have the advantage that we make explicit use of the underlying  $H$ -Toda structure, and we naturally incorporate the supersymmetry through the extension to the affine  $G$  Toda structure. From the previous section we see immediately that the S-matrix of the  $\mathcal{M}_{N,k}$  models with  $k > 1$  has the tensor product structure that was advanced in [7]. We therefore only need to examine the factor  $\tilde{S}_{nm}^{(\widehat{G}/H);g-h}(\theta)$  of (3.32) in more detail, or equivalently, analyze the scattering matrices for the  $\mathcal{M}_{N,1}$  models.

For later convenience, consider the  $\check{R}$ -matrix for the fundamental vector ( $(M+N)$ -dimensional) representation of  $SU(M+N)$ . In the homogeneous gradation this is given by[31]

$$\begin{aligned} \check{R}(x, q) = & (xq - x^{-1}q^{-1}) \sum_{\alpha=1}^{M+N} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + (x - x^{-1}) \sum_{\substack{\alpha \neq \beta \\ \alpha, \beta=1}}^{(M+N)} E_{\alpha\beta} \otimes E_{\beta\alpha} \\ & + (q - q^{-1}) \left[ x \sum_{\alpha > \beta} + x^{-1} \sum_{\alpha < \beta} \right] E_{\alpha\alpha} \otimes E_{\beta\beta} , \end{aligned} \quad (4.1)$$

where  $E_{\alpha\beta}$  is an  $(M+N) \times (M+N)$  matrix whose entries  $(E_{\alpha\beta})_{ij}$  are equal to  $\delta_{\alpha i} \delta_{\beta j}$ . The spectral parameter  $x$  is determined from (3.10); at the  $N=2$  supersymmetric point it is  $x = \exp(\theta/g)$ , where  $g = M+N$ . Let the indices  $a, b$  and  $i, j$  run from 1 to  $M$  and from  $M+1$  to  $M+N$  respectively. Let  $\check{R}_{(1)}(x, q)$  and  $\check{R}_{(2)}(x, q)$  be the diagonal  $M \times M$  and  $N \times N$  blocks in  $\check{R}$ . Note that the sub-matrices  $\check{R}_{(1)}$  and  $\check{R}_{(2)}$  are simply the  $\check{R}$ -matrices for  $SU(M)$  and  $SU(N)$  respectively. Now perform the conjugation operation by  $\sigma_G^{-1} \otimes \sigma_G^{-1}$  to pass to the principal gradation, and then perform the conjugation (3.15). Let  $\check{R}'$  denote the matrix that results from these two conjugations. The combined effect of the two conjugation operations does not modify the structure of the sub-matrices  $\check{R}_{(1)}$  and  $\check{R}_{(2)}$ , except that  $x \rightarrow x^{g/2}$ . One also finds that the off-diagonal blocks of  $\check{R}'$  become extremely simple:

$$\begin{aligned} \check{R}'(x, q)_{ai, bj} &= \check{R}'(x, q)_{ia, jb} = (q - q^{-1}) \delta_{ab} \delta_{ij} \\ \check{R}'(x, q)_{ia, bj} &= \check{R}'(x, q)_{ai, jb} = (x^{g/2} - x^{-g/2}) \delta_{ab} \delta_{ij} . \end{aligned} \quad (4.2)$$

It is elementary to convert this to the required RSOS matrix. The Fock space,  $\mathcal{F}$ , in (3.17) is obtained by tensoring together fundamental representations of  $G$  and decomposing into highest weights of  $H$ . These  $H$  representations are then restricted to the affine highest

weights of  $H_{g-h}$ . Let  $\Lambda \equiv (\lambda, \nu; q)$  denote a weight of  $SU(M+N)$  that decomposes into an affine highest weight  $\lambda$  of  $SU_N(M)$ , an affine highest weight  $\nu$  of  $SU_M(N)$  and let  $q = q_{U(1)} = \frac{2}{g}(\rho_G - \rho_H) \cdot \Lambda$  be the  $N = 2$ ,  $U(1)$  charge of the vector  $\Lambda$ . Now consider the matrix elements  $(\check{R}')_{\Lambda_4, \Lambda_3}^{\Lambda_1, \Lambda_2}$  that are involved in the exchange relation (3.19). Let  $\Lambda_i \equiv (\lambda_i, \nu_i; q_i)$ .

