Asymptotic Analysis, Approximations and Simulations of the Compensation Problem in Hyperbolic Systems

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Abstract
This work concern an extension of the asymptotic compensation problem in hyperbolic systems. We introduce and characterize the notion of asymptotic remediability, which consists to compensate asymptotically a disturbance by a convenient choice of actuators (efficient actuators).
We also study the relationship between these different notions and asymptotic controllability as well as strategic actuators. Then, we characterize the set of remediable disturbances, and we show how to find the optimal control which compensates asymptotically the disturbance
of the system. As an application, we consider the case where the domain is one or two dimension. Approximations and numerical simulations are also presented.

**Keywords:** Hyperbolic systems, remediability, controllability, actuators, sensors, numerical simulations

## 1 Introduction

This work concern the problem of asymptotic compensation for a class of hyperbolic systems. It is an extension of previous works on finite time horizon for parabolic systems [1, 3, 4, 5, 9, 10].

We consider, without loss of generality, the system described by the following hyperbolic equation

\[
\begin{cases}
\frac{\partial^2 x}{\partial t^2}(\xi,t) = \Delta x(\xi,t) + B u(t) + f(\xi,t) \\
\text{in } \Omega \times ]0, +\infty[ \\
x(\xi,0) = \frac{\partial x}{\partial t}(\xi,0) = 0 \text{ on } \Omega \\
x(\eta,t) = 0 \text{ on } \partial \Omega \times ]0, +\infty[ 
\end{cases}
\]  

(1)

where $\Omega$ is an open and bounded subset of $\mathbb{R}^n$, with a sufficiently regular boundary $\Gamma = \partial \Omega$. $B \in \mathcal{L}(U, L^2(\Omega))$, $u \in L^2(0, +\infty; U)$; $U$ is a Hilbert space representing the control space and $\Delta$ is the Laplacian operator. The disturbance term $f \in L^2(0, +\infty; L^2(\Omega))$ is generally unknown. The system (1) is augmented by the output equation

\[
y(t) = \begin{pmatrix} C_1 x(.,t) \\ C_2 \frac{\partial x}{\partial t}(.,t) \end{pmatrix}
\]

(2)

where $C_1 \in \mathcal{L}(L^2(\Omega), Y_1)$ (and then $C_1 \in \mathcal{L}(H^1_0(\Omega), Y_1)$), $C_2 \in \mathcal{L}(L^2(\Omega), Y_2)$, $Y_1$ and $Y_2$ are two Hilbert spaces (observation spaces).

Let $A$ be the operator defined by $A \psi = \Delta \psi$ for $\psi \in D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, and $z = \begin{pmatrix} x \\ \frac{\partial x}{\partial t} \end{pmatrix} \in L^2(0, +\infty; \mathcal{E})$ where $\mathcal{E} = H^1_0(\Omega) \times L^2(\Omega)$. The system (1) is equivalent to

\[
(S) \begin{cases}
\dot{z}(t) = Az(t) + Bu(t) + \tilde{f}(t) \quad t > 0 \\
z(0) = 0
\end{cases}
\]

(3)
and the output equation can be written

\[ y(t) = Cz(t) \]  \hspace{1cm} (4)

where the operator \( A = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \), with \( D(A) = D(A) \times H^1_0(\Omega) \). The adjoint \( A^* \) of \( A \) is given by \( A^* = -A \).

The operator \( B \) is defined by \( B = \begin{pmatrix} 0 & B \end{pmatrix} \) and its adjoint is defined by \( B^* = 0 \begin{pmatrix} B \end{pmatrix} \). \( \tilde{f} = \begin{pmatrix} 0 \\ f \end{pmatrix} \) and \( \mathcal{C} \in \mathcal{L}(\mathcal{E}, Y) \) is defined by \( \mathcal{C} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \), with \( Y = Y_1 \times Y_2 \).

