GENERALIZED THEORY OF MICROPOLAR-FRACTIONAL-ORDERED THERMOELASTICITY WITH TWO-TEMPERATURE

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ABSTRACT

In this paper, a new theory of generalized micropolar thermoelasticity is derived by using fractional calculus. The generalized heat conduction equation in micropolar thermoelasticity has been modified with two distinct temperatures, conductive temperature and thermodynamic temperature by fractional calculus which depends upon the idea of the Riemann- Liouville fractional integral operators. A uniqueness theorem of this model is proved. A variational principle and a reciprocity theorem are derived.

Keywords: generalized micropolar thermoelasticity, fractional calculus, two temperature theory.

1 INTRODUCTION

It is well established that the thermoelasticity theory is a fusion of the theory of heat conduction and the theory of elasticity. In nineteenth century, the uncoupled thermoelasticity was introduced by Duhamel (Duhamel 1837) and Neumann (Neumann 1885). In the year 1956, Biot (Biot 1956) advocated the coupled thermoelasticity. The generalization of the earlier stated thermoelasticity theories have been developed with the aim of removing the paradox, inherent in the classical theory of thermoelasticity, that thermal signal propagates with an infinite speed. The extended thermoelasticity theory, with one relaxation time, was proposed by Lord and Shulman (Lord and Shulman 1967) and the temperature dependent thermoelasticity, with two relaxation times, was introduced by Green and Lindsay (Green and Lindsay 1972). These non-classical theories are termed as generalized theory of thermoelasticity. In both of these theories the governing equations are of hyperbolic type and eliminate the paradox of infinite speed of thermal signals. In 1980, Dhaliwal and Sherief (Dhaliwal and Sherief 1980) extended the generalized theory for anisotropic media. Later on, during the year 1991-1993 Green and Naghdi (Green and Naghdi 1991; Green and Naghdi 1992; Green and Naghdi 1993) introduced a new theory of thermoelasticity and divide their theory into three parts, referred as types I, II and III. In an extensive review work on the development of the subject of generalized/ hyperbolic thermoelasticity till 1998 is available in the review article of Chandrasekharai (Chandrasekharai 1998).
The theory of heat conduction on a deformable body, formulated by Chen and Gurtin (Chen and Gurtin 1968) and Chen et al. (Chen et al. 1968; Chen et al. 1969), depends on two temperatures: conductive temperature $\phi$ and thermodynamic temperature $\theta$. The difference between these two temperatures is proportional to the heat supply and in absence of heat supply these two temperatures are identical for time-independent situation. In the time-dependent case these two temperatures are in general different, regardless of heat supply. The uniqueness and reciprocity theorems in two temperature thermoelasticity theory for homogeneous, isotropic solids were done by Iesan (Iesan 1970). In recent years, the two temperature thermoelasticity theory (2TT) is revisited once again. The theory of two temperature generalized thermoelasticity with uniqueness theorem introduced by Youssef (Youssef 2006), propagation of harmonic plane waves under two temperature theory by Puri and Jordan (Puri and Jordan 2006), effects of thermal relaxation time on plane wave propagation under 2TT by Kumar and Mukhopadhyay (Kumar and Mukhopadhyay 2010).

The linear theory of micropolar thermoelasticity was developed by extending the theory of micropolar elasticity including thermal effect by Eringen (Eringen 1966; Eringen 1999) and Nowacki (Nowacki 1970; Nowacki 1977). Minagawa et al. (Minagawa et al. 1981) discussed on plane harmonic wave propagation in a cubic micropolar medium. Kumar and Rani (Kumar and Rani 2003) studied the response of time harmonic sources in a thermally conducting cubic crystal, the Mechanical/thermal sources in a micropolar thermoelastic medium with cubic symmetry were investigated by Kumar and Aliawalia (Kumar and Aliawalia 2007). In the year 2006, Kumar and Aliawalia (Kumar and Aliawalia 2006) studied on deformation due to time harmonic sources in micropolar thermoelastic medium with two relaxation times. Electro-magneto-thermoelastic plane waves in micropolar solids’ involving two temperatures was discussed by Ezzat et al. (Ezzat et al. 2010).

Fractional calculus has been used successfully to modify many existing model of physical process. In the formulation of tautochrone problem, Abel applied fractional calculus to solve the integral equation and that was the first application of fractional derivatives. In the second half of nineteenth century, Caputo (Caputo 1967), Caputo and Mainardi (Caputo and Mainardi 1971) found an agreement between the experimental results with theoretical one when using fractional derivatives for the description of viscoelastic materials.

The current manuscript is an attempt to combine the previous results with generalized theory of micropolar-fractional thermoelasticity with two-temperatures.

2 DERIVATION OF GOVERNING EQUATIONS

The Riemann-Liouville fractional integral is introduced as a natural generalization of the convolution type integral (Miller and Ross 1993; Podlubny 1999; Povstenko 2005; Povstenko 2009)

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0) \quad (2.1)$$

The Laplace transform rule for this fractional integral is
The Riemann-Liouville derivative of fractional order \( \alpha \) is defined as the left-inverse of the fractional integral \( I^\alpha \) as

\[
D_{RL}^\alpha f(t) = D^\alpha I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad n-1 < \alpha < n
\]  

(2.3)

and for Laplace transform, the initial values of the fractional integral \( I^{n-\alpha} f(t) \) and the derivatives of its of order \( k = 1,2,3,\ldots,n-1 \) are required, where

\[
L[D_{RL}^\alpha f(t)] = s^\alpha L[f(t)] - \sum_{k=0}^{n-1} s^{n-1-k} D^k I^{n-\alpha} f(0), \quad n-1 < \alpha < n.
\]  

(2.4)

An alternative definition of fractional derivative was proposed by Caputo (Caputo 1967) as,

\[
D_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} f(\tau) d\tau, \quad n-1 < \alpha < n
\]  

(2.5)

and for the Laplace transform, this definition have an advantage, the initial values of \( f(t) \) and its integer derivatives of the order \( k = 1,2,3,\ldots,n-1 \) are required unlike the fractional ordered derivatives for the definition given by the equation (2.3), so that

\[
L[D_c^\alpha f(t)] = s^\alpha L[f(t)] - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0), \quad n-1 < \alpha < n.
\]  

(2.6)

Here we shall use the definition of fractional derivatives of order \( \alpha \in (0,1] \), according to (Caputo 1967), for an absolute continuous function \( f(t) \) given by

\[
\frac{d^\alpha}{dt^\alpha} f(t) = I^{1-\alpha} f'(t)
\]  

(2.7)

where \( I^\beta \) is the fractional integral of order \( \beta \ (> 0) \) of the Lebesgue integrable function \( f(t) \), in equation (2.1), defined by Miller and Ross (Miller and Ross 1993).

In the case of absolutely continuous function \( f(t) \),

\[
\lim_{\alpha \to 1} \frac{d^\alpha}{dt^\alpha} f(t) = f'(t)
\]  

(2.8)

The constitutive equations for isotropic, micropolar thermoelastic solids are (Eringen 1999),

\[
L[I^\alpha f(t)] = \frac{1}{s^\alpha} L[f(t)].
\]  

(2.2)
\[ t_{ij} = \lambda \varepsilon_{ik} \delta_j + (\mu + \kappa)\varepsilon_{ij} + \mu \varepsilon_{ji} - \beta_1 \partial_i \delta_{ij} \]  
(2.9)

\[ m_{ij} = \alpha \gamma_{ik} \delta_j + \beta \gamma_{ij} + \gamma_{ji} \]  
(2.10)

\[ \rho T_0 \dot{S} = \rho C_E \dot{\theta} + \beta T_0 \varepsilon_{ik} \]  
(2.11)

and the linear equations of balance law are

\[ t_{\beta \alpha \beta} + \rho F_\alpha = \rho \ddot{u}_\alpha, \]  
\[ m_{\beta \alpha \beta} + \varepsilon_{\alpha \mu \nu} t_{\mu \nu} + \rho t^1_\alpha = \rho \ddot{j}_\alpha. \]  
(2.12)

The linearized form of heat conduction is

\[ \rho T_0 \dot{S} = -q_{i,i} + W. \]  
(2.13)

Now, in isotropic media, we assume a new generalized heat conduction equation of the form

\[ q_i + \tau_0 \frac{\partial^a}{\partial t^a} q_i = -K \varphi_{i,i} \]  
(2.14)

where \( \varphi \) is the conductive temperature and satisfies the relation

\[ \varphi - \theta = a \varphi_{i,i} \]  
(2.15)

in which \( a > 0 \) is the two temperature parameter, \( K \) is the thermal conductivity and \( \alpha \) is the constant such that \( 0 < \alpha \leq 1 \).

Now using divergence theorem and the equations (2.11), (2.13), from the equation (2.14) we obtain

\[ K \varphi_{i,i} = \rho C_E \left( 1 + \tau_0 \frac{\partial^a}{\partial t^a} \right) \frac{\partial \theta}{\partial t} + T_0 \beta_1 \left( 1 + \tau_0 \frac{\partial^a}{\partial t^a} \right) \frac{\partial \varepsilon_{ik}}{\partial t} - \left( 1 + \tau_0 \frac{\partial^a}{\partial t^a} \right) W. \]  
(2.16)

We may consider equation (2.16) as the fractional ordered generalized heat conduction equation in micropolar, isotropic, elastic solids in two temperatures.

Now from the equations (2.7)-(2.8) it can be written that,

\[ \frac{\partial^a}{\partial t^a} f(x,t) = \begin{cases} 
  f(x,t) - f(x,0), & \alpha \to 0 \\
  l^{1-a} \frac{\partial f(x,t)}{\partial t}, & 0 < \alpha < 1 \\
  \frac{\partial f(x,t)}{\partial t}, & \alpha \to 1 
\end{cases} \]

when \( \alpha \to 1 \), equation (2.16) reduces to
This is the generalized heat conduction equation with two temperatures.

Again, when $\alpha \to 0$, equation (2.16) transform to

$$K'\varphi_{x_i} = \rho C_F \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \frac{\partial \theta}{\partial t} + T_0 \beta_1 \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \frac{\partial e_{ik}}{\partial t} - \left(1 + \tau_0 \frac{\partial}{\partial t}\right)W.$$  \hspace{1cm} (2.17)

in which, $K' = \frac{K}{1 + \tau_0}$,

which is the heat conduction equation for micropolar-coupled thermoelasticity.

3 VARIATIONAL THEOREM

The variational theorem in classical thermoelasticity first derived by Biot (Biot 1959) and explained their applications by means of several examples. In micropolar thermoelasticity the variational principle and uniqueness theorem was done by Eringen (Eringen 1966, 1999). Recently, a variational principle of fractional order generalized thermoelasticity was done by Youssef and AI-Lehaibi (Youssef and AI-Lehaibi 2010). Now we shall present a compact derivation of the variational theorem on generalized micropolar-fractional thermoelasticity under 2TT.

We consider,

$$W = \frac{1}{2} \int_v \left[ A_{klmn} e_{mi} e_{mj} + B_{klmn} \gamma_{kli} \gamma_{lij} \right] dV$$

where the integrand is homogeneous quadratic form of strain tensor and microrotation tensors.

We now consider a virtual displacement i.e. for a neighboring state in which the displacement, strain tensor, microrotation tensors are changed by the quantities $\delta u_i, \delta e_{ij}, \delta \gamma_{ij}$ respectively to obtain,

$$\delta W = \frac{1}{2} \int_v \left[ A_{klmn} (\delta e_{mi}) e_{mj} + A_{klmn} e_{mi} (\delta e_{mj}) + B_{klmn} (\delta \gamma_{kli}) \gamma_{lij} + B_{klmn} \gamma_{kli} (\delta \gamma_{lij}) \right] dV$$

Using the constitutive equations;

$$t_{ij} = \lambda e_{ik} \delta_{ij} + (\mu + \kappa) e_{ij} + \mu e_{ji} - \beta_1 \theta \delta_{ij},$$

$$m_{ij} = \alpha \gamma_{ik} \delta_{ij} + \beta \gamma_{ij} + \gamma \gamma_{ji}.$$  \hspace{1cm} (3.3)

taking into account the equations of motion given by the equation (2.12) and the corresponding boundary conditions

\[ p_i = t_{ji} n_j \]  
\[ m_i = m_{ji} n_j \]  

and using divergence theorem, we obtain,

\[
\int_V \rho F_i \delta u_i \, dV + \int_A p_i \delta u_i \, dA + \int_V \rho \delta \phi_t \, dV + \int_A m_i \delta \phi_t \, dA - \int_V \rho \dot{u}_i \delta u_i \, dV - \rho \int_V \delta \phi_\delta \, dV
\]

\[ = \delta W - \beta_1 \int_V \theta \delta \varepsilon_{ik} \, dV. \]  

(3.7)

This is the first variational equation and it would be complete for uncoupled thermoelasticity if the mechanical temperature \( \theta \) in the right-hand side integration of the equation (3.7) is known. Now taking into account the coupling between the strain field and the mechanical temperature it is observed that \( \theta \) is unknown. Hence it is necessary to introduce other relations considering the phenomena of heat conduction.

According to Biot (Biot 1956), related with entropy, we introduce one vector \( \tilde{H} \) by,

\[ \rho S = -\text{div}\tilde{H} = -H_{i,i}. \]  

(3.8)

Again we know the relations,

\[ \rho T_0 S = \rho C_E \theta + \beta_1 T_0 \varepsilon_{ik}, \]  

(3.9)

\[ q_{i,i} = -\rho T_0 \dot{\theta}, \]  

(3.10)

\[ q_i + T_0 \frac{\partial}{\partial t} q_i = -K \varphi_{i,i}. \]  

(3.11)

Now using the equation (3.8), from the relations given by the equation (3.9) – (3.11), we obtain,

\[ -H_{i,i} = \frac{\rho C_E}{T_0} \theta + \beta_1 \varepsilon_{ik} \]  

(3.12)

\[ q_i = T_0 \dot{H}_i \]  

(3.13)

\[ T_0 \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^{+\alpha}}{\partial t^{+\alpha}} \right) H_i = -K \varphi_{i,i}. \]  

(3.14)
Multiplying both sides of the equation (3.14) by \( \delta H_i \) and then integrating over the region \( V \) of the body we obtain,

\[
\int_V \left[ \phi_{i,j} + \frac{T_0}{K} \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \right) H_i \right] \delta H_i \, dV = 0
\]

(3.15)

Now, \( \int_V \phi_{i,j} \delta H_i \, dV = \int_v \phi_{i,j} \delta H_i \, dA + \frac{\rho C_p}{T_0} \int_v \phi \delta \theta \, dV + \beta \int_v \phi \delta \varepsilon_{ik} \, dV \)

(3.16)

Using the equations (3.16) and (2.15), from the equation (3.15) we obtain,

\[
\int_A \phi_{n,j} \delta H_i \, dA + \frac{\rho C_p}{T_0} \int_v \theta \delta \theta \, dV + \beta \int_v \theta \delta \varepsilon_{ik} \, dV + \frac{T_0}{K} \int_v \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \right) H_i \, \delta H_i \, dV
\]

\[-a \int_v \delta H_i \, \phi_{i,j} \, dV = 0. \]

(3.17)

We introduce heat potential function

\[
P = \frac{\rho C_p}{2T_0} \int_v \theta^2 \, dV
\]

(3.18)

and the heat dissipation function \( ID \), where

\[
\delta ID = \frac{T_0}{K} \int_v \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \right) H_i \, \delta H_i \, dV.
\]

(3.19)

Therefore, from equation (3.17) we obtain,

\[
\int_A \phi_{n,j} \delta H_i \, dA + \delta (P + ID) + \beta \int_v \theta \delta \varepsilon_{ik} \, dV - a \int_v \delta H_i \, \phi_{i,j} \, dV = 0.
\]

(3.20)

This is the second variational equation connected with heat conduction.

Now from the equations (3.7) and (3.20) we obtain,

\[
\delta \left[ W + P + ID \right] = -\int_A \phi_{n,j} \delta H_i \, dA + \int_A \rho F_i \delta u_i \, dV + \int_A \rho \phi_i \delta \phi_i \, dV
\]

\[
+ \int_A m_i \delta \phi_i \, dA - \rho \int_v \delta \dot{u}_i \, dV - \int_A \rho \phi_i \delta \phi_i \, dV + a \int_v \delta H_i \, \phi_{i,j} \, dV
\]

(3.21)

Again, \( \delta u_i = \dot{u}_i \, dt \), \( \delta \theta = \dot{\theta} \, dt \), \( \delta H_i = \dot{H}_i \, dt \), \( \delta \phi = \dot{\phi} \, dt \) etc.
Therefore, equation (3.21) reduces to,
\[
\frac{d}{dt} \left[ \mathbf{W} + \mathbf{S} + P + ID \right] = \int_V \rho F_i \dot{u}_i \, dV + \int_V \rho I_i \dot{\phi}_i \, dV + \int_A p_i \dot{u}_i \, dA + \int_A m_i \dot{\phi}_i \, dA - \int_A \varphi \dot{n} \, dA + a \int_V \dot{H}_{i,j} \varphi_{i,j} \, dV
\]

(3.22)

This is the variational equation of fractional micropolar thermoelasticity in 2TT.

4 UNIQUENESS THEOREM

Statement: There is only one solution of the equations (2.12) and (2.16) subject to the boundary conditions
\[
p_i = t_{ij} n_j, \quad m_i = m_{ji} n_j, \quad \theta = \theta_i(x,t), \quad \varphi = \varphi_i(x,t) \quad x \in A, \ t > 0
\]

and the initial conditions
\[
u_i(x,0) = u_{i0}(x), \quad \dot{u}_i(x,0) = \dot{u}_{i0}(x), \quad \phi_i(x,0) = \phi_{i0}(x), \quad \dot{\phi}_i(x,0) = \dot{\phi}_{i0}(x), \quad \theta(x,0) = T_0(x),
\]
\[
\varphi(x,0) = T_0(x), \quad x \in V, \ t = 0,
\]

where the body occupies the region V bounded by the surface A.

Proof: We consider, if possible, there exist two sets of solutions \((u_i, \phi_i, \theta, \varphi)\) and \((u_i', \phi_i', \theta', \varphi')\).

We take, \(\hat{u}_i = u_i \neq u_i'\), \(\hat{\phi}_i = \phi_i \neq \phi_i'\), \(\hat{\theta}_i = \theta_i \neq \theta_i'\), \(\hat{\varphi} = \varphi \neq \varphi'\).

Then the difference function \((\hat{u}_i, \hat{\phi}_i, \hat{\theta}, \hat{\varphi})\) must satisfy the following equations of motion and heat conduction equation with no body forces, body couples and without heat sources term:
\[
t_{\beta \alpha, \beta} = \rho \dddot{u}_\alpha
\]

(4.1)

\[
m_{\beta \alpha} + \varepsilon_{\alpha \gamma \mu} \tau_{\gamma \mu} = \rho \dddot{\phi}_\alpha
\]

(4.2)

\[
K_{\varphi, \alpha} = \rho C_E \left(1 + \tau_0 \frac{\partial}{\partial t} \right) \dot{\theta} + \beta T_0 \left(1 + \tau_0 \frac{\partial}{\partial t} \right) \dot{\varepsilon}_{ik}
\]

(4.3)

with homogeneous boundary and initial conditions.
Thus we arrive with a system for which the displacement $u_i$, microrotation $\hat{\phi}_i$, mechanical temperature $\hat{\Theta}$ and conductive temperature $\hat{\phi}$ vanishes inside the body initially and the surface traction $\hat{p}_i$, surface couple $\hat{m}_i$ along with mechanical as well as conductive temperatures vanishes on surface $A$. It is enough to prove that the measure of strain tensor, microrotation tensor and mechanical as well as conductive temperatures vanishes inside the body.

Now from the equation (3.22) we obtain,

$$\frac{d}{dt} \left[ \mathbf{W}^+ + \mathbf{K} + \mathbf{P} + \mathbf{ID} \right] = a \int_{V} \mathbf{H}_{ij} \dot{\phi}_{ij} dV$$

Or,

$$\frac{d}{dt} \left[ \mathbf{W}^+ + \mathbf{K} + \mathbf{P} \right] = -\frac{aT_0}{K} \int_{V} \mathbf{H}_{ij} \left( 1 + \tau_0 \frac{\ddot{\Theta}^\alpha}{\partial t^\alpha} \right) \dot{H}_{ij} dV - \frac{T_0}{K} \int_{V} \mathbf{H} \left( 1 + \tau_0 \frac{\ddot{\Theta}^\alpha}{\partial t^\alpha} \right) \dot{H} dV \leq 0$$

(4.4)

The integral in the left hand side of equation (4.4) is zero initially. On the other hand the inequality proves that the left hand side of the equation is either zero or decreases taking negative values. Since the integrand in the left hand side is a sum of squares and vanishes at $t = 0$, therefore, first possibility holds.

Hence, $\mathbf{W}^+ + \mathbf{K} + \mathbf{P} = 0$,

and

$$\dot{u}_i = \ddot{\phi}_i = \ddot{\Phi} = \ddot{\phi} = 0, \quad \dot{\epsilon}_{ij} = 0, \quad \dot{\gamma}_{ij} = 0 \quad \text{for } t \geq 0.$$  

Since $\dot{i}_{ij}$ and $\dot{m}_{ij}$ are the linear functions of $\dot{\epsilon}_{ij}, \dot{\gamma}_{ij}$ and $\ddot{\Theta}$ which are zero for $t \geq 0$, therefore, $\dot{i}_{ij} = 0 = \dot{m}_{ij}, \ t \geq 0$.

Hence, $\dddot{u}_i = \dddot{\phi}_i = \dddot{\phi} = 0 \quad \text{for } t \geq 0$.

This completes the proof of uniqueness theorem.

5 RECIPROCITY THEOREM

A reciprocity theorem in fractional order theory of thermoelasticity was proved by Sherief et al. (Sherief et al. 2010). In micropolar elasticity the reciprocity theorem was discussed by Iesan (Iesan 1967), Chandrasekharaih (Chandrasekharaih 1987), and in the year 1990 by Scalia (Scalia 1990).
We consider a region $V$ bounded by a surface $A$ of a homogeneous, isotropic, micropolar elastic body.

We assume that the stress components $t_{ij}$, components of strain measure $\varepsilon_{ij}$, microrotation strain measure $\gamma_{ij}$ are continuous whereas the displacements $u_i$, microrotation $\phi_i$, mechanical temperature $\theta$, conductive temperature $\varphi$ have their second order derivatives are also continuous, for $x \in A + V$, $t > 0$. These functions satisfy the equations of motion,

$$t_{\beta\alpha,\beta} + \rho F_{\alpha} = \rho \dddot{u}_{\alpha}$$  \hspace{1cm} (5.1)

$$m_{\beta\alpha,\beta} + \varepsilon_{\alpha\alpha\alpha} t_{\alpha\alpha} + \rho l_{\alpha} = \rho y_{\beta}$$ \hspace{1cm} (5.2)

Moreover, they must satisfy the generalized heat conduction equation,

$$K \varphi_{,\alpha,\alpha} = \rho C_{E} \left( 1 + \tau_{0} \frac{\partial^{\alpha}}{\partial t^{\alpha}} \right) \frac{\partial \theta}{\partial t} + \beta_{1} T_{0} \left( 1 + \tau_{0} \frac{\partial^{\alpha}}{\partial t^{\alpha}} \right) \frac{\partial \varepsilon_{\alpha\alpha}}{\partial t} - \left( 1 + \tau_{0} \frac{\partial^{\alpha}}{\partial t^{\alpha}} \right) W$$  \hspace{1cm} (5.3)

$$\varphi - \theta = a \varphi_{,\alpha,\alpha}$$ \hspace{1cm} (5.4)

Also we have the constitutive equations,

$$t_{ij} = \lambda \varepsilon_{ik} \delta_{ij} + (\mu + \kappa) \varepsilon_{ij} + \mu \varepsilon_{ji} - \beta_{1} \theta \delta_{ij}$$ \hspace{1cm} (5.5)

$$m_{ij} = \alpha \gamma_{ik} \delta_{ij} + \beta_{1} \gamma_{ij} + \gamma'_{ji} ,$$ \hspace{1cm} (5.6)

with boundary conditions;

$$t_{ji} n_{j} = p_{i}(x,t), \quad m_{ji} n_{j} = m_{i}(x,t), \quad \theta(x,t) = k_{1}(x,t), \quad \varphi(x,t) = k_{2}(x,t), \quad x \in A, \quad t > 0$$  \hspace{1cm} (5.7)

and homogeneous initial conditions;

$$u_{i}(x,0) = \ddot{u}_{i}(x,0) = 0, \quad \phi_{i}(x,0) = \phi_{i}(x,0) = 0, \quad \theta(x,0) = \theta(x,0) = 0, \quad \varphi(x,0) = \varphi(x,0) = 0,$$

$$W(x,0) = \dot{W}(x,0) = 0, \quad x \in V, \quad t = 0.$$  \hspace{1cm} (5.8)

Let the body be subjected to the action of body force $F_{i}$, body couple $l_{i}$, surface traction $p_{i}$, surface couple $m_{i}$, heat source $W$ and two surface heating temperatures $k_{1}$ (mechanical), $k_{2}$ (conductive). These causes are written symbolically as,

$$I = \{F_{i}, l_{i}, p_{i}, m_{i}, W, k_{1}, k_{2} \}$$ \hspace{1cm} (5.9)

These causes produce displacement $u_{i}$, microrotation $\phi_{i}$ and two temperature increment $\theta$, $\varphi$. We write these effects symbolically in the form,
$$G = \{u_i, \phi_j, \theta, \varphi\}$$

(5.10)

Now we assume that apart from the above cause and effects there is another system of cause and effects and we denote them by primes,

$$I' = \{F'_i, l'_i, p'_i, m'_i, W', k'_i, k'_2\}$$

(5.11)

$$G' = \{u'_i, \phi'_j, \theta', \varphi'\}$$

(5.12)

The reciprocity theorem establishes a relation between the above two system of causes and effects.

Taking Laplace transform in both sides of the equations (5.5) and (5.6) we obtain,

$$\tilde{i}_{ij} = \lambda \tilde{e}_{ik} \delta_{ij} + (\mu + \kappa)\bar{e}_{ij} + \mu \tilde{e}_{ij} = \beta_i \bar{e}_{ij}$$

(5.13)

$$\tilde{m}_{ij} = \alpha \bar{f}_{ik} \delta_{ij} + \beta \bar{f}_{ij} + \gamma \tilde{f}_{ij}$$

(5.14)

Where

$$\tilde{f}(x, s) = L[f(x, t); t \to s] = \int_0^\infty e^{-st} f(x, t) dt$$

Now the analogous equations for the system (5.11), (5.12) are,

$$\tilde{i}'_{ij} = \lambda \tilde{e}'_{ik} \delta_{ij} + (\mu + \kappa)\bar{e}'_{ij} + \mu \tilde{e}'_{ij} = \beta_i \bar{e}'_{ij}$$

(5.15)

$$\tilde{m}'_{ij} = \alpha \bar{f}'_{ik} \delta_{ij} + \beta \bar{f}'_{ij} + \gamma \tilde{f}'_{ij}$$

(5.16)

Multiplying equation (5.13) by \( \tilde{e}_{ij} \) and equation (5.15) by \( \bar{e}'_{ij} \) then integrating the difference over the region V we obtain,

$$\int_V (\tilde{i}_{ij} \tilde{e}_{ij} - \tilde{i}'_{ij} \bar{e}_{ij}) dV = \mu \int_V (\tilde{e}_{ij} \bar{e}'_{ij} - \bar{e}_{ij} \tilde{e}'_{ij}) dV - \beta_i \int_V (\bar{e}_{ik} \bar{e}'_{ik} - \bar{e}_i \bar{e}'_{ik}) dV$$

(5.17)

Again, multiplying equation (5.13) by \( \tilde{e}_{ij} \) and equation (5.15) by \( \bar{e}'_{ij} \) then integrating the difference over the region V we obtain,

$$\int_V (\tilde{i}_{ij} \tilde{e}_{ij} - \tilde{i}'_{ij} \bar{e}_{ij}) dV = (\mu + \kappa) \int_V (\bar{e}_{ij} \tilde{e}'_{ij} - \tilde{e}_{ij} \tilde{e}'_{ij}) dV - \beta_i \int_V (\bar{e}_{ik} \tilde{e}'_{ik} - \tilde{e}_i \bar{e}'_{ik}) dV$$

(5.18)

Now,

$$\int_V \tilde{i}_{ij} \tilde{e}_{ij} dV = \int_A \tilde{p}_j \tilde{u}'_j dA + \int_V \rho \tilde{F}_j \tilde{u}'_j dV - \rho \int_V s^2 \tilde{u}'_j \tilde{u}'_j dV - \epsilon_{ij} \int_V \tilde{t}_{ij} \tilde{A}' dV$$

(5.19)

Eliminating \( \int_V (\tilde{e}_{ij} \tilde{e}'_{ij} - \bar{e}_{ij} \tilde{e}'_{ij}) dV \) from the equations (5.17), (5.18) and then using the equation (5.19) we obtain,
\begin{align}
(\mu + \kappa) \left[ \int_A (\mathbf{p}_j \cdot \mathbf{p}_j' - \mathbf{p}_j' \cdot \mathbf{p}_j) dA + \int \rho (F_j \mathbf{p}_j' - F_j' \mathbf{p}_j) dV \right] + \mu \int \left[ (\tilde{t}_{ij} \mathbf{p}_{i,j} - \tilde{t}_{ij}' \mathbf{p}_{i,j}) dV - \kappa \varepsilon_{ijk} \int (\tilde{t}_{ij} \phi_k' - \tilde{t}_{ij}' \phi_k) dV \right] \\
= -\beta_1 (2 \mu + \kappa) \int (\mathbf{\bar{t}}_i, \mathbf{\bar{t}}_i' - \mathbf{\bar{t}}_i' \mathbf{\bar{t}}_i) dV 
\end{align}

Similarly, multiplying equation (5.14) by \( \mathbf{\tilde{f}}'_{ij} \) and equation (5.16) by \( \mathbf{\tilde{f}}_{ij} \) then subtracting and integrating over the region \( \mathbf{V} \) we obtain

\begin{align}
\int_V (\mathbf{m}_{ij} \mathbf{\tilde{f}}'_{ij} - \mathbf{m}_{ij}' \mathbf{\tilde{f}}_{ij}) dV = \gamma \int_V (\mathbf{\tilde{f}}'_{ij} - \mathbf{\tilde{f}}_{ij}) dV 
\end{align}

Again, multiplying equation (5.14) by \( \mathbf{\tilde{f}}'_{ij} \) and equation (5.16) by \( \mathbf{\tilde{f}}_{ij} \) then subtracting and integrating over the region \( \mathbf{V} \) we obtain,

\begin{align}
\int_V (\mathbf{m}_{ij} \mathbf{\tilde{f}}'_{ij} - \mathbf{m}_{ij}' \mathbf{\tilde{f}}_{ij}) dV = \beta \int_V (\mathbf{\tilde{f}}'_{ij} - \mathbf{\tilde{f}}_{ij}) dV 
\end{align}

Now, \( \int_V \mathbf{m}_{ij} \mathbf{\tilde{f}}_{ij} dV = \int \mathbf{m}_j \mathbf{\tilde{f}}_{ij} dA + \int \rho \mathbf{\tilde{f}}_{ij} \mathbf{d}V - \int \rho \mathbf{\tilde{f}}_{ij} \mathbf{d}V + \varepsilon_{ijmn} \int \mathbf{\tilde{f}}_{ij} \mathbf{d}V 
\)

Eliminating \( \int_V (\mathbf{\tilde{f}}_{ij} \mathbf{\tilde{f}}_{ij} - \mathbf{\tilde{f}}_{ij}' \mathbf{\tilde{f}}_{ij}) dV \) from the equations (5.21), (5.22) and then using the equation (5.23) we obtain,

\begin{align}
\gamma \left[ \int_A (\mathbf{m}_j \mathbf{\tilde{f}}_{ij} - \mathbf{m}_j' \mathbf{\tilde{f}}_{ij}) dA + \rho \int (\mathbf{\tilde{t}}_{ij} \mathbf{\tilde{f}}_{ij} - \mathbf{\tilde{t}}_{ij}' \mathbf{\tilde{f}}_{ij}) dV + \varepsilon_{ijmn} \int (\mathbf{\tilde{f}}_{ij} \mathbf{\tilde{f}}_{ij} - \mathbf{\tilde{f}}_{ij}' \mathbf{\tilde{f}}_{ij}) dV \right] \\
+ \beta \int_V (\mathbf{m}_{ij} \mathbf{\tilde{f}}_{ij}' - \mathbf{m}_{ij}' \mathbf{\tilde{f}}_{ij}) dV = 0 
\end{align}

Now eliminating \( \int_V (\mathbf{\tilde{t}}_{ij} \mathbf{\tilde{f}}_{ij}' - \mathbf{\tilde{t}}_{ij}' \mathbf{\tilde{f}}_{ij}) dV \) from the equations (5.20) and (5.24) we obtain,

\begin{align}
(\mu + \kappa) \left[ \int_A (\mathbf{p}_j \cdot \mathbf{p}_j' - \mathbf{p}_j' \cdot \mathbf{p}_j) dA + \int \rho (\mathbf{f}_j \mathbf{p}_j' - \mathbf{f}_j' \mathbf{p}_j) dV \right] + \kappa \left[ \int_A (\mathbf{m}_k \mathbf{f}_k' - \mathbf{m}_k' \mathbf{f}_k) dA + \rho \int (\mathbf{\tilde{t}}_{ij} \mathbf{f}_k' - \mathbf{\tilde{t}}_{ij}' \mathbf{f}_k) dV \right] \\
+ \mu \int \left[ (\tilde{t}_{ij} \mathbf{p}_{i,j} - \tilde{t}_{ij}' \mathbf{p}_{i,j}) dV + \frac{\kappa \rho}{\gamma} \int (\mathbf{m}_{ik} \mathbf{f}_{j,k}' - \mathbf{m}_{ik}' \mathbf{f}_{j,k}) dV \right] \\
= -\beta_1 (2 \mu + \kappa) \int (\mathbf{\bar{t}}_{ik}, \mathbf{\bar{t}}_{ik} - \mathbf{\bar{t}}_{ik} \mathbf{\bar{t}}_{ik}) dV 
\end{align}

which is the first part of reciprocity theorem and it contains only causes of mechanical nature.

To obtain the second part of the reciprocity theorem, we take Laplace transform of the equation (5.3) and using the appropriate initial conditions we obtain,

$$K\tilde{\varphi}_{ij} = \left(s + \tau_0 s^{1+\alpha}\right)\left(\rho C_E \tilde{\varphi} + \beta_i T_0 \tilde{e}_{ik}\right) - \left(1 + \tau_0 s^\alpha\right)\overline{W}$$ \hspace{1cm} (5.26)

The analogous equation of the equation (5.26) is

$$K\tilde{\varphi}'_{ij} = \left(s + \tau_0 s^{1+\alpha}\right)\left(\rho C_E \tilde{\varphi}' + \beta_i T_0 \tilde{e}_{ik}\right) - \left(1 + \tau_0 s^\alpha\right)\overline{W}'$$ \hspace{1cm} (5.27)

Now multiplying the equation (5.26) by $\tilde{\varphi}'$ and the equation (5.27) by $\tilde{\varphi}$ and then integrating the difference over the region $V$ we obtain,

$$K \int_V \left(\tilde{\varphi}'_{ij} \tilde{\varphi}_{ij} - \tilde{\varphi}_{ij} \tilde{\varphi}'_{ij}\right) dV = \beta_i T_0 \left(s + \tau_0 s^{1+\alpha}\right) \int_V \left[ \tilde{e}_{ik} \tilde{\varphi}' - \tilde{e}_{ik} \tilde{\varphi} \right] dV - \left(1 + \tau_0 s^\alpha\right) \int_V \left(\overline{W} \tilde{\varphi}' - \overline{W}' \tilde{\varphi}\right) dV \hspace{1cm} (5.28)$$

Now, \[ \int_V \tilde{\varphi}_{ij} \tilde{\varphi}_{ij} dV = \int_A \left(\kappa_{ij} \frac{\partial \tilde{\varphi}}{\partial n} - \kappa_{ij} \frac{\partial \tilde{\varphi}'}{\partial n}\right) dA = \int_V \left(\tilde{\varphi}_{ij} \tilde{\varphi}'_{ij} - \tilde{\varphi}_{ij} \tilde{\varphi}_{ij}'\right) dV \hspace{1cm} (5.29) \]

Using equation (5.29), equation (5.28) reduces to,

$$K \left[ \int_A \left(\kappa_{ij} \frac{\partial \tilde{\varphi}}{\partial n} - \kappa_{ij} \frac{\partial \tilde{\varphi}'}{\partial n}\right) dA - \int_V \left(\tilde{\varphi}_{ij} \tilde{\varphi}'_{ij} - \tilde{\varphi}_{ij} \tilde{\varphi}_{ij}'\right) dV \right] = \beta_i T_0 \left(s + \tau_0 s^{1+\alpha}\right) \int_V \left[ \tilde{e}_{ik} \tilde{\varphi}' - \tilde{e}_{ik} \tilde{\varphi} \right] dV - \left(1 + \tau_0 s^\alpha\right) \int_V \left(\overline{W} \tilde{\varphi}' - \overline{W}' \tilde{\varphi}\right) dV \hspace{1cm} (5.30)$$

Eliminating \[ \int_V \left(\tilde{e}_{ik} \tilde{\varphi}' - \tilde{e}_{ik} \tilde{\varphi} \right) dV \] from the equations (5.25) and (5.30) we obtain,

$$\left(2\mu + \kappa\right) \left[ K \left[ \int_A \left(\kappa_{ij} \frac{\partial \tilde{\varphi}}{\partial n} - \kappa_{ij} \frac{\partial \tilde{\varphi}'}{\partial n}\right) dA - \int_V \left(\tilde{\varphi}_{ij} \tilde{\varphi}'_{ij} - \tilde{\varphi}_{ij} \tilde{\varphi}_{ij}'\right) dV \right] + \left(1 + \tau_0 s^\alpha\right) \int_V \left(\overline{W} \tilde{\varphi}' - \overline{W}' \tilde{\varphi}\right) dV \right]$$

$$= T_0 \left(s + \tau_0 s^{1+\alpha}\right) \left[ \int_A \left(\eta_{ij} \tilde{\sigma}' + \eta_{ij} \tilde{\sigma}\right) dA + \int_V \rho \left(F' \tilde{\sigma}' - F' \tilde{\sigma}\right) dV \right]$$

$$+ \kappa \left[ \int_A \left[m_i \tilde{\phi}'_i - m_i \tilde{\phi}_i\right] dA + \int_V \rho \left[\tilde{l}_j \tilde{\phi}'_j - \tilde{l}_j \tilde{\phi}_j\right] dV \right] + \mu \left[ \int_A \left(m_i \tilde{\phi}'_{1,i} - m_i \tilde{\phi}_{1,i}\right) dA + \int_V \rho \left[\tilde{l}_j \tilde{\phi}'_{1,j} - \tilde{l}_j \tilde{\phi}_{1,j}\right] dV \right]$$

$$+ \frac{\kappa \theta}{\gamma} \left[ \int_A \left[m_i \tilde{\phi}'_{1,i} - m_i \tilde{\phi}_{1,i}\right] dA + \int_V \rho \left[\tilde{l}_j \tilde{\phi}'_{1,j} - \tilde{l}_j \tilde{\phi}_{1,j}\right] dV \right]$$ \hspace{1cm} (5.31)

This is the reciprocity theorem containing the $I, I'$ and the effects $G, G'$ in Laplace transform domain. Taking inverse Laplace transform we can obtain the reciprocity theorem in physical domain.

The main difference of this theorem with any other theorems that this theorem differentiates between the wave propagation of the temperatures that comes from the thermal process (heat

conduction) and which comes from the mechanical process (mechanical/thermodynamic temperature).

Special Case:

When \( \kappa \rightarrow 0, a \rightarrow 0 \), then from the equation we obtain,

\[
K \int_A \left( k_i \frac{\partial \phi}{\partial n} - k_i \frac{\partial \phi'}{\partial n} \right) dA + \left( 1 + \tau_0 s^a \right) \left( W \phi' - W \phi \right) dV
\]

\[
= T_0 \left( s + \tau_0 s^{1+\alpha} \right) \left\{ \int_A \left( \bar{F}_j \bar{u}_j' - \bar{F}_j \bar{u}_j \right) dA + \int_V \rho \left( \bar{F}_j \bar{u}_j' - \bar{F}_j \bar{u}_j \right) dV \right\}
\]

which is the reciprocity theorem of fractional order thermoelasticity (in classical theory) as was done by Sherief et al. (Sherief et al. 2010).

NOMENCLATURE:

\( U \) : Internal energy per unit mass,
\( \dot{\phi} \) : Kinetic energy per unit mass,
\( \mathcal{L} \) : Power of external force,
\( Q \) : Heat absorbed by the material body,
\( W \) : Quantity of heat generated in unit time in unit volume,
\( \bar{q} \) : Heat flux vector,
\( \bar{F} \) : External body force,
\( \bar{\mathcal{L}} \) : Body Couple,
\( \sigma_{ij} \) : Microstretch rotatory inertia,
\( \rho \) : Constant mass density of the medium,
\( \bar{u} \) : Displacement vector.
\( \tilde{\xi} \) : Microdisplacement vector,
\( \tilde{\phi} \) : Microrotation vector,
\( j \) : Microinertia,
\( \tilde{n} \) : Outward drawn normal vector,
\( \mathcal{I}_{ij} \) : Stress tensor,
\( m_{ij} \) : Couple stress tensor,
\( \epsilon_{ij} \) : Micropolar strain tensor,
\( \nu_i = \dot{\phi}_i \) : Time rate change of microrotation component,
\( \gamma_{ij} = \phi_i,j \) : Microrotation tensor
\( \theta \) : Mechanical temperature,
\( \varphi \) : Conductive temperature,
\( S \) : Entropy per unit mass, \( \lambda, \mu \) : Lame’ constants,
Micropolar elastic constants,

\[ \beta_i = (3\lambda + 2\mu + \kappa)\alpha_i, \]

\( \alpha_i \): Coefficient of linear thermal expansion,

\( C_E \): Specific heat at constant strain,

\( \tau_0 \): Thermal relaxation time,

\( K \): Coefficient of thermal conductivity,

\( \varepsilon_{ijk} \): Permutation tensor,

\( \delta_{ij} \): Kronecker delta.

REFERENCES