For such a matrix element to be non-zero there are three possibilities:

- (i) The exchange does not involve the  $SU(N)$  factor; *i.e.*  $\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu$ .
- (ii) The exchange does not involve the  $SU(M)$  factor; *i.e.*  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$ .
- (iii) The off-diagonal terms of (4.2) act.

The expression for  $\check{R}'$  in situations (i) and (ii) is well known ( see, for example, [44] ). In situation (ii), one can write

$$\check{R}' = \sinh\left(\frac{i\pi}{M+N} - \frac{\theta}{2}\right) + \sinh\left(\frac{\theta}{2}\right) \mathcal{U}, \quad (4.3)$$

where we have taken  $q = -e^{-i\pi/(M+N)}$  and used  $x^{g/2} = e^{\theta/2}$ . To define  $\mathcal{U}$ , introduce the vectors  $e_1 \equiv \xi_1$ ,  $e_N \equiv -\xi_{N-1}$  and  $e_j \equiv \xi_j - \xi_{j-1}$  for  $j = 2, \dots, N-1$ , where  $\xi_j$  is the  $j^{\text{th}}$  fundamental weight of  $SU(N)$ , and define a function

$$s_{jk}(\nu) \equiv \sin\left(\frac{\pi}{M+N}(e_j - e_k) \cdot (\nu + \rho)\right). \quad (4.4)$$

where  $\rho$  is the Weyl vector of  $SU(N)$ . The operator,  $\mathcal{U}$ , then has the form:

$$\mathcal{U} \equiv (1 - \delta_{jl}) \frac{\left(s_{jl}(\nu + e_j) s_{jl}(\nu + e_k)\right)^{\frac{1}{2}}}{s_{jl}(\nu)}, \quad (4.5)$$

where  $\Lambda_1 \equiv (\lambda, \nu; q)$ ,  $\Lambda_2 \equiv (\lambda, \nu + e_j; q + \frac{M}{M+N})$ ,  $\Lambda_3 \equiv (\lambda, \nu + e_j + e_i; q + \frac{2M}{M+N})$  and  $\Lambda_4 \equiv (\lambda, \nu + e_k; q + \frac{M}{M+N})$ . The RSOS reduction of the  $\check{R}'$  matrix in the  $SU(M)$  factor is much the same. Note that for  $M = 1$ , the  $\check{R}'$  matrix in the “ $SU(1)$ ” direction involves only the  $U(1)$  charge and, as can be seen from (4.1), the  $\check{R}'$  matrix reduces to a simple multiplicative factor of  $\sinh\left(\frac{\theta}{2} - \frac{i\pi}{M+N}\right)$  (*i.e.*  $\mathcal{U}$  vanishes).

In the foregoing components of  $\check{R}'$  one either had  $\lambda_1 = \lambda_2 = \lambda_3$  or  $\nu_1 = \nu_2 = \nu_3$ . The other non-zero components come from the off diagonal blocks, and one has

$$\check{R}' = \begin{cases} i \sin\left(\frac{\pi}{M+N}\right) & \text{or} \\ \sinh\left(\frac{\theta}{2}\right) \end{cases} \quad (4.6)$$

where one has  $i \sin(\pi/(M+N))$  for either a)  $\Lambda_1 = (\lambda_1, \nu_1; q)$ ,  $\Lambda_2 = \Lambda_4 = (\lambda_1, \nu_2; q + \frac{M}{M+N})$  and  $\Lambda_3 = (\lambda_3, \nu_2; q + \frac{M-N}{M+N})$ , or b)  $\Lambda_1 = (\lambda_1, \nu_1; q)$ ,  $\Lambda_2 = \Lambda_4 = (\lambda_2, \nu_1; q - \frac{N}{M+N})$  and

$\Lambda_3 = (\lambda_2, \nu_3; q + \frac{M-N}{M+N})$ ; and one has  $\sinh(\theta/2)$  for either c)  $\Lambda_1 = (\lambda_1, \nu_1; q)$ ,  $\Lambda_2 = (\lambda_1, \nu_2; q + \frac{M}{M+N})$ ,  $\Lambda_3 = (\lambda_3, \nu_2; q + \frac{M-N}{M+N})$  and  $\Lambda_4 = (\lambda_3, \nu_1; q - \frac{N}{M+N})$ , or d)  $\Lambda_1 = (\lambda_1, \nu_1; q)$ ,  $\Lambda_2 = (\lambda_2, \nu_1; q - \frac{N}{M+N})$ ,  $\Lambda_3 = (\lambda_2, \nu_3; q + \frac{M-N}{M+N})$  and  $\Lambda_4 = (\lambda_1, \nu_3; q - \frac{N}{M+N})$ .

One should also note that the fundamental,  $(M+N)$ -dimensional representation of  $SU(M+N)$  decomposes into the  $(M, 1)(-\frac{N}{M+N}) \oplus (1, N)(+\frac{M}{M+N})$  of  $H = SU(M) \times SU(N) \times U(1)$ . Moreover, these two representations of  $H$  are mapped into one another by the generators  $X_{\pm\gamma}$  and  $X_{\pm\psi}$  that extend  $U_q(H)$  to  $U_q(\widehat{G})$ . This means that kink operators corresponding to  $(M, 1)(-\frac{N}{M+N})$  and  $(1, N)(+\frac{M}{M+N})$  should be a doublet of the superalgebra.

Consider now the  $\mathbf{CP}^N$  models (*i.e* set  $M = 1$ ). Let  $u_1$  and  $d_1$  denote the kinks corresponding to the  $N(\frac{1}{N+1})$  and  $1(-\frac{N}{N+1})$  of  $SU(N) \times U(1)$ . As in [7] these are a doublet of solitons of mass  $M_1$ . Note that  $u_1$  has fermion number  $\frac{1}{N+1}$  and  $d_1$  has fermion number  $-\frac{N}{N+1}$ . The RSOS heights are restricted to affine highest weights of  $SU_1(N) \times U(1)$  and one can easily see from (4.4) and (4.5) that  $\mathcal{U}$  is non-zero if and only if  $\nu = \xi_{j-1}$ ,  $l = j + 1 \pmod{N}$  and  $\Lambda_2 = \Lambda_4$ . One then has  $\mathcal{U} = 2 \cos(\pi/(N+1))$  and hence (4.3) gives a factor of  $-\sinh(\theta/2 - i\pi/(N+1)) + 2 \cos(\pi/(N+1)) \sinh(\theta/2) = \sinh(\theta/2 + i\pi/(N+1))$ . Consequently the scattering matrix is:

$$\begin{aligned}
u_1 u_1 &\rightarrow u_1 u_1 & Z_{1,1} \sinh\left(\frac{\theta}{2} + \frac{i\pi}{N+1}\right) \\
d_1 d_1 &\rightarrow d_1 d_1 & - Z_{1,1} \sinh\left(\frac{\theta}{2} - \frac{i\pi}{N+1}\right) \\
\left. \begin{aligned} u_1 d_1 &\rightarrow u_1 d_1 \\ d_1 u_1 &\rightarrow d_1 u_1 \end{aligned} \right\} & i Z_{1,1} \sin\left(\frac{\pi}{N+1}\right) \\
\left. \begin{aligned} u_1 d_1 &\rightarrow d_1 u_1 \\ d_1 u_1 &\rightarrow u_1 d_1 \end{aligned} \right\} & Z_{1,1} \sinh\left(\frac{\theta}{2}\right)
\end{aligned}$$

where  $Z_{1,1} = X_{1,1} v_{1,1}(\theta, q)$ , and

$$X_{1,1} = \frac{\sin\left(\frac{\theta}{2i} + \frac{\pi}{N+1}\right)}{\sin\left(\frac{\theta}{2i} - \frac{\pi}{N+1}\right)}$$

$$v_{1,1} = \frac{1}{2\pi i} \Gamma\left(\frac{i\theta}{2\pi} + \frac{1}{N+1}\right) \Gamma\left(1 - \frac{i\theta}{2\pi} - \frac{1}{N+1}\right) \prod_{j=1}^{\infty} \frac{\Gamma\left(1 + \frac{i\theta}{2\pi} + j - 1\right)}{\Gamma\left(1 - \frac{i\theta}{2\pi} + j - 1\right)}$$

$$\times \frac{\Gamma\left(\frac{i\theta}{2\pi} + j\right)}{\Gamma\left(-\frac{i\theta}{2\pi} + j\right)} \frac{\Gamma\left(-\frac{i\theta}{2\pi} + j - 1 + \frac{1}{N+1}\right)}{\Gamma\left(\frac{i\theta}{2\pi} + j - 1 + \frac{1}{N+1}\right)} \frac{\Gamma\left(1 - \frac{i\theta}{2\pi} + j - \frac{1}{N+1}\right)}{\Gamma\left(1 + \frac{i\theta}{2\pi} + j - \frac{1}{N+1}\right)}.$$

This S-matrix agrees with the results in [7]. The rest of the scattering amplitudes can be deduced by conjugating and making RSOS reductions of the  $\check{R}$  matrices for other fundamental representations of  $G = SU(N+1)$ . Alternatively, they can be obtained by fusing the S-matrix given above.

While one can directly relate the scattering matrices of the affine  $\widehat{G}$  Toda theory to the scattering matrices of the Landau-Ginzburg solitons, it is important to point out some of the apparent differences between the two theories and how these differences could possibly be resolved in mapping one theory onto the other. The most obvious difference is that the solitons of the Landau-Ginzburg theory interpolate between the finite number of Landau-Ginzburg vacua, whereas the kinks of the affine Toda theory interpolate between the infinite set of affine  $H_{g-h}$  highest weights that appear in tensor products of  $G$  representations. In particular these highest weights are unrestricted in the range of charges in the free  $U(1)$  factor of  $H$ . The simplest example of this is the  $\mathcal{M}_{1,1}$  model with  $c = 1$ . The Landau-Ginzburg potential is  $W(x) = \frac{1}{3}x^3 - a^2x$ , with vacua at  $x = \pm a$ . The associated affine Toda theory is simply sine-Gordon theory at choice of the coupling constant that yields a supersymmetric model. The latter theory has infinitely many distinct vacua. However, if one considers the allowed scattering between solitons in both theories one comes up with the same scattering matrix, and a natural identification of Landau-Ginzburg solitons and sine-Gordon kinks. One should also remember that in the  $c = 1$ ,  $N = 2$  superconformal theory the supersymmetry generators are:

$$G^\pm(z) = e^{\pm i\sqrt{3}\phi(z)},$$

while the chiral primary field  $x(z)$  is given by

$$x(z) = e^{\frac{i}{\sqrt{3}}\phi(z)}.$$

In the ultra-violet limit of the sine-Gordon theory the fundamental kink operator reduces to:

$$K^\pm(z) = e^{\pm \frac{i\sqrt{3}}{2}\phi(z)}. \quad (4.7)$$

The kink fields corresponding to (4.7) has  $\Delta\phi \equiv \phi(+\infty) - \phi(-\infty) = \pm\pi\sqrt{3}$ . Observe that if one shifts  $\phi$  by this amount then one maps  $x(z) \rightarrow -x(z)$ . A shift of  $\pm 2\pi\sqrt{3}$  (which

corresponds to the supercharge since  $K^\pm(z)K^\pm(w) \sim (z-w)^{3/4}G^\pm(z)$  maps  $x(z)$  back to itself. Thus, for the choice of the coupling constant  $\beta = \sqrt{2/3}$ , one can view the “double-kink” operators (*i.e.* the kink operators with  $\Lambda = \pm\sqrt{2}$ ) as a supercharges. Incorporating these supercharges into the chiral algebra of the conformal model thus has an off-critical counterpart in which one can replace the sine-Gordon theory by a supersymmetric effective field theory of the operator  $x(z)$ . The sine-Gordon kinks then map onto Landau-Ginzburg solitons running between  $-a$  and  $+a$ . One can then use the vertex operator realization of the supercharges to see how the sine-Gordon kinks fall into supermultiplets of Landau-Ginzburg solitons.

The generalization of the foregoing to less trivial models is made much more difficult by the necessity of screening. One can try to use the vertex operator realization of the supercurrents to map the infinitely many vacua labelled by the highest weights of affine  $H_{g-h}$  to the finitely many vacua of the Landau-Ginzburg theory. One can then attempt to map the Toda kinks onto Landau-Ginzburg solitons. This can be carried out successfully in the  $\mathcal{M}_{N,1}$  models. The basic difficulty is that the screenings and BRST reduction make it difficult to determine exactly how certain Toda kink operators, or perhaps combinations of Toda kink operators, interpolate between Landau-Ginzburg vacua. There are, however, certain elementary reductions that can be made in general. For example observe that shifting  $\phi$  by  $\frac{2\pi}{\beta}g\Lambda = 2\pi\sqrt{g(g+1)}\Lambda$  for  $\Lambda \in M^*(G)$  will not change the chiral primary fields. The RSOS reduction of the S-matrix will also not change under such a shift. This means that if one incorporates suitable  $U(1)$  translations on the weight lattice of  $G$  into the chiral algebra then one can reduce the kink spectrum of the Toda theory to a finite subset. Then, if one can properly incorporate the supercharge one should be able to further reduce this finite subset to the Landau-Ginzburg solitons. We have already shown how to extract the correct Landau-Ginzburg soliton scattering matrix from the affine Toda scattering matrix in the  $\mathbb{C}\mathbb{P}^N$  models, and this suggests it is indeed possible to make the desired identifications of Toda and Landau-Ginzburg solitons.

It is desirable however to find an *a priori* argument based on the supersymmetry that shows how to deduce the Landau-Ginzburg spectrum from the Toda spectrum. The ensuing difficulties in doing this are best illustrated by a simple example. Consider the model

$$\frac{SU_1(4) \times SO_1(8)}{SU_2(3) \times SU_3(2) \times U(1)} . \tag{4.8}$$

This is the  $c = 12/5$  minimal model, but with the type  $D$  modular invariant. The perturbed Landau-Ginzburg superpotential is

$$W(x, y) = \frac{1}{5}x^5 + xy^2 - a^4x .$$

The critical points are at  $x = 0, y = \pm a^2$  and at  $y = 0, x = \pm a, \pm ai$ . A schematic representation of the soliton polytope [3] is shown in figure 1, and the mass projection [2] is shown in figure 2.

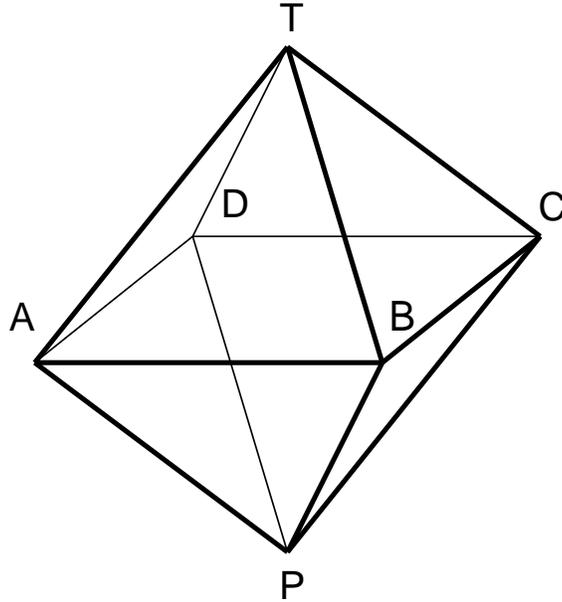


Figure 1. The soliton polytope for the integrable model with Landau-Ginzburg potential  $W(x, y) = \frac{1}{5}x^5 + xy^2 - a^4x$ .

The top (T) and bottom (P) vertices of the soliton polytope project to the center of the square in the mass projection. Each line from the center of the square to a corner thus represents two distinct types of soliton supermultiplet: ones that start (or finish) at T and ones that start (or finish) at P. Let  $M_1$  be the mass of these solitons and let  $M_2$  be the mass of the solitons that run along the edges (from corner to corner) of the square in figure 1b. From the results of [2] we know that  $M_2/M_1 = \sqrt{2}$ . Label the corners of the square by A, B, C and D. The geometry suggests that scattering the soliton that runs from A to T against a soliton that runs from T to B should give a resonance at  $\theta = i\pi/2$  for creating, at rest, a soliton that runs from A to B. These facts lead us to associate the solitons of mass  $M_1$  with the 4 or  $\bar{4}$  of  $SU(4)$  and those of mass  $M_2$  with the 6 of  $SU(4)$ .

It was noted in [2] that the foregoing solitons could not form a closed scattering theory since scattering a soliton running from T to A against a soliton from A to B must yield an outgoing state of mass  $M_2$  that starts at T. There are no fundamental Landau-Ginzburg solitons that satisfy this, but if one added particle or breather states of mass  $M_2$  that are localized at P and T then one could probably close the scattering theory.

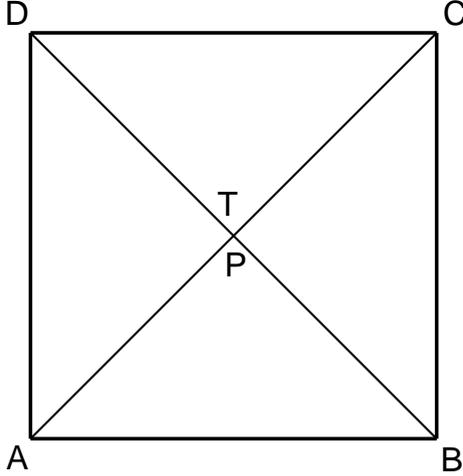


Figure 2. The mass projection of the soliton polytope for the Landau-Ginzburg potential  $W(x, y) = \frac{1}{5}x^5 + xy^2 - a^4x$ . The lines from the center to the corners correspond to two types of solitons: those that connect to T and those that connect to P. Both these types of soliton have mass  $M_1$ . The solitons running between the corners have mass  $M_2 = \sqrt{2}M_1$ .

Our analysis provides a candidate S-matrix for the foregoing model. The problem is to see how the affine Toda theory relates to the Landau-Ginzburg theory. The soliton masses certainly agree. One can also verify that the kinks have the correct anomalous fermion numbers. The 4 of  $SU(4)$  decomposes into  $(\frac{1}{2}, 0)(-\frac{1}{2}) \oplus (0, \frac{1}{2})(+\frac{1}{2})$  of  $SU(2) \times SU(2) \times U(1)$ , while the 6 decomposes into  $(0, 0)(-1) \oplus (\frac{1}{2}, \frac{1}{2})(0) \oplus (0, 0)(+1)$ . One can check this against the formula given in [7][45]:

$$f = \frac{1}{2\pi} \Delta \text{Im} \left( \det \left( \frac{\partial^2 W}{\partial x_i \partial x_j} \right) \right) \pmod{1}, \quad (4.9)$$

which yields the fermion numbers of  $\frac{1}{2} \pmod{1}$  for the 4 and  $\bar{4}$  and  $0 \pmod{1}$  for the 6. The peculiarities start to arise when one considers the supermultiplet structure. The 6 appears to decompose into an irreducible *three* dimensional supermultiplet, which is impossible in

the standard representations of supersymmetry. One might hope that this problem would be resolved by considering the representation of the supersymmetry on the full set of RSOS kinks (3.18), but we find that the problem persists. One tends to find that completely different kinks want to have a third kink as a common superpartner. This surprise can be elucidated somewhat by taking the ultra-violet limit of the perturbed model and looking at the limit of the kink operators in the conformal theory. The affine Toda kink operators limit to operators of the form

$$\exp\left(-\frac{i}{\beta}\Lambda \cdot \phi(z)\right)$$

for  $\Lambda$  a fundamental weight of  $SU(4)$ . The original superconformal model restricted the labels  $\Lambda$  to be of the form  $w(\rho_G) - \rho_G$ , which is always at least a root of  $G = SU(4)$ . This restriction amounted to choosing a special, and indeed exceptional, modular invariant for the numerator factor of  $H = SU(2) \times SU(2) \times U(1)$ . The choice of this exceptional modular invariant made a very important change to the fusion rules. For a general choice of modular invariant, the supercurrents, which correspond to the operators  $\Phi_0^{0,-\gamma}$  and  $\Phi_0^{0,\psi}$ , would have non-trivial braidings with other operators. For the exceptional modular invariant, the operators corresponding to the supercurrents become simple currents and can therefore be incorporated into an extended chiral algebra for the theory, *i.e.* they become supercurrents in the standard sense of the word. The kink operators in the ultra-violet limit correspond to conformal fields that are excluded from the special modular invariant, and indeed have non-abelian braiding relationships with the operators corresponding to the supercurrents. Thus the new feature that we encounter is that the affine Toda kinks have non-abelian braiding relationships with the supercurrents and so generate a highly non-standard supermultiplet structure. The only class of models where this does not happen is, in fact, in the  $\mathbb{C}\mathbb{P}^N$  models, where the supercurrents are still simple currents even when the kink operators are included. However, even in this situation the kinks have anomalous fermion numbers which lead to non-trivial, though abelian, braidings of the kinks with the supercharges.

The foregoing discussion raises two possibilities: either there is non-abelian braiding of the supercharges with the Landau-Ginzburg solitons, or the non-abelian braiding of the supercharges is purely an artifact of the Toda description and is not present in the Landau-Ginzburg formulation. One possible method for getting around the non-abelian braiding is to attempt to identify several distinct Toda kinks with a single Landau-Ginzburg soliton so that the supercharges are once again well-behaved members of an extended

chiral algebra. In the example (4.8) we could find no such identifications that were also consistent with the scattering matrix. As regards being an artifact of our formulation, we note that anomalous abelian braiding relationships are already present in the Landau-Ginzburg formulation of the  $\mathbb{C}\mathbb{P}^N$  models [7] as a consequence of the anomalous fermion number of the kinks. Moreover, if one considers the Landau-Ginzburg formulation of the model (4.8), one can see from (4.9) that the solitons of mass  $M_1$  have fermion number  $\frac{1}{2}$ . If we also make the plausible assumption that the kink field should be local with respect to the perturbing operators, one can then argue that in the ultra-violet limit the resulting operator in the conformal field theory will have non-abelian braidings with the supercharges. We therefore suspect that once one goes beyond the  $\mathbb{C}\mathbb{P}^N$  models, the kinks of the  $N = 2$  supersymmetric  $\widehat{G}$ -affine Toda models will exhibit non-abelian braiding with the supercurrents. If this is the case, we expect that one should also encounter this phenomenon in the Landau-Ginzburg formulation.

## 5. Conclusions

There are two major roles for  $\check{R}$ -matrices in two-dimensional models: either they appear in scattering matrices or they appear in Boltzmann weights of exactly solvable models. Perturbed conformal theories give a method of interrelating both of these applications. Given an exactly solvable model one can take its continuum limit, at criticality, and obtain a conformal field theory. For such conformal models there is always a relevant perturbation that leads to a quantum integrable model. Another  $\check{R}$ -matrix can then be obtained from the scattering theory of this integrable model. For the non-supersymmetric minimal series, the energy perturbation of the  $p^{\text{th}}$  minimal model yields an integrable model whose scattering matrix involves an RSOS  $\check{R}$ -matrix[8], and this  $\check{R}$ -matrix defines the Boltzmann weights of a lattice model whose continuum limit is the  $(p - 1)^{\text{th}}$  minimal model[46]. One can also see this sort of progression in the  $N = 2$  superconformal models, their lattice counterparts and the related quantum integrable models. In [47], the lattice analogues of the  $N = 2$  superconformal models were described, and exactly the same  $\check{R}$ -matrices were used there as have been used employed in this paper, except that in [47] one took  $q = e^{i\pi/(g+1)}$  in order to get the RSOS lattice model Boltzmann weights.

One way to understand the distinction between the  $\check{R}$ -matrices of the lattice model and the soliton S-matrix is that, for the former, one is working with the quantum group structure associated with the denominator of the coset, whereas for the latter, one is

working with the quantum group structure of the numerator factors, as was observed in [37]. This means that for the quantum integrable perturbations of the  $G \times H/H$  models, the extension of  $U_q(H)$  to  $U_q(\widehat{G})$  involves requiring that the S-matrix commutes with the supersymmetry charges (2.17), whereas in the lattice model the same extension means that the Boltzmann weights commute with the perturbation operators (2.18).

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