The operator \( A \) is linear, closed with a dense domain in the state space \( \mathcal{E} \), and generates on \( \mathcal{E} \) a strongly continuous semi-group \( (S_t)_{t \geq 0} \) defined by

\[
S(t) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \\
\sum_{n \geq 1} \sum_{j=1}^{r_n} \frac{1}{\sqrt{-\lambda_n}} \begin{pmatrix} < z_1, \varphi_{nj} >_\Omega \cos(\sqrt{-\lambda_n}t) + \\ < z_2, \varphi_{nj} >_\Omega \sin(\sqrt{-\lambda_n}t) \varphi_{nj} + \\ < z_2, \varphi_{nj} >_\Omega \cos(\sqrt{-\lambda_n}t) \varphi_{nj} \end{pmatrix} \\
\sum_{n \geq 1} \sum_{j=1}^{r_n} \frac{1}{\sqrt{-\lambda_n}} \begin{pmatrix} < z_1, \varphi_{nj} >_\Omega \sin(\sqrt{-\lambda_n}t) + \\ < z_1, \varphi_{nj} >_\Omega \cos(\sqrt{-\lambda_n}t) \varphi_{nj} \end{pmatrix}
\]  \hspace{1cm} (5)

where \( < .., .. >_\Omega \) is the inner product in \( L^2(\Omega) \) and \( (\varphi_{nj})_{n \geq 1, r_n} \) is an orthonormal basis of eigenvectors of \( A \), associated to eigenvalues \( \lambda_n < 0 \) with a multiplicity \( r_n \). The adjoint semi-group is defined by \( S^*(t) = S(-t) \forall t \geq 0 \). \( \mathcal{E} \) is a Hilbert space for the inner product \( < z, z' >_\mathcal{E} = < (-A)^{1/2} z_1, (-A)^{1/2} z_1' >_\Omega + < z_2, z_2' >_\Omega \), for \( z = (z_1, z_2) \) and \( z' = (z_1', z_2') \in \mathcal{E} \).

If \( f = 0 \), the observation is disturbed and the problem is to find an input operator \( B \) (actuators), with respect to the output operators \( C_1 \) et \( C_2 \) (sensors), ensuring the asymptotic compensation of any disturbance, i.e. for any \( f \in L^2(0, +\infty; L^2(\Omega)) \), there exists \( u \in L^2(0, +\infty; U) \) such that

\[
\int_0^{+\infty} \mathcal{C}S(s)\tilde{f}(s)ds + \int_0^{+\infty} \mathcal{C}S(s)Bu(s)ds = 0 \]  \hspace{1cm} (6)

or for any \( f \in L^2(0, +\infty; L^2(\Omega)) \) and any \( \epsilon > 0 \), there exists \( u \in L^2(0, +\infty; U) \)
such that
\[
\| \int_0^\infty \mathcal{C} \mathcal{S}(s) \tilde{f}(s) ds + \int_0^\infty \mathcal{C} \mathcal{S}(s) Bu(s) ds \| < \epsilon
\]  
(7)

This is the basic notion of asymptotic remediability, which is, as it will be shown, weaker than the asymptotic controllability.

This paper is organized as follows: In paragraph 2, we briefly recall the notions of asymptotic controllability, strategic actuators and sensors. In paragraph 3, we define and characterize the notion of exact and weak asymptotic remediability, as well as asymptotically efficient actuators. Then, we study in paragraph 4, the relationship between asymptotic controllability and asymptotic remediability, and hence between strategic actuators and efficient actuators. In paragraph 5, using an extension of Hilbert Uniqueness Method (H.U.M.), we examine the problem of exact asymptotic remediability with minimum energy. We also characterize the set of exactly remediable disturbances, and we give the optimal control which compensate spatially an arbitrary disturbance. In paragraph 6, we give an application in one and two space dimensions. In the last paragraph, approximations and numerical results are presented.

2 Asymptotic controllability

We consider the linear system
\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bu(t) \quad ; \quad t > 0 \\
z(0) &= 0
\end{align*}
\]  
(8)

and the operator $H^\infty$ defined on $L^2(0, +\infty; U)$ by

\[
H^\infty u = \left( \sum_{n \geq 1} \sum_{j=1}^{r_n} \frac{1}{\sqrt{-\lambda_n}} \int_0^\infty <Bu(s), \varphi_{nj}>_X \sin(\sqrt{-\lambda_n} s) ds \varphi_{nj} \right) \in X
\]  
(9)

where $X = L^2(\Omega) \times L^2(\Omega)$.

We assume that $H^\infty$ is well defined. Let us note that if $(\mathcal{S}(t))_{t \geq 0}$ is exponentially stable, then $H^\infty$ is well defined on $L^2(0, +\infty; U)$. This hypothesis concern the dynamics $A$ of the system through the semi-group $(\mathcal{S}(t))_{t \geq 0}$, and also the input operator $B$. An example illustrating this situation will be given
later. Let us note equally that if $H^\infty$ is not well defined on the whole space $L^2(0, +\infty; U)$, one can replace $L^2(0, +\infty; U)$ by the subspace

$$\mathcal{D}(H^\infty) = \{u \in L^2(0, +\infty; U) \text{ such that } H^\infty u \text{ is well defined}\}$$

Hereafter, we introduce and we characterize the notion of asymptotic controllability.

**Definition 2.1** The system (8) is said to be
i) exactly controllable asymptotically if for every $z_d \in \mathcal{E}$, there exists $u \in L^2(0, +\infty; U)$ such that $H^\infty u = z_d$, or equivalently $\text{Im}(H^\infty) = \mathcal{E}$
ii) weakly controllable asymptotically if for every $z_d \in \mathcal{E}$, and every $\epsilon > 0$, there exists $u \in L^2(0, +\infty; U)$ such that

$$\| H^\infty u - z_d \| < \epsilon$$

or

$$\text{Im}(\overline{H^\infty}) = X$$

Let $\mathcal{E}', \mathcal{U}'$ be the dual spaces of $\mathcal{E}$ and $\mathcal{U}$ respectively, then we have the following result

**Proposition 2.2** The system (S) is
i) exactly controllable asymptotically

$$\exists \gamma > 0 \text{ such that, } \forall z^* \in \mathcal{E}' \quad \| z^* \|_{\mathcal{E}'} \leq \gamma \| (H^\infty)^* z^* \|_{L^2(0, +\infty; \mathcal{U}')}$$

(10)

ii) weakly controllable asymptotically

$$\ker[(H^\infty)^*] = \{0\}$$

(11)

The exact asymptotic controllability implies the weak asymptotic controllability, the converse is not true.

In the case of $p$ actuators $(\Omega_i, g_i)_{i=1,p}$ [9], we have $\mathcal{U} = \mathbb{R}^p$ and

$$\mathcal{B}: \mathbb{R}^p \rightarrow \mathcal{E}$$

$$u(t) \mapsto \mathcal{B}u(t) = (0, \sum_{i=1}^{p} g_i u_i(t))^t$$

(12)

where $u = (u_1, ..., u_p)^t \in L^2(0, T; \mathbb{R}^p)$ and $g_i \in L^2(\Omega_i), \Omega_i = \text{supp}(g_i) \subset \Omega$ for $i = 1, p$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, we have
\[ \mathcal{B}^* z = (0, < g_1, z_2 >_\Omega, ..., < g_p, z_2 >_\Omega)^{tr} \]

for \( z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathcal{E}' \).

**Definition 2.3** We say that the actuators \((\Omega_i, g_i)_{i=1,p}\) are asymptotically strategic (or just strategic), if the corresponding system (3) is weakly controllable asymptotically.

We have the following characterization result [8][9].

**Proposition 2.4** The actuators \((\Omega_i, g_i)_{i=1,p}\) are strategic, if and only if

\[
\begin{cases}
  i) & p \geq r_n, \ \forall \ n \geq 1 \\
  ii) & \text{rank}(M_n) = r_n, \ \forall \ n \geq 1 \\
  \text{where} & M_n = (\langle g_i, \varphi_{nj} >_\Omega)_{i=1,p}^{j=1,r_n} 
\end{cases}
\]

(13)

The characterization of strategic pointwise actuators is similar to (13), with \( M_n = (\varphi_{nj}(b_i))_{i=1,p}^{j=1,r_n} \), where \( b_i \) are the actuators locations.

3 Asymptotic remediability

3.1 Asymptotic remediability

We consider the system (3) augmented by (4). Let \( K^\infty_C \) and \( R^\infty_C \) be the operators defined by

\[
K^\infty_C : L^2(0, +\infty; \mathcal{U}) \longrightarrow \mathcal{E} \\
u \longrightarrow K^\infty_C u = \int_0^{+\infty} CS(s)Bu(s)ds
\]

(14)

and

\[
R^\infty_C : L^2(0, +\infty; \{0\} \times L^2(\Omega)) \longrightarrow \mathcal{E} \\
\tilde{f} \longrightarrow R^\infty_C \tilde{f} = \int_0^{+\infty} CS(s)\tilde{f}(s)ds
\]

(15)

We assume that the operators \( K^\infty_C \) and \( R^\infty_C \) are well defined. This hypothesis concern the choice of the output operators \( C_1 \) and \( C_2 \) (an example will be given later) and the dynamics \( \mathcal{A} \) of the system (and also the input operator \( \mathcal{B} \) concerning \( K^\infty_C \)). Here also, let us note that if \( K^\infty_C \) and \( R^\infty_C \) are not
well defined on the whole corresponding spaces, i.e. for \( u \in L^2(0, +\infty; \mathcal{U}) \) and \( f \in L^2(0, +\infty; L^2(\Omega)) \), one can consider their restrictions respectively to

\[
\mathcal{D}(K_C^\infty) = \{ u \in L^2(0, +\infty; \mathcal{U}) \text{ such that } K_C^\infty u \text{ is well defined } \}
\]

and

\[
\mathcal{D}(R_C^\infty) = \{ f \in L^2(0, +\infty; L^2(\Omega)) \text{ such that } R_C^\infty \left( \begin{array}{c} 0 \\ f \end{array} \right) \text{ is well defined } \}
\]

In the considered case, we have

\[
K_C^\infty u = \left( \sum_{n \geq 1} \sum_{j=1}^{r_n} \frac{1}{\sqrt{-\lambda_n}} \int_{0}^{+\infty} <Bu(s), \varphi_{nj}> X \sin(\sqrt{-\lambda_n}s)ds C_1 \varphi_{nj} \right)
\]

and

\[
R_C^\infty \tilde{f} = \left( \sum_{n \geq 1} \sum_{j=1}^{r_n} \frac{1}{\sqrt{-\lambda_n}} \int_{0}^{+\infty} <f(s), \varphi_{nj}> X \sin(\sqrt{-\lambda_n}s)ds C_1 \varphi_{nj} \right)
\]

In the "normal" case, i.e. without disturbance and control, the system (3) becomes

\[
\begin{aligned}
\dot{z}(t) &= \mathcal{A}z(t) \quad t > 0 \\
z(0) &= 0
\end{aligned}
\]

the state \( z(.) \) is given by \( z(t) = 0 \), then the observation is \( y(t) = 0 \). But if the system is disturbed by \( \tilde{f} \), then the observation becomes

\[
y(t) = \int_{0}^{t} \mathcal{CS}(t-s)\tilde{f}(s)ds \neq 0
\]

Then, we introduce a control term \( Bu \) in order to compensate asymptotically this disturbance. We have the following definitions.
Definition 3.1  
i) We say that the system (3) augmented by the output equation (4) (or (3)+(4) is exactly remediable asymptotically, if for any \( f \in L^2(0, +\infty; L^2(\Omega)) \), there exists \( u \in L^2(0, +\infty; U) \) such that
\[
K_C^\infty u + R_C^\infty \hat{f} = 0
\]  
(19)
i) We say that \((1) + (E)\) is weakly remediable asymptotically, if for any \( \epsilon > 0 \) and \( f \in L^2(0, +\infty; L^2(\Omega)) \), there exists \( u \in L^2(0, +\infty; U) \) such that
\[
\| K_C^\infty u + R_C^\infty \hat{f} \| < \epsilon
\]
(20)

The regularity of the solution \( z_{u,f}(\cdot) \) of the system \((3)\) depends on that of \( f \) and on the control term \( Bu \). In the general case, we have \( z_{u,f}(\cdot) