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Adding Apples and Oranges: Alignment of Semantic and Formal Knowledge

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We show that the same mechanism that mediates analogical reasoning (i.e., structural alignment) leads to interpretive ''content effects'' in reasoning about arithmetic word problems. Specifically, we show that both college students and textbook writers tend to construct arithmetic word problems that maintain systematic correspondence between the semantic relations that people infer from pairs of real-world objects (e.g., apples and baskets support the semantic relation CONTAIN [content, container]) and mathematical relations between arguments of arithmetic operations (e.g., DIVIDE [dividend, divisor]). Such relational alignments, to which we refer here as *semantic alignments*, lead to selective and sensible application of abstract formal knowledge. For example, people usually divide apples among baskets rather than baskets among apples, and readily add apples and oranges but refrain from adding apples and baskets.

Flexible transfer of knowledge, be it knowledge of Newton's laws, linear equations, or social rules, requires that people be able to recognize structural

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similarities between situations that differ in appearance. This ability is measured on most intelligence tests. For example, we credit people with intelligence points on tests of analogical reasoning when they realize that the pair ''apples:baskets'' is analogous to the pair ''cars:parking lots'' although apples differ from cars and baskets differ from parking lots. In other words, we reward people for heeding similarities between semantic relations, such as CONTAIN [content, container], which they infer from knowledge about specific objects.

In this paper we show that such inferred semantic relations between objects help people decide when and how to apply their abstract formal knowledge of arithmetic. Specifically, we asked college students to construct simple addition or division word problems for various pairs of object sets we provided. We found that they aligned the mathematical relations between arguments of arithmetic operations with the semantic relations that were evoked by the given pairs of object sets. For example, division involves an asymmetric mathematical relation between dividend and divisor ($a/b \neq b/a$). Students who completed our mathematical task applied this asymmetric operation to object sets that readily evoked functionally asymmetric semantic relations (e.g., *a* apples and *b* baskets), but refrained from dividing object sets that did not evoke functionally asymmetric relations (e.g., *a* apples and *b* oranges). Instead, they usually related such functionally symmetric sets with the mathematically symmetric operation of addition ($a + b = b + a$).[1]

In the next section, we situate our work in the context of research on analogical transfer. We point out that this research ignores the interpretive process by which people use content to understand the structure of the stimuli they align. Our studies were designed to show that the interpretive process that mediates understanding of mathematical word problems is similar to the process that mediates analogical reasoning (i.e., structural alignment). We argue and show that this process leads to interpretive ''content effects'' that reflect intelligent adults' ability (indeed, preference) to apply their formal knowledge of mathematics in a way that is consistent with their world knowledge.

How Content Affects Analogical Transfer

Researchers who study analogical transfer aim to identify regularities in the processes by which people retrieve analogous problems from memory (*access*) and align the representations of analogous problems (*mapping*). In a typical experimental paradigm, researchers present participants with a solu-

[1] We use ''apples-oranges'' for the purpose of exposition, but did not use this pair in our experiments. This pair alludes to the common saying ''It's like comparing apples and oranges,'' which captures the gist of our argument that processing has to fit the things being processed. Interestingly, the saying exemplifies inappropriate comparison with objects that can be readily and meaningfully compared (or added). For a more detailed discussion of this point see Bassok (1997).

tion to one or more *base* problems and after a short delay, during which participants may be required to work on an intervening task, ask them to solve one or more *target* problems. In most studies, the base and target problems are constructed such that they share a similar structure but differ in their content instantiations. Some studies also include base and target problems that share similar content instantiations but differ in structure. This orthogonal design enables researchers to separate the impact of similarities and differences in content from those of structure on access and mapping.

Although people can notice and exploit structural similarities between analogous stimuli, they are very sensitive to similarities and differences between the specific content instantiations of these stimuli (for a review see Reeves & Weisberg, 1994). For example, without a hint from the experimenter, many participants fail to notice that a solution learned in the context of a military problem can be applied to an analogous medical problem (Gick & Holyoak, 1980) or that a statistical principle learned from a sample problem about weather forecasting can be applied to a problem about arrangements of pizza toppings (Ross, 1987). Similarities between the content instantiations of analogous stimuli often impair access without affecting mapping. That is, people often fail to spontaneously access solutions to analogous problems that differ in content, but readily use these solutions after receiving a hint from the experimenter (e.g., Perfetto, Bransford, & Franks, 1983; Gentner & Landers, 1985; Gick & Holyoak, 1980). Nevertheless, the objects that happen to serve as arguments of analogous stimuli have significant effects on mapping performance (Gentner & Toupin 1986; Ross, 1987, 1989).

Content effects in analogical transfer are less pronounced when people have the ability, knowledge, and cognitive resources needed for abstracting the relevant structure from its specific content instantiation. That is, experts, good learners, and older children are more likely than novices, poor learners, and younger children to exploit structural similarities between stimuli that differ in content, context, and phrasing (e.g., Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Chi, Feltovich, & Glaser, 1981; Gentner & Rattermann, 1991; Gentner & Toupin, 1986; Novick, 1988, 1992; Silver, 1981; Schoenfeld & Herrmann, 1982). Similarly, training that encourages construction of more abstract representations for the base stimuli increases the probability of successful transfer (e.g., Bassok & Holyoak, 1989; Brown, 1989; Catrambone & Holyoak, 1989; Gick & Holyoak, 1983; Needham & Begg, 1991; Reed, 1993; Ross & Kennedy, 1990).

The dominant explanation for this pattern of content effects in analogical transfer is that people retain the specific content of the base and target stimuli in their representations and rely on matches and mismatches in content, in addition to matches and mismatches in structure, for retrieval and application of analogous solutions. This explanation is captured by various computational models of analogical access and mapping (e.g., Falkenhainer, Forbus, & Gentner, 1989; Forbus, Gentner, & Law, 1995; Hofstadter, Mitchell, & French, 1987; Holyoak & Thagard, 1989; Hummel & Holyoak, 1997; Thagard, Holyoak, Nelson, & Gochfeld, 1990). Within this view, abstraction of structure from content leads to deletion of the specific content from the representation of the base. For example, this is how Holyoak and Thagard (1995) describe abstraction of a joint schema from two base analogs that differ in content: ‘‘The resulting schema will therefore lay bare the structure of the analogs, stripping away the specifics of the individual examples’’ (p. 134).

There is little doubt that analogical transfer may fail due to mismatches in various aspects of content that people retain in their representations of the base and target stimuli. However, this is not the only way in which content affects transfer performance. Importantly, because abstraction of structure from content is an inferential process (e.g., apples and baskets evoke the CONTAIN relation), content may affect how people understand the structures of the base and target stimuli. Unfortunately, the issue of structure interpretation remains outside the scope of research on analogical transfer. Because researchers ignore interpretation, they risk drawing erroneous conclusions about the impact of content and structure on transfer performance. Specifically, while researchers believe that they are presenting participants with base and target stimuli that share the same structure and differ only in content, the participants may infer from content that the base and target actually differ in their structures. When this happens, both access and mapping may fail due to mismatches in the inferred (or interpreted) structures of the stimuli rather than, or in addition to, mismatches in undeleted aspects of content.

Bassok and her colleagues have shown such interpretive content effects on transfer performance (for a review, see Bassok, 1996). Specifically, they found that when different objects served as arguments of mathematically isomorphic word problems college students inferred that the problems had different mathematical structures. For example, Bassok, Wu, and Olseth (1995) asked college students to solve mathematically isomorphic permutation problems that involved a person who randomly assigned three elements from one set (*a*) to three elements from a different set (*b*) and asked for the probabilities of such random assignments. Because these problems were unfamiliar to participants, it is not surprising that their spontaneous solutions were incorrect. However, these spontaneous solutions varied systematically with the symmetry of the inferred semantic relation between the objects in the given sets. Most participants (87%) who received problems in which the paired objects bore an asymmetric semantic relation to each other (e.g., *a* students get *b* prizes) constructed equations in which the two sets played asymmetric mathematical roles (e.g., $a^3/b!$; $1/b^3$). By contrast, most participants (78%) who received problems in which the paired objects bore a symmetric semantic relation (e.g., *a* doctors work with *b* doctors) constructed equations in which the two sets played symmetric mathematical roles (e.g., $(a + b)/(ab)^3$; $3/(a + b)!$).

Interpretation is especially important to the understanding of transfer performance when the base and target stimuli are word problems from formal domains (e.g., mathematics, logic, probability). This is because the *formal structures* that determine the mathematical or logical solutions to such problems, and which experimenters typically consider the relevant structures (e.g., Bassok & Holyoak, 1989; Ross, 1987), do not coincide with the *semantic structures* of the situations described in the cover stories. To illustrate, consider two addition word problems that share the same mathematical structure (e.g., $a + b = c$) but differ in their semantic structures or ‘‘situation models’’ (e.g., Kintsch & Greeno, 1985). In one problem, which entails the ‘‘combine’’ semantic structure, *c* represents the number of marbles owned jointly by Tom (*a*) and Jim (*b*); in another problem, which entails the ‘‘change’’ semantic structure, *c* represents the number of Tom’s marbles after he got *b* marbles from Jim and added them to his original set of *a* marbles.

Just as the formal structure of addition can be mapped either to the ‘‘combine’’ or ‘‘change’’ semantic relation, the mathematical structure of the permutation problems in Bassok et al. (1995) can be aligned with either the asymmetric GET [givens, receivers] or the symmetric WORK WITH [workers, workers] semantic structures. Interestingly, in trying to understand the mathematical structures of these unfamiliar word problems, the students aligned the symmetry of the semantic and the mathematical structures. We refer to such interpretive effects as *semantic alignments*.

As in other studies that document the existence of content effects in analogical transfer, the participants who performed semantic alignments in Bassok et al. (1995) did not understand the mathematical structure of the permutation problems. Yet it appears that their performance was guided by a valid assumption about meaningful application of abstract mathematical concepts to real-life situations: that mathematical and world knowledge should be brought into correspondence. If semantic alignments are indeed mediated by such an assumption, then people should engage in semantic alignments when they apply familiar and well understood mathematical knowledge. To test this prediction, the present studies examined whether college students engage in semantic alignments when reasoning about arithmetic word problems involving addition and division.

Semantic Alignments for the Operations of Addition and Division

We asked undergraduate students with extensive experience in using arithmetic operations to construct simple addition or division word problems for pairs of object sets we provided. The solution that meets the minimal requirements of the construction task is to relate the given pair of object sets directly by the required arithmetic operation of addition or division. However, as we describe below, the object pairs were designed such that half afforded semantic alignments with direct addition but not with direct division (e.g., tulips–daffodils), whereas the other half afforded semantic alignments with direct division but not with direct addition (e.g., tulips–vases). When the mathematical and semantic structures conflicted, participants were forced either to sacrifice semantic alignments, to defy one or more of the task requirements, or to construct a problem with a more complex mathematical structure that preserves semantic alignment.

Because our stimuli were pairs of object sets (i.e., countable entities), we restrict our discussion of addition and division to the case involving positive integers, a and b, which denote the number of elements in two distinct nonempty sets, S and T, such that $a = n\{S\}$ and $b = n\{T\}$. The numerosity of the sets, and therefore the mathematical properties of addition and division, are independent of the specific elements in the sets. In *The Structure of Arithmetic,* Campbell (1970, p. 35) demonstrates this point using the set $S =$ {cat, dog, boat}. However, we argue and show that people tend to align the mathematical relation between the numerosities of the paired sets (a and b) with the semantic relation that they infer from the elements in the reference sets ($s_i \in S$ and $t_j \in T$). Figure 1 presents a schematic representation of a partial semantic network for the pairs tulips–daffodils and tulips–vases and shows how the objects in these pairs (represented by ovals) are aligned with the arguments of the addition and division operations (represented by squares).

Addition. The operation of addition, $a + b = c$, relates two *addends* (a and b) and their *sum* (c), where c is also a positive integer that denotes the number of elements in the union of sets S and T: $c = n\{S \cup T\}$. That is, the operation of addition relates the positive integers (a, b, and c) that denote the numerosities of three object sets (S, T, and S ∪ T, respectively). The fact that the sum of two positive integers is also a positive integer is consistent with the mathematical property of ‘‘closure’’ under addition. Another property of addition that is important for our discussion of semantic alignments is commutativity ($a + b = b + a$): the addends play interchangeable or *symmetric* structural roles in the additive relation.

As can be seen in Fig. 1, the mathematically symmetric roles of addends (a and b) with respect to their sum (c) can be readily aligned with the semantically symmetric roles of two subsets (e.g., tulips and daffodils) with respect to their joint taxonomic category (e.g., flowers): tulips and daffodils (or daffodils and tulips) can be placed in a one-to-one correspondence with the addends a and b, and the set of flowers can be placed in a one-to-one correspondence with the sum c.[2] When the paired sets are not subsets of the same

[2] Mapping the union set of tulips and daffodils onto flowers is not one-to-one in the sense that the union set is subsumed by the set ‘‘flowers,’’ which also includes other flowers (e.g., roses).

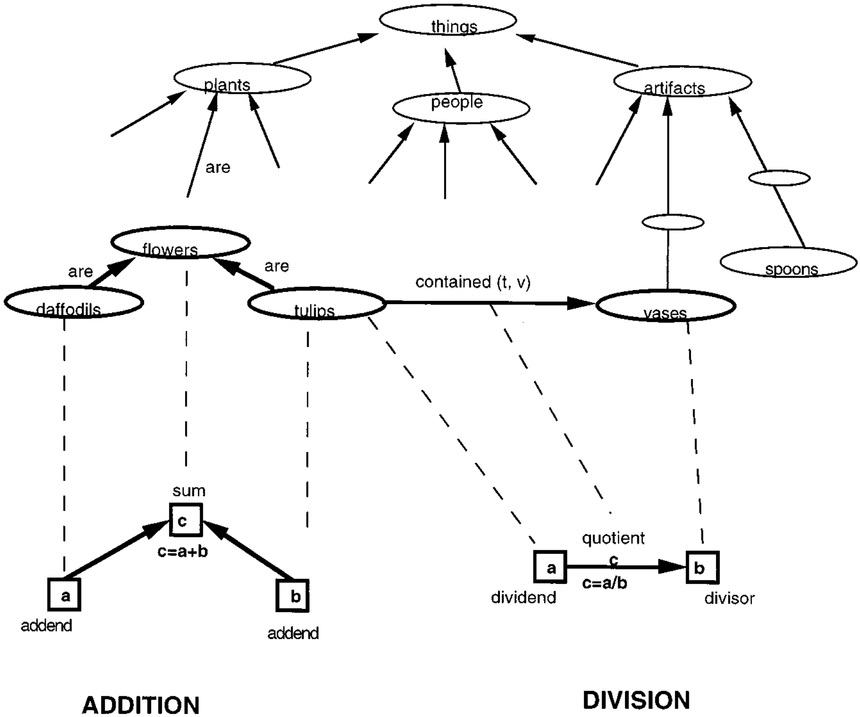

FIG. 1. An illustration of semantic alignments for the mathematical operations of addition (tulips–daffodils) and division (tulips–vases).

taxonomic category (e.g., tulips–spoons), semantic alignment should be quite difficult for at least two reasons. First, alignment of such sets demands traversing several levels of the semantic net to find a joint category that corresponds to their union (‘‘things’’). Second, each link in the semantic structure that must be traversed in order to achieve semantic alignment introduces a relational mismatch into the mapping with the additive mathematical structure, which lacks such intermediate links. Semantic alignment with addition for object sets from distinct taxonomic categories should be even more difficult when such sets are known to be related by an asymmetric semantic relation (e.g., tulips–vases).[3] An asymmetric semantic relation between elements of such sets creates not only a relational mismatch with addition (where there is no corresponding relation between the addends), but also a relational match with division the alternative and therefore potentially competing arithmetic operation of division.

[3] Because arithmetic operations take the numerosities of the sets as arguments, we restrict ourselves to asymmetric semantic relations that imply a difference in the relative numerosities of the paired sets.

Division. The operation of division, $a/b = c$, relates a *dividend* (a), a *divisor* (b), and their *quotient* (c), such that $a = cb$. Because positive integers are not closed under division, the quotient (c) does not denote the numerosity of a set whose elements are of the same type as the elements in S and T (and may be any rational number). That is, c does not denote the numerosity of some object set. Rather, it denotes a one-to-one pairing between distinct subsets of elements from the S and T sets (e.g., c elements from the S set are paired with one element from the T set). Hence, unlike in addition, there are only two rather than three reference object sets to be aligned (S and T). Importantly, and again unlike addition, division is not commutative ($a/b \neq b/a$). That is, the two arguments of division—the dividend (a) and the divisor (b)—play *asymmetric* structural roles.

As can be seen in Fig. 1, the mathematically asymmetric roles of dividend (a) and divisor (b) can be readily aligned with object sets that are arguments of asymmetric semantic relations. In particular, when the implied semantic relation is CONTAIN (contents, containers), the dividend (a) is aligned with contents (e.g., tulips) and the divisor (b) with containers (e.g., vases). Note that, unlike in the symmetric addition structure, semantic considerations constrain the structural roles that the paired sets assume in the asymmetric division structure (e.g., a tulips are contained in and therefore divided by b vases rather than vice versa).[4] When the paired sets lack an asymmetric semantic relation (e.g., tulips–spoons), semantic alignment should be hindered by a relational mismatch. When the paired sets lack an asymmetric semantic relation and are also subsets of a joint taxonomic category (e.g., tulips–daffodils), alignment with division will be further hindered by the relational match with an alternative and therefore potentially competing operation of addition.

Let us summarize the above analysis for the two types of paired object sets we used in the construction task.[5] Object sets from the same taxonomic category, to which we refer as *Symmetric sets* because of their symmetric relation to their joint superset (e.g., both tulips and daffodils are flowers), allow for a good semantic alignment with the mathematically symmetric operation of addition and a poor semantic alignment with the mathematically asymmetric operation of division. At the same time, object sets that play asymmetric structural roles in semantic relations (e.g., vases contain tulips but not vice versa), to which we refer as *Asymmetric sets*, permit a good semantic alignment with the mathematically asymmetric operation of division and a poor semantic alignment with the mathematically symmetric operation of addition.

The construction task was designed to test whether people's reasoning about arithmetic word problems is affected by such potentially different se-

[4] These semantic considerations remain outside the scope of the present analysis.

[5] In Experiment 2 we also examined semantic alignments for object pairs that supported the Set–Subset relation (e.g., flowers–tulips).

mantic alignments. If people ignore considerations of semantic alignment, then the solution that fulfills the minimal requirements of the construction task is direct addition ($a + b$) or direct division (a/b) of the numbers that denote the numerosities of members in the given sets regardless of whether the sets are Symmetric or Asymmetric. We will refer to such solutions as *Mathematically Direct* (MD). If, however, people perform semantic alignments as we predict, then the frequency of MD solutions should be higher for semantically alignable than nonalignable object pairs. That is, matches between the mathematical and semantic relations should facilitate construction of MD problems for Symmetric pairs in the Addition condition and Asymmetric pairs in the Division condition, while mismatches between the mathematical and semantic relations should hinder construction of MD problems for Asymmetric pairs in the Addition condition and Symmetric pairs in the Division condition.

Instead of constructing MD problems for nonalignable object pairs, people should construct problems that reflect a variety of what we call *Semantic Escape* strategies (SE). In one type of SE strategy, participants might violate the task requirements. For example, they might include only one of the given sets in the computation (either a or b) and introduce a third set (p) that can be added or divided with the retained set in a semantically compatible way (e.g., $a + p$; b/p). In another type of SE strategy, they might construct problems with a more complex mathematical structure (e.g., $[a + p]/b$) that fulfill the task requirements and, at the same time, achieve semantic alignment.

Experiments 1 and 2 investigate whether semantic alignments affect how college students perform the construction task. Experiment 3 examines students' expectations about the existence of semantic alignments in textbook word problems involving addition and division of either Symmetric or Asymmetric object sets.

EXPERIMENT 1

Method

Participants. Participants were 80 University of Chicago undergraduate students recruited to participate in a problem-solving study. They were tested individually or in small groups and were paid for their participation.

Materials. We used 16 pairs of object sets, 8 Symmetric and 8 Asymmetric. To ensure generality, we manipulated the animacy of the paired sets. Half of the Symmetric and Asymmetric pairs involved people (e.g., boys–girls; boys–teachers) and the other half inanimate objects (e.g., tulips–daffodils; tulips–vases). Symmetric sets were always drawn from the same-level taxonomic category; Asymmetric sets of people and objects were related by the functional relations SERVE and CONTAIN, respectively.

We constructed the materials such that one set (e.g., tulips) appeared in both a Symmetric pair (e.g., tulips–daffodils) and an Asymmetric pair (e.g., tulips–vases). By making this *common* set a member of both a Symmetric and an Asymmetric pair, we were able to examine another prediction derived from the semantic-alignment hypothesis. Specifically, because participants had to invent the numerosities for the sets we provided (i.e., to invent *a* and *b*), we were able to examine whether they assigned relative numerosities to the sets that are consistent with their world knowledge. In Asymmetric pairs the common set was the one that we expected participants to make larger (e.g., more tulips than vases),[6] whereas in Symmetric pairs we did not expect a systematic tendency to make the common set either smaller or larger than the other set (e.g., the number of tulips might be made equal, smaller, or larger than the number of daffodils).

The 16 pairs of object sets (Table 2, leftmost column) were divided into two equivalent construction booklets, each consisting of 8 pairs—two Symmetric pairs of people, two Symmetric pairs of objects, two Asymmetric pairs of people, and two Asymmetric pairs of objects—in a randomized order. Each common set never appeared more than once in the same booklet. Each pair was typed at the top of a separate page, and each booklet had two versions that reversed the left/right position of the sets in each pair (e.g., tulips–daffodils; daffodils–tulips).

On the cover page participants were told to construct a simple addition or division word problem involving each pair of sets. These general instructions were followed by a sample problem. The sample problem exhibited the MD structure and involved object sets that were semantically alignable with the MD structure (i.e., Symmetric in the Addition condition and Asymmetric in the Division condition). The sample problem in the Addition condition involved (*a*) trucks and (*b*) cars and entailed the $a + b$ solution: ‘‘A supermarket had 4 trucks and 10 cars in its parking lot. How many vehicles were there all together?’’ The sample problem in the Division condition involved (*a*) books and (*b*) shelves and entailed the a/b solution: ‘‘A bookcase has an equal number of books on each shelf. There are 20 books and 4 shelves. How many books are on each shelf?’’

Procedure. Participants were randomly assigned to receive either an Addition ($N = 40$) or a Division ($N = 40$) construction booklet. They worked at their own pace without any intervention from the experimenter.

Results

We developed a coding scheme for the problems participants constructed in the Addition and Division conditions that captured differences in the mathematical structures of the problems[7] and the relative sizes assigned to the sets. Two independent judges coded the data with an average agreement of over 90%. Disagreements were resolved by discussion between the judges such that all data were included in the analysis. Below we first report results pertaining to the mathematical structure of the constructed problems (i.e., whether participants constructed MD, SE, or other problems). We then report results pertaining to the relative numerosity of the sets (i.e., whether participants made the size of the common set larger than the other set).

Mathematical structure. The two judges constructed the equation necessary for solving each word problem. Based on these equations, they classified the problems into three main categories: mathematically direct (MD), semantic escape (SE), and other (O).

[6] The two exceptions were the common sets ‘‘doctors’’ and ‘‘priests,’’ each of which was expected to be the smaller set in its respective Asymmetric pair. The analyses were adjusted accordingly.

[7] In Experiment 1 we did not distinguish between MD division problems in which the structural roles of the paired sets were semantically consistent (e.g., tulips/vases) and inconsistent (e.g., vases/tulips). We kept track of such role assignments in Experiment 2.

1. *Mathematically Direct*. Problems were coded as MD if, as in the sample problem, the equation related the given sets directly by the required arithmetic operation ($a + b = c$ for addition; $a/b = c$ for division) and did not involve any other mathematical operation. In Experiment 1, we did not distinguish between problems involving the mathematically complementary operations of addition and subtraction ($a + b = c$ and $c - b = a$), or between problems involving the mathematically complementary operations of division and multiplication ($a/b = c$ and $bc = a$). An example of an addition MD problem constructed for the pair doctors–lawyers is: ‘‘If there are two doctors and three lawyers in a room, how many people are there altogether?’’ An example of a division MD problem constructed for the pair boys–teachers is: ‘‘Three teachers want to evenly divide a class of 60 boys. How many boys should go with each teacher?’’

Problems that related the sets directly by the requested operation but included further computation were coded separately as *complex MD*. This category enabled us to examine whether participants introduced variation into their problems that was unrelated to semantic alignments. The following addition problem constructed for the symmetric pair drummers–guitarists, for example, involved addition of the given sets and multiplication by a third set ($p \cdot (a + b)$): ‘‘In a contest there are three bands performing. If each band has only one drummer and two guitarists, how many people are participating in this contest in all?’’

2. *Semantic Escape*. We identified three types of SE problems: alignable operation, unrelated sets, and double violations. In alignable-operation SE problems, the sets were related by the semantically alignable operation instead of being related by the requested but semantically nonalignable operation: division instead of addition or addition instead of division. An example of an alignable-operation SE problem is a direct division problem (a/b) constructed in the Addition condition for the Asymmetric pair peaches–baskets: ‘‘Two baskets hold 30 peaches, how many peaches does 1 basket hold?’’; or a direct addition problem ($a + b$) constructed in the Division condition for the Symmetric pair peaches–plums: ‘‘If there is a basket with peaches and plums in it, and we know that the total number of pieces of fruit is 20, and that there are 5 peaches, how many plums must there be in the basket?’’ Note that the above examples achieve semantic alignments at the expense of including the requested arithmetic operation. Other alignable-operation SE problems satisfied this requirement by making the mathematical structure more complex. For example, a problem constructed for the Symmetric pair tulips–daffodils in the Division condition also involved addition of the two given sets of flowers ($(a + b)/p$): ‘‘Wilma planted 250 tulips and 250 daffodils and it took 20 days to plant them. How many flowers did she plant per day?’’

In unrelated-sets SE problems, the given sets were not related in the computation by any arithmetic operation. The following problem constructed for

TABLE 1
Percentages of MD, Complex MD, SE, and Other Problems Constructed in Addition and Division Conditions in Experiment 1

	MD	Complex MD	SE	Other
Addition				
Symmetric (N = 160)	82	6	8	4
Asymmetric (N = 160)	61	1	36	2
Division				
Symmetric (N = 150)	56	14	27	3
Asymmetric (N = 158)	79	8	9	4

the Asymmetric pair doctors–patients in the Addition condition is an example ($b_1 + b_2$): ''One doctor sees 5 patients on Monday and 6 on Wednesday. How many patients has she seen all together?'' This problem achieves semantic alignment at the expense of meeting the task requirement of including both sets in the computation (i.e., the number of doctors is irrelevant to the computation). Other unrelated-sets SE problems fulfilled this requirement via increased complexity: they consisted of two separate problems, one for each given set. For example, the following problem was constructed for the Symmetric pair peaches–plums in the Division condition ($\max\{a/p, b/q\}$): ''Every year Grandma's plum tree produces 45 less plums and her peach tree produces 110 less peaches. If she had 220 plums and 330 peaches this year, how long will it be before she has no produce?'' Another example is a problem constructed for the Symmetric pair guitarists–drummers in the Division condition (a/p; b/p): ''For 6 bands there are 30 guitarists and 12 drummers. How many of each are in each band?''

Finally, in double-violation SE problems, the computation involved only one of the given sets and a nonarithmetic operation (e.g., a^p).

3. *Other*. This category included problems that were unsolveable (e.g., jokes). For example, the following problem was constructed for the pair priests–parishioners in the Addition condition: ''One priest plus 2000 parishioners = one very stressed-out priest.''

Table 1 presents the percentage distribution of MD, complex MD, SE, and O problems, broken down by mathematical relation (Addition and Division) and semantic relation (Symmetric and Asymmetric). The percentages reported in Table 1 are based on a total of 320 problems constructed by participants in the Addition condition and 308 problems constructed by participants in the Division condition. The smaller number of problems in the Division condition reflects the fact that 8 of the 40 participants in this condition left a total of 12 blanks (i.e., did not construct a problem for at least one pair). Ten of these blanks occurred when the implied relation between object sets was nonalignable with division. Although leaving a blank could reflect an extreme type of escape strategy, we did not include blanks in our analyses. The data are collapsed across animate and inanimate pairs because a preliminary analysis did not reveal any significant effects of this factor.

Overall, a majority of the problems (69%) exhibited the MD structure ($a+b$ in the Addition condition; a/b in the Division condition). However, as predicted, problems were much more likely to have the MD structure when the semantic relation supported by the given sets was alignable with the requested mathematical operation. As shown in Table 1, the percentages of MD problems constructed for the alignable pairs, Symmetric Addition and Asymmetric Division, were 82 and 79%, respectively. By contrast, the percentages of MD problems constructed for the nonalignable pairs, Asymmetric Addition and Symmetric Division, were 61 and 56%, respectively. Semantic escapes showed the complementary pattern. Out of the 71 SE problems in the Addition condition, 82% were for Asymmetric pairs, whereas out of the 55 SE problems in the Division condition, 75% were for Symmetric pairs.

Semantic alignments exceeded any baseline tendency to construct problems that differ in their structure from the structure of the MD problems. This is reflected in the relative proportions of SE and complex MD problems constructed for the semantically alignable and nonalignable pairs. For alignable pairs, there was no difference between the proportions of SE and complex MD problems (8 and 7%, respectively). By contrast, for nonalignable pairs, SE problems were four times more frequent than complex MD problems (32 vs. 8%, respectively). It might also be argued that because of a task demand to introduce variety into the mathematical structures, participants would construct fewer MD problems for pairs that appeared later rather than earlier in the booklet. To test for this possibility, we compared the percentage of MD problems constructed in the first and second halves of each booklet and found no difference (69% in each half). There was still no difference when we compared the percentages of MD problems constructed in each half for Symmetric and Asymmetric pairs and within the Addition and Division conditions.

Because each participant constructed six to eight problems, one might ask if only a few participants were responsible for constructing the non-MD (i.e., SE and O) problems. This was not the case. Fifty-nine of the 80 participants (74%) constructed at least one non-MD problem, with a mean of 2.9 and a median of 2 non-MD problems constructed per participant in this group. To test for the relation between object symmetry and problem structure at the level of participants, we calculated the difference between the number of MD and non-MD problems each participant constructed for Symmetric and Asymmetric pairs. In the Addition condition, the mean difference was 2.6 for Symmetric pairs and .9 for Asymmetric pairs (Mann-Whitney $U = 1062$,

TABLE 2
Percentages of MD Problems Constructed in Addition and Division Conditions for Each Pair in Experiment 1

	Addition	Division
Symmetric		
boys, girls	89	61
doctors, lawyers	90	79
guitarists, drummers	78	63
priests, ministers	95	44
muffins, brownies	70	33
tulips, daffodils	89	47
crayons, markers	79	65
peaches, plums	95	61
Asymmetric		
boys, teachers	65	88
doctors, patients	80	84
guitarists, agents	74	74
priests, parishioners	72	79
muffins, trays	65	79
tulips, vases	65	79
crayons, boxes	35	90
peaches, baskets	40	89

Note. Each percentage based on 17–20 observations.

$p < .01$). In the Division condition, the mean difference was .4 for Symmetric pairs and 2.3 for Asymmetric pairs (Mann-Whitney $U = 474$, $p < .001$). These results confirm at the level of individual participants what the overall percentages of MD problems already showed: MD problems were more frequent when the requested mathematical operation and the semantic relation implied by the object sets were alignable rather than nonalignable.

Table 2 presents the percentages of MD problems generated for each of the pairs in the Addition and Division conditions. Although the magnitude of semantic alignments varied across items, for the Symmetric pairs (top panel of Table 2) the percentage of MD problems was always higher when the required operation was addition rather than division whereas, for the Asymmetric pairs (bottom panel of Table 2), the percentage of MD problems was always higher when the required operation was division rather than addition (except for the guitarists–agents pair, for which the percentages were equal). That is, the difference in percentage of MD problems for alignable versus nonalignable pairs was positive for all but one pair. The mean difference was 25%, the median difference was 19%, and only three of the 16 pairs yielded differences of less than 10%.

Set size. Out of 602 problems in which participants specified the size of both sets (a, b), we excluded 11 Asymmetric problems (4%) and 34 Symmetric problems (12%) in which the two sets were of equal size and only checked to see whether there was a differential preference for the common set to be the larger one. As predicted, the common set was the larger in 81% of the 296 remaining Asymmetric pairs (e.g., more muffins than trays) and in only 51% of the 261 remaining Symmetric pairs (e.g., more muffins than brownies).

As mentioned earlier, alignment of the asymmetric semantic relation between the paired object sets with the mathematically asymmetric relation between the dividend and the divisor constrains the structural roles each set plays (e.g., muffins/trays rather than trays/muffins). This, in turn, ensures that the quotient is larger than 1. One would predict a preference to construct problems that yield quotients greater than 1 from people's general tendency to avoid proper fractions (e.g., Fischbein, Deri, Nello, & Marino, 1985). Interestingly, however, the tendency of our participants to constrain the relative numerosity of the paired sets was not limited to division problems. In fact, the common set was generally larger irrespective of whether participants constructed division or addition problems (70 and 64%, respectively). Thus, even when participants constructed nonalignable MD problems, they preserved at least one aspect of semantic alignment by making, for example, muffins more numerous than trays: ‘‘If there are sixty muffins and eight trays, how many muffins and trays are there altogether?’’

Discussion

The results of Experiment 1 show that when adults with many years of mathematical schooling reason about arithmetic word problems they tend to align semantic relations that are afforded by pairs of object sets with mathematical relations between arguments of arithmetic operations. As predicted, participants were significantly more likely to relate a given pair of object sets by direct addition or direct division (i.e., to use the MD structure) when the semantic relation between the paired sets was alignable rather than nonalignable with the mathematical relation between the arguments of the target arithmetic operation. Even when they constructed nonalignable MD problems (e.g., problems that required adding muffins and trays), they heeded semantic content in assigning numerosities to the sets (e.g., more muffins than trays). That is, participants applied their formal mathematical knowledge in a way that was consistent with their world knowledge.

The construction of alignable-operation SE problems (e.g., involving division rather than addition of tulips and vases) falls directly out of our analysis of semantic alignments: When there is a structural mismatch between the semantic structure and the requested mathematical structure and a structural match with a competing mathematical structure, people will sometimes defy the task requirements by constructing a problem involving the alternative operation. Perhaps more interesting are those semantic escapes in which participants did not violate the task requirements, but rather achieved semantic alignments by constructing word problems that had a more complex mathematical structure than the MD problems (e.g., problems that involved three rather than two variables, such as [(p) roses + (a) tulips]/(b) vases. Such complex alignments seem to reflect investment of extra cognitive effort to achieve semantic alignments while also meeting the task requirements. Assuming this is so, complex semantic escapes suggest that participants believe that one should avoid meaningless application of arithmetic operations.

Although semantic alignments may reflect the belief that mathematical and semantic knowledge should be brought into correspondence, only 32% of the problems constructed for the nonalignable pairs were semantic escapes. Put differently, a substantial proportion (64%) of problems constructed for the nonalignable pairs in Experiment 1 fulfilled the task requirements in what appears to be semantically meaningless way (e.g., direct addition of (a) tulips and (b) vases).[8] This suggests that participants also believed the opposite: that in order to prove their mathematical ability, they should treat the paired sets as if they were arbitrary variables. We discuss these potentially conflicting beliefs about mathematics word problems in the context of Experiments 2 and 3.

EXPERIMENT 2

In Experiment 2 we examined the generality and robustness of the results obtained in Experiment 1 by (1) eliminating the sample problems and (2) including pairs of object sets that had a set–subset semantic structure (e.g., tulips–flowers; guitarists–musicians). In all other respects, Experiment 2 was a replication of Experiment 1.

Sample problems. In Experiment 1, participants received sample problems that had the MD structure and that were semantically alignable (i.e., adding cars and trucks; dividing books among shelves). Given that people have a strong tendency to rely on examples (e.g., Chi & Bassok, 1989; LeFevre & Dixon, 1986; Pirolli & Anderson, 1985; Ross, 1984; VanLehn, 1986; Zhu & Simon, 1987), each problem constructed for a semantically nonalignable pair in Experiment 1 probably reflected a conflict resolution between constructing a semantically nonalignable problem that was mathematically isomorphic to the sample problem and a semantically alignable problem with a nonisomorphic mathematical structure. Participants in Experiment 1 may thus have constructed either more MD or more semantically alignable problems for nonalignable pairs than they would have without the sample problem. In

[8] Whether or not these are truly meaningless problems depends on the context. For example, if you're deciding whether or not to get into the express lane at the supermarket, it makes sense to add up muffins and trays and toilet paper to figure out how many ''things'' you have. Because our coding scheme only captured the mathematical relations between the given sets, we do not know what proportion of the nonalignable MD problems had a potentially meaningful context.

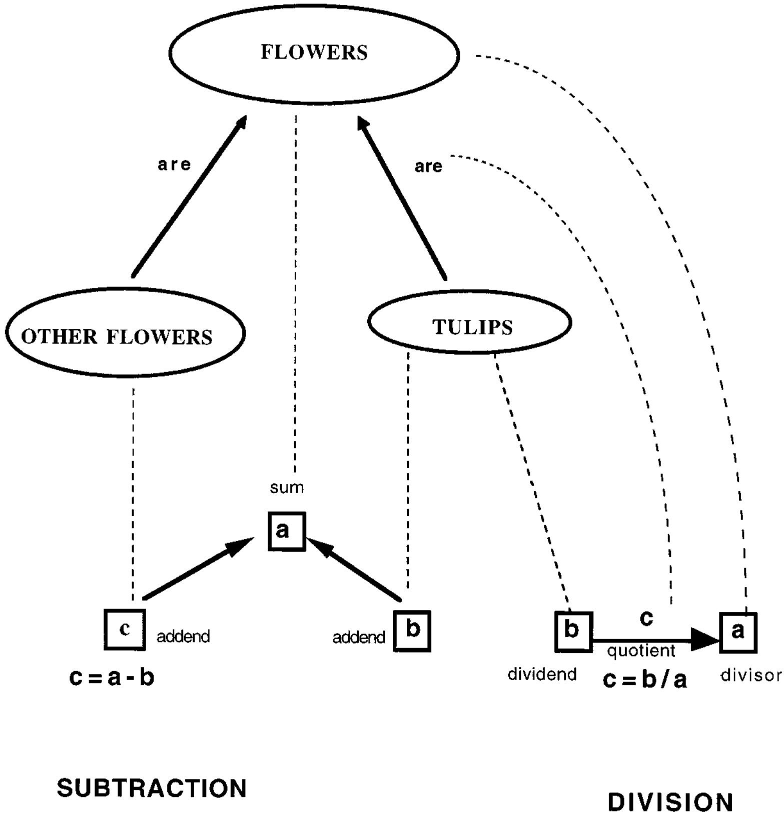

FIG. 2. Alternative semantic alignments for the Set–Subset semantic relation (tulips–flowers) with the mathematical operations of subtraction and division.

order to examine people's spontaneous tendency to construct MD and semantically alignable problems, in Experiment 2 we did not give participants a sample problem. If, as we believe, people's tendency to strive for semantic alignment was damped by the provision of an MD sample problem, then participants in Experiment 2 should construct fewer MD nonalignable problems than those in Experiment 1.

The Set–Subset relation. In order to further validate our analysis of semantic alignments for addition and division, in Experiment 2 we included pairs that support the Set–Subset semantic relation (e.g., tulips–flowers). The Set–Subset relation is an interesting intermediate case because it is an asymmetric semantic relation (e.g., all tulips are flowers, but some flowers are not tulips) that affords semantic alignment with the MD structure of both addition and division. Figure 2 presents a schematic representation of alignment between the Set–Subset relation and the MD addition and division structures for the pair tulips–flowers. Object sets are represented by ovals and the arguments of the addition and division operations are represented by squares.

As can be seen in Fig. 2, the Set-Subset relation affords an alignable MD solution to the Addition construction task if one subtracts the subset (a = tulips) from the set (b = flowers). In this alignment, c is the complementary subset of flowers that are not tulips. At the same time, the Set–Subset relation affords an alignable MD solution to the Division construction task. In this case, semantic considerations dictate that the subset, which is contained in the inclusive set, be aligned with the dividend (a) and the inclusive set with the divisor (b) rather than vice versa (a tulips/ b flowers = c proportion of tulips in the set of flowers).

We predicted that the dual status of the Set–Subset relation with respect to alignablity with the MD addition and division structures would lead to performance that falls between that for Symmetric sets (i.e., same taxonomic category), which are alignable only with addition, and Asymmetric sets (i.e., functionally asymmetric relation, such as CONTAIN), which are alignable only with division (see Fig. 1). Specifically, we predicted that the proportion of MD problems in the Addition condition would be highest for Symmetric pairs, intermediate for Set–Subset pairs, and lowest for Asymmetric pairs. Conversely, the proportion of MD problems in the Division condition should be highest for Asymmetric pairs, intermediate for Set–Subset pairs, and lowest for Symmetric pairs.

In our discussion of Experiment 1, we observed that participants' tendency to align the larger set with the dividend rather than with the divisor is consistent with fraction avoidance (Fischbein et al., 1985) as well as considerations of semantic alignment. Unlike functionally Asymmetric sets (e.g., contain), semantic alignment for the taxonomically asymmetric Set–Subset pairs dictates that the smaller rather than the larger set be aligned with the dividend (e.g., a tulips/b flowers, not vice versa). Because such semantic alignments result in proper fractions ($a/b < 1$), the Set–Subset pairs enabled us to examine whether semantic considerations are powerful enough to override fraction avoidance.

Method

Participants. Participants were 91 University of Chicago undergraduate students. As in Experiment 1, they were tested individually or in small groups and were paid for their participation.

Materials. We used 18 pairs of object sets: 6 Symmetric pairs, in which the sets came from the same taxonomic category (e.g., tulips–daffodils), 6 Asymmetric pairs, in which the sets bore a ''contain'' relation to one another (e.g., tulips–vases), and 6 Set–Subset pairs, in which the sets came from adjacent taxonomic levels with one set subsuming the other (e.g., tulips–flowers). As in Experiment 1, half of the pairs involved sets of people and the other half sets of inanimate objects. However, unlike in Experiment 1, the semantic relation in all Asymmetric pairs was ''contain.'' Hence, Asymmetric pairs involving people did not involve two sets of people, but rather people and locations (e.g., doctors–hospitals).

TABLE 3
Percentages of MD, Complex MD, SE, and Other Problems Constructed in Addition and Division Conditions in Experiment 2

	MD	Complex MD	SE	Other
Addition				
Symmetric (N = 92)	55	15	17	12
Set–Subset (N = 92)	36	8	43	13
Asymmetric (N = 92)	9	3	80	8
Division				
Symmetric (N = 90)	29	12	38	21
Set–Subset (N = 89)	48	18	18	16
Asymmetric (N = 89)	72	12	7	9

The 18 pairs of object sets were randomly divided into three equivalent construction booklets, each of which consisted of 6 different pairs: one Symmetric pair of people, one Symmetric pair of objects, one Asymmetric pair of people and locations, one Asymmetric pair of objects, one Set–Subset pair of people, and one Set–Subset pair of objects. Each common set appeared only once in each booklet, and each pair was typed at the top of a separate page. The pairs in each booklet were presented in one of two randomized orders. The 18 pairs appear in the leftmost column of Table 4. The cover page of the construction booklets was identical to that used in Experiment 1 except that it did not include a sample problem.

Procedure. The procedure was identical to that in Experiment 1, with 46 participants in the Addition and 45 in the Division condition.

Results

We slightly modified the coding scheme used in Experiment 1 to allow for tests of our predictions about Set–Subset pairs. Specifically, in problems involving the given sets we kept track of whether the sets were related by addition or subtraction in addition problems and recorded which set was assigned to the dividend role in division problems. Two independent judges coded the data with an average agreement of over 80%. As in Experiment 1, disagreements were resolved by discussion between the judges such that all data were included in the analysis.

Strategies. Table 3 shows the percentages of MD, complex MD, SE, and O problems constructed by participants in the Addition and Division conditions for each of the three semantic relations. These percentages are based on a total of 276 addition and 268 division problems. (Two pairs in the Division condition were left blank and were omitted from further analysis.) Because animacy did not have a systematic effect on the percentage of MD problems constructed for the Symmetric and Set–Subset pairs, results for people and objects were collapsed in the analysis.

As Table 3 shows, the distribution of MD problems both replicates the semantic alignment effects documented in Experiment 1 and validates our predictions about the dual status of the Set–Subset relation. Specifically, the relative frequency of MD problems in the Addition condition was highest for Symmetric pairs (55%), intermediate for Set–Subset pairs (36%), and lowest for Asymmetric pairs (9%). The reverse pattern held in the Division condition: 72, 48, and 29% for the Asymmetric, Set–Subset, and Symmetric pairs, respectively. As in Experiment 1, SE problems showed the complementary pattern. Considering Symmetric and Asymmetric pairs only, 82% of the 90 SE problems in the Addition condition were for Asymmetric pairs, and 85% of the 40 SE problems in the Division conditions were for Symmetric pairs. Also, as in Experiment 1, participants constructed comparable frequencies of SE and complex MD problems for alignable pairs (12 and 14%, respectively), but constructed more than seven times as many SE as complex MD problems for nonalignable pairs (59 vs. 8%, respectively).

There were some interesting differences between the results in Experiments 1 and 2. First, only 41% of the problems exhibited the MD structure compared with 69% in Experiment 1. (The difference is the same if one omits Set–Subset pairs, which were not used in Experiment 1.) Second, semantic alignment effects in Experiment 2 were substantially larger than in Experiment 1. If one considers only Symmetric and Asymmetric pairs, the difference between the frequencies of MD problems constructed for alignable and nonalignable pairs was 45% in Experiment 2 and only 25% in Experiment 1. That is, when participants were not provided with alignable MD sample problems, they constructed fewer MD problems and more SE problems.

The third interesting difference between Experiments 1 and 2 concerns the relative frequencies of MD addition and division problems. Again, considering only Symmetric and Asymmetric pairs, participants constructed significantly fewer MD addition than division problems in Experiment 2 (32 vs. 51%, respectively), but constructed similar proportions of MD addition and division problems in Experiment 1 (69 and 70%, respectively). Whatever the reason for the differential decrease in MD addition and division problems from Experiment 1 to Experiment 2, it did not lead to corresponding differences in the magnitude of semantic alignment effects in the Addition and Division conditions. As in Experiment 1, the difference between the frequencies of MD problems constructed for Symmetric and Asymmetric pairs was similar in the Addition (46%) and Division (43%) conditions. There was a slight order effect in Experiment 2, with a higher percentage of MD problems generated in the first than in the second half of the booklets (45 vs. 38%, respectively, including Set–Subset pairs), but it cannot account for the large semantic alignment effects found in this experiment.

As in Experiment 1, the non-MD problems in this experiment cannot be

TABLE 4
Percentages of MD Problems Constructed in Addition and Division Conditions for Each Pair in Experiment 2

	Addition	Division
Symmetric		
boys, girls	67	31
doctors, lawyers	50	29
guitarists, drummers	53	33
cantaloupes, honeydew melons	60	31
washers, refrigerators	38	29
tulips, daffodils	67	20
Set–Subset		
boys, children	53	53
doctors, professionals	47	53
guitarists, musicians	7	31
cantaloupes, melons	38	43
washers, appliances	47	53
tulips, flowers	25	57
Asymmetric		
boys, classrooms	0	50
doctors, hospitals	7	81
guitarists, bands	6	79
cantaloupes, crates	20	79
washers, laundromats	7	73
tulips, vases	13	69

Note. Each percentage based on 14–16 observations.

attributed to a small number of participants. Every participant constructed a minimum of two non-MD problems. The mean number of such problems constructed per participant (out of 6 possible problems) was 3.9; the median was 4. The mean difference between the number of MD and non-MD problems each participant constructed in the Addition condition was .2 for the Symmetric pairs and −1.65 for the Asymmetric pairs (Mann-Whitney $U = 1667$, $p < .001$); the mean difference for the Set–Subset pairs was small (.6) and close to that for Symmetric pairs. In the Division condition, the mean difference was −.9 for the Symmetric pairs and .8 for the Asymmetric pairs (Mann-Whitney $U = 414$, $p < .001$); the difference for the Set–Subset pairs (−.1) fell between those for Symmetric and Asymmetric pairs. To summarize, individual participants in both conditions were more likely to construct MD problems when the requested operation was alignable with the semantic relation between sets, and the Set-Subset relation led to intermediate performance.

Table 4 presents the percentages of MD problems generated for each pair in the Addition and Division conditions. As can be seen in Table 4, the proportion of MD problems was higher in the Addition than in the Division condition for all six Symmetric pairs and higher in the Division than in the

Addition condition for all six Asymmetric pairs. The results for Set–Subset pairs were intermediate: For two of them the proportion of MD problems was higher in the Division condition, and for the other four there was practically no difference between the Addition and Division conditions. Consistent with the relative frequencies of MD problems, in Experiment 2 the pair effects were larger than in Experiment 1. Whereas in Experiment 1 the differences for 8 of the 16 Symmetric and Asymmetric pairs were under 20%, in Experiment 2 the difference for only one out of the 12 Symmetric and Asymmetric pairs was under 20%.

Addition vs. subtraction. We predicted that subtraction would be more common for Set–Subset pairs than for the other pairs. Indeed, 50% of all problems that related the Set–Subset pairs by an additive structure involved subtraction, whereas none of the problems that related Symmetric or Asymmetric sets by an additive structure involved subtraction.

Set size. Out of the 469 problems in which numerical values were assigned to both sets (a, b), we excluded one Asymmetric (under 1%), 21 Symmetric (4%), and 11 Set–Subset (2%) problems in which the two sets were of equal size, and only checked to see whether there was a differential preference for the common set to be the larger one. As in Experiment 1, the common set was the larger one in 94% of the 163 Asymmetric pairs (e.g., more tulips than vases) and in only 58% of the 136 Symmetric pairs (e.g., more tulips than daffodils). Moreover, and consistent with participants' attempts to achieve semantic alignments, the common set was the larger one in only 11% of the 137 Set–Subset pairs (e.g., more tulips than flowers). This pattern of semantic alignments held in both the addition and the division problems.

Importantly, and consistent with the differences in the relative size of the common set in the three types of pairs, the percentages of proper fractions (i.e., $a/b < 1$) in problems that related the sets with the operation of division was highest for Set–Subset pairs (81% of 47 problems), intermediate for Symmetric pairs (41% of 29 problems), and lowest for Asymmetric pairs (5% of 102 problems). In other words, considerations of semantic alignment were strong enough to overcome fraction avoidance.

Discussion

Experiment 2 replicated the two main results of Experiment 1: (1) the proportion of MD problems that participants constructed was significantly higher when the semantic relation between the given sets was alignable with the mathematical relation than when it was not and (2) participants assigned numerosities to the sets in accordance with their semantic knowledge. The results of Experiment 2 also extend the findings of Experiment 1. First, they show that semantic alignments cannot be explained by people's tendency to reproduce the structure of sample problems. Second, they validate our analysis of semantic alignment for addition and division by demonstrating semantic alignments for the Set–Subset semantic relation. Below we discuss these two extensions in greater detail.

Sample problems. People constructed a smaller proportion of MD problems in Experiment 2 than in Experiment 1. Although the two experiments differed in several other respects (type and number of object pairs), the most likely explanation for the higher proportion of MD problems in Experiment 1 is that participants tried to construct problems that had a mathematical structure isomorphic to that of the sample problem. Without a sample problem, participants in Experiment 2 had more freedom to construct non-MD problems. At the same time, the lack of a sample problem in Experiment 2 did not decrease the frequency of semantic alignments, indicating that semantic alignments cannot be explained by people's tendency to mimic examples. Indeed, the proportion of semantic-escape problems constructed for nonalignable pairs was much higher in Experiment 2 (59%) than in Experiment 1 (32%). This pattern of results suggests that adults with extensive experience in solving mathematical word problems assume that it is more important to reproduce the mathematical structure of examples than to reproduce their semantic alignment. (We return to this point in discussing the results of Experiment 3.) Freed from having to choose between reproducing the mathematical structure or the semantic alignment of a sample problem (Experiment 1), participants in Experiment 2 constructed more semantically alignable SE problems for nonalignable pairs.

Alignments with the Set–Subset relation. The frequency of MD problems constructed for Set–Subset pairs fell between the frequencies of MD problems constructed for Symmetric and Asymmetric pairs in both the Addition and the Division conditions. These results are consistent with our analysis of semantic alignments, according to which the taxonomically asymmetric semantic relation in the Set–Subset pairs can be aligned with the asymmetric mathematical relation between the sum and addend (subtraction) and dividend and divisor (division).

Alignment of the Set–Subset relation with the additive structure demands subtraction rather than addition, while alignment with division demands placing the smaller rather than the larger set in the numerator (i.e., constructing proper fractions). It is possible that these constraints conflicted with participants' beliefs about the task requirement for constructing addition and division word problems, thereby contributing to the intermediate frequency of MD problems constructed for the Set–Subset pairs. Specifically, the fact that participants did not construct subtraction problems for either Symmetric or Asymmetric pairs suggests that they did not interpret the instructions to construct addition problems as a permission to relate the sets by the mathematically equivalent operation of subtraction. Hence, in trying to avoid subtraction, participants may have been driven to construct non-MD ''true'' addition problems for the Set–Subset pairs (50% of the problems in which

Set–Subset pairs were aligned with the addition structure). Similarly, fraction avoidance may have led some participants to construct non-MD ‘‘true’’ division problems in which the result of division was larger than 1 (11% of the problems in which Set–Subset pairs were aligned with the division structure).

In general, there appears to be a complex interplay among people’s (1) tendency to align semantic and mathematical structures, (2) beliefs about the relative importance of reproducing the mathematical structure versus reproducing the mapping between mathematical and semantic structures sample problems’, and (3) understanding of task requirements and of mathematical concepts and conventions (e.g., the difference between ‘‘addition’’ and ‘‘subtraction’’ problems). This interplay depends critically on people’s mathematical training.

EXPERIMENT 3

It is likely that the pattern of semantic alignments found in the construction task partly reflects a learning history in which participants encountered more semantically alignable than nonalignable arithmetic word problems. An analysis of chapter review and test word problems for grades 1 through 8 in a popular textbook series (Eicholz, O’Daffer, Fleener, Charles, Young, & Barnett, 1987), which was in use when the participants in Experiments 1 and 2 attended elementary school, provides strong support for this conjecture. Using a coding scheme similar to that employed in the first two experiments, we analyzed a total of 421 word problems.[9] We began by parsing them into subproblems ($N = 507$), each involving a single arithmetic operation that related either two sets of objects (e.g., frogs and ponds) or two object measurements (e.g., lengths of two rods). We then examined the relative proportion of Symmetric entities (e.g., red and blue marbles) and Asymmetric entities (e.g., cookies and jars) that had to be related by either addition or division, treating Set–Subset pairs as Asymmetric.

In the overwhelming majority of problems, the semantic relation between the entities could be aligned with the arithmetic operation. Less than 3% of the 258 pairs that entailed addition involved Asymmetric entities. (All of these problems related a subset and a set by subtraction and can therefore be considered semantically aligned.) Similarly, less than 6% of 249 pairs that entailed division involved Symmetric entities. Thus, it appears that mathematics educators (i.e., experts in this domain) very rarely expose their students to nonalignable word problems, although this conclusion may not apply to other aspects of semantic compatibility (e.g., Greer, 1993).

[9] This number represents all review and test word problems that involved arithmetic operations, with the exception of 19 problems that involved scalar changes in a single variable (e.g., a metal rod that shrinks in length) and 28 probability problems.

This textbook analysis validates our initial analysis of semantic alignment and is consistent with the pattern of semantic alignments in the construction task. However, whereas nonalignable operations in the textbook problems were very rare (less than 5%), nonalignable MD problems for the Symmetric and Asymmetric pairs in Experiments 1 (64%) and 2 (26%) were still common. Our experimental finding is consistent with claims that mathematics teachers emphasize the abstractness of mathematics more than its modeling role (e.g., De Corte & Verschaffel, 1985; Greer, 1993; Freudenthal, 1991; Hatano, 1988; Schoenfeld, 1991). That is, although the participants in our experiments were exposed to very few nonalignable problems in textbooks, they may have behaved as they were taught to do in school: to heed the mathematical rather than the semantic structure of word problems when the two types of structure conflict (but again, see footnote 8). This speculation is consistent with our suggestion that the sample problems were responsible for the high proportion of potentially meaningless MD problems generated for nonalignable pairs in Experiment 1 (e.g., muffins + trays = things).

Experiment 3 was designed to examine participants' beliefs about the prevalence of semantically alignable arithmetic word problems in textbooks. Specifically, participants were asked to judge the likelihood that Symmetric and Asymmetric pairs of entities encountered in arithmetic word problems would have to be related by addition or division. Based on the textbook analysis and participants' performance on the construction task, we predicted that participants would give higher likelihood ratings to alignable than to nonalignable operations.

Method

Participants. Participants were 27 University of Chicago undergraduate students drawn from the same population used in Experiments 1 and 2. As in the previous experiments, they were recruited to participate in a problem-solving experiment, tested individually or in small groups, and paid for their participation.

Materials. We transformed the two construction booklets used in Experiment 1 into two questionnaires, each consisting of four Symmetric pairs (two pairs of people and two pairs of objects) and four Asymmetric pairs (two pairs of people and two pairs of objects). As in Experiment 1, each questionnaire had two versions that reversed the left/right position of the sets (e.g., tulips–daffodils; daffodils–tulips). Under each pair appeared two seven-point rating scales, one for addition and the other for division. Participants were asked to imagine that the pairs came from arithmetic word problems and to indicate on the corresponding scale for each pair how likely the entities were to be added and how likely they were to be divided. The endpoints of each scale were labeled ''very unlikely'' (1) and ''very likely'' (7).

Procedure. Participants were randomly assigned to receive one of the two equivalent questionnaires. They worked at their own pace without any intervention from the experimenter.

Results

Ratings obtained from the two questionnaires were combined and collapsed across Symmetric and Asymmetric pairs of objects and people following a preliminary analysis that did not yield either main effects or interactions

TABLE 5
Mean Addition and Division Likelihood Ratings for Symmetric and Asymmetric Pairs in Experiment 3

	Addition	Division
Symmetric		
boys, girls	5.64	4.29
doctors, lawyers	4.43	3.43
guitarists, drummers	5.85	1.92
priests, ministers	6.23	2.25
muffins, brownies	5.29	2.92
tulips, daffodils	5.14	3.15
crayons, markers	6.00	1.85
peaches, plums	6.31	1.92
Asymmetric		
boys, teachers	4.00	4.92
doctors, patients	3.46	5.46
guitarists, agents	4.36	4.71
priests, parishioners	4.43	4.50
muffins, trays	2.92	5.31
tulips, vases	3.15	5.08
crayons, boxes	4.36	4.93
peaches, baskets	4.50	4.07

Note. Each rating based on 12–14 observations.

due to these factors. The combined data were analyzed using a repeated measures ANOVA with Symmetry (2 levels) and Operation (2 levels) as factors. Overall, addition was rated to be a somewhat more likely operation than division (means 4.75 and 3.81, respectively; $F(1, 26) = 4.40$, $MS_e = 4.96$, $p < .04$), possibly because it is a more commonly encountered operation. There was no main effect of symmetry. As predicted, there was a very strong interaction between Symmetry and Operation ($F(1, 26) = 29.00$, $MS_e = 3.60$, $p < .000$). Symmetric pairs were rated more likely to be related by the operation of addition than by the operation of division (means 5.57 and 2.67, respectively), whereas Asymmetric pairs were rated less likely to be related by the operation of addition than by the operation of division (means 3.93 and 4.95, respectively).

All participants contributed to the overall differences in likelihood ratings. Out of the 216 pairs judged (27 participants $\times$ 8 pairs per questionnaire), only 20 were ever assigned identical ratings for addition and division (10 Symmetric and 10 Asymmetric). These identical ratings were given by 13 participants with no more than three identical ratings per participant in this group. Moreover, the Symmetry-by-Operation effect held true for 15 of the 16 pairs. Table 5 presents the mean likelihood ratings for the operations of addition and division for each of the 16 pairs.

As can be seen by comparing the top and bottom panels of Table 5, for

Symmetric pairs (top panel) the operation of addition always received higher likelihood ratings than the operation of division. For Asymmetric pairs (bottom panel), the operation of division received higher likelihood ratings than the operation of addition for 7 of the 8 pairs. *T* tests were performed on the differences between each participant's mean addition and division ratings separately for Symmetric and Asymmetric pairs. This analysis showed that ratings for the two operations were significantly different for Symmetric pairs (mean difference of 2.86, $t = 6.40$, df = 26, $p < .001$) but did not approach significance for Asymmetric pairs (mean difference of −0.95, $t = -1.41$, df = 26, $p = .167$).

Interestingly, the ratings for nonalignable operations were not extremely low. As can be seen in Table 5, only 3 of the 16 nonalignable operations received ratings under 2 (which was labeled ''unlikely''). To the extent that the sample of problems in our textbook analysis is representative of those participants encountered in school, the likelihood judgments may reflect a biased belief that nonalignable textbook problems are more common than they actually are.

Discussion

The textbook analysis strongly suggests that mathematics educators treat semantic alignment as a standard of correct performance, although they may not emphasize it explicitly. The results of our experiments indicate that people appreciate the importance of aligning mathematical and semantic structure at least to some degree. The likelihood judgments verified our speculation that people with extensive experience in solving word problems are sensitive to semantic alignment in problems presented to them by others. However, the results of Experiment 3 also suggest that participants underestimated the actual frequency with which mathematical and semantic structures are aligned in textbook problems.

GENERAL DISCUSSION

We have shown that when intelligent adults apply their well-learned abstract knowledge of arithmetic to concrete entities, they tend to do so in ways that are consistent with their semantic knowledge. Specifically, in Experiments 1 and 2, we showed that when undergraduate students construct arithmetic word problems they (1) infer a semantic relation by which a given pair of object sets is likely to be related (i.e., use content to infer semantic structure) and (2) align the inferred semantic relation with the mathematical relation between the arguments of an arithmetic operation (i.e., align the semantic and mathematical structures). These semantic alignments were reflected both in the mathematical operations by which participants related the entities in the problems they constructed (e.g., they divided tulips by vases rather than adding them) and in the relative numerical values they assigned to the sets (e.g., more tulips than vases). In Experiment 3, we showed that undergraduate students rightly expect that word problems presented to them by mathematics educators will exhibit similar semantic alignments.

The semantic alignments uncovered here suggest that the structural alignment approach to analogical transfer can be extended to help us understand selective and sensible use of abstract formal knowledge—the process by which people ensure that they add apples and oranges but do not add apples and baskets. At the same time, semantic alignments demonstrate the importance of the interpretive component of analogical transfer. They show that people use content (e.g., object attributes) to understand the structure of problem situations and to decide which formal rules (e.g., addition or division) can assist them in reasoning about the problem solutions.

There are many intriguing questions about the mechanism that mediates semantic alignments that the present studies cannot address. In particular, we do not know whether and to what extent semantic alignments are automatic or strategic. For the stimuli used in our studies, where people's responses were clearly affected by schooling, it is reasonable to assume both an automatic and a strategic component. For example, it might be argued that semantic alignments reflect automatic responses built up over a history of selective associations between arithmetic operations and semantically alignable situations (Rothkopf & Dashen, 1995). However, this account cannot by itself explain why teachers select alignable rather than nonalignable examples or why a failure to suppress powerful associations between semantic and mathematical structures leads to construction of more complex alignable problems. In the next section we discuss how schooling might enhance a ''belief system'' (Schoenfeld, 1985) that, at least for more competent students, may encourage strategic attempts to achieve semantic alignments.

Using Formal Rules as Modeling Tools

A common instructional practice in formal domains is to present students with examples of real-life situations that connect formal concepts and rules to students' experience. By selecting semantically alignable word problems (e.g., a division problem in which there are more flowers than vases rather than vice versa), mathematics educators encourage students to align their semantic and formal knowledge. This practice might sometimes lead students to misinterpret the meaning and the generality of the formal rules (Bassok & Olseth, 1995; Bassok et al., 1995; Fischbein et al., 1985; Goswami, 1992; Nesher, 1989). Nevertheless, instruction that combines explicit explanations of formal concepts with extensive exposure to a variety of word problems enables many students to achieve a level of abstraction at which they can reliably apply mathematical concepts to cases of many distinct semantic schemas (e.g., Bassok & Holyoak, 1993; Carpenter & Moser, 1983; Kintsch & Greeno, 1985; Nesher, Greeno, & Riley, 1982).

One could argue that the process of inferring formal mathematical structure from content may be facilitated by exposure to both semantically alignable and nonalignable examples. However, educators' aim of making formal rules meaningful to students discourages them from using semantically nonalignable word problems as examples. Importantly, teaching students a set of abstract formal rules is not the only purpose of mathematical training. Educators also aim to impart formal knowledge that can assist students in reasoning about real-life situations—to provide them with concepts and rules that can be used as *modeling tools*. Accordingly, they present students with semantically alignable problems not only to exemplify abstract concepts and rules but also to demonstrate potential domains of application (for an insightful discussion of these two distinct uses of word problems in mathematics education, see Nesher, 1989). In particular, because the nature of the entities that instantiate the variable roles is extremely important in the modeling context, teachers encourage students not to apply formal tools in nonsensical ways (e.g., not to add speed to distance) and to ensure that their solutions make real-world sense (e.g., that their solution does not entail that 4.12 buses be rented for a school trip; Silver, Shapiro, & Deutsch, 1993).

The fact that many teachers value the modeling role of mathematics helps to explain why participants in Experiments 1 and 2 were ready to invest cognitive effort to construct semantically alignable complex problems. At the same time, the relatively high proportion of potentially meaningless MD problems participants constructed suggests that mathematics teachers succeed in conveying the abstract nature of mathematics but fail to convey the modeling role of mathematics (e.g., Verschaffel, De Corte, & Lasure, 1994). Some students find the message that abstract formal rules can and should be used as modeling tools confusing. For example, a teacher might say that the operation of division can be applied to any two arbitrary variables (X and Y) and that the rules of mathematics do not dictate whether X should be in the numerator and Y in the denominator or vice versa. At the same time, she might claim that the correct way to express the solution in a problem involving a group of children sharing a pizza is ''pizzas/children'' and not ''children/pizzas.'' Such apparently contradictory statements lead some students to give up on their attempts to understand the relation between content and structure in word problems, even though they achieve reasonable alignments when solving realistic problems (Nunes, Schliemann, & Carraher, 1993). Hence, they often mechanically translate problem texts into equations, using syntactic cues and key words to decide which abstract solution procedure they should apply (e.g., Nesher & Teubal, 1974).

Unlike other content effects, sensitivity to semantic distinctions implied by problem cover stories is more common in students who have good rather than poor mathematical understanding (e.g., Greer, 1993; Hinsley, Hayes, & Simon, 1977; Nathan, Kintsch, & Young, 1992; Paige & Simon, 1966). The content effects documented in our experiments rest on the ability to select a mathematical structure that can be aligned with the semantic structure implied by content. Hence, one would predict that children who are in the process of learning arithmetic, especially those with poor mathematical understanding, would construct more arbitrary MD problems for semantically nonalignable pairs than the mathematically sophisticated participants in our experiments. That is, they would exhibit smaller rather than larger content effects.

Interestingly, Gelman and Gallistel (1978) arrived at a similar conclusion in their analysis of children's preschool counting competence. Whereas ''adults . . . know full well that we cannot add apples and oranges'' (p. 216), children initially tend to count heterogeneous arrays of elements, disregarding differences in color, size, or type (see again footnote 1). Gelman and Gallistel speculate that children ignore differences in object attributes (e.g., differences between flowers and chairs) because they can refer to them by the same generic term ''things.'' Only later do they learn to apply semantic distinctions to decide which things can and cannot be counted together. That is, according to Gelman and Gallistel, some semantic distinctions (e.g., taxonomic organization of conceptual knowledge) may develop later than some basic mathematical distinctions. Their argument complements our claim because semantic alignments demand that people possess both types of relevant knowledge, semantic and formal. Whether acquisition of abstract, general-purpose procedures precedes or follows acquisition of semantic knowledge, the ability to use such procedures as useful modeling tools appears to be a significant intellectual achievement.

Selection of Modeling Tools

Our analysis of semantic alignments for addition and division may explain why teachers select semantically alignable rather than nonalignable problems and how students decide which pairs of object sets are alignable or nonalignable with direct addition and division. Importantly, the distinction between semantically symmetric and asymmetric object sets crosses the boundaries between quite general models or schemas that serve to exemplify addition and division (e.g., Greer, 1992). Judging from our textbook analysis, the participants in our experiments probably encountered significantly more instances of division problems that involved asymmetric functional relations, such as CONTAIN (crayons, boxes) or SERVE (doctors, patients), than of purely proportional relations between semantically symmetric members from the same taxonomic category (e.g., peaches/plums; priests/ministers). At the same time, they probably encountered significantly more instances of addition problems that involved symmetric rather than asymmetric object sets.

The psychological validity of the distinction between symmetric and asymmetric sets has received further support from recent studies showing that it affects people's performance on other cognitive tasks (Bassok, 1997; Wisniewski, 1995; Wisniewski & Bassok, 1996). For example, Wisniewski (1996) found that pairs of nouns that correspond to our Symmetric and

Asymmetric sets induce two qualitatively different mechanisms of conceptual combination. When combining nouns from the same taxonomic category, such as ''skunk–squirrel,'' people tend to substitute an attribute of one noun with an attribute of the other noun (e.g., a bad-smelling squirrel). By contrast, when combining functionally related nouns, such as ''rabbit–box,'' people tend to form functional conceptual combinations (e.g., a box for holding rabbits).

The distinction between symmetric and asymmetric sets needs further refinement before it can account for cases in which, for example, object sets from the same taxonomic category are also related by a functionally asymmetric relation (e.g., cats–dogs, cars–tow trucks), or in which a given pair of object sets implies more than one semantic relation (e.g., people either eat or bake pizzas, and therefore are either more or less numerous than pizzas). Moreover, this distinction is obviously not the only one that mediates selective application of mathematical rules. For example, Bassok and Olseth (1995) found that the distinction between continuous and discrete change (e.g., ice deliveries to a restaurant vs. ice melting off a glacier, respectively) mediated analogical transfer between arithmetic progressions (discrete) and linear functions (continuous). Future research could inform both psychologists and mathematics educators by identifying semantic distinctions that guide, via a mechanism of structural alignment, sensible application of formal rules.

Although in the present research we focused on application of arithmetic knowledge, the discrepancy between formal and semantic knowledge and the challenge of aligning the two is not limited to arithmetic. For example, there is substantial evidence that people who reason about problem situations that have the formal logical structure of material implication (if *p* then *q*) are guided by the rules of various pragmatic and social schemas (e.g., Cheng & Holyoak, 1985, 1989; Cheng, Holyoak, Nisbett, & Oliver, 1986; Cosmides, 1989; Cosmides & Tooby, 1994; Cummins, 1995; Gigerenzer & Hug, 1992). In particular, according to Cheng and Holyoak (1985), people apply different reasoning rules to formally isomorphic statements such as ''if there are clouds, then it rains'' and ''if you drink beer, then you must be at least 21 years old.'' In the first case, they seem to be guided by knowledge that clouds are a necessary albeit insufficient cause for rain (e.g., causation schema), whereas in the second they are guided by knowledge that drinking age is established by a law that might be disobeyed (e.g., permission schema).

As in the case of arithmetic operations, the rules of formal logic are consistent with the rules of some schemas and inconsistent with others. For example, modus tollens (if *not q* then *not p*) is consistent with the rules of the permission schema (e.g., if you are under 21 (*not q*), then you are not permitted to drink alcohol (*not p*)), but sometimes conflicts with the rules of the causation schema (e.g., if it doesn't rain (*not q*), it does not necessarily follow that there are no clouds (*not p*)). Cheng et al. (1986) found that college students who received training in application of modus tollens nevertheless committed logical errors on conditional statements that induced incompatible schemas, much as the participants in our experiments sometimes did not fulfill the task requirements in order to escape having to construct semantically nonalignable problems. Cheng et al. (1986) argued convincingly that adherence to semantic and pragmatic constraints (i.e., content effects) protects people from arbitrary and anomalous conclusions.

Consistent with the gist of their and similar arguments, our results strongly suggest that when application of formal rules conflicts with people's semantic and pragmatic knowledge, people who have good understanding of formal rules may prefer arriving at logically invalid but reasonable conclusions to arriving at valid but anomalous conclusions. In general, semantic alignments can be viewed as a mechanism by which people exploit their rich and highly organized world knowledge to ensure sensible application of abstract concepts and rules. In particular, because different stimuli ''afford'' different physical and mental operations (Greeno, Moore, & Smith, 1993), semantic alignments allow people to find the best fit between their processing tools (e.g., addition) and the constraints implied by the stimuli they encounter (e.g., apples and oranges).

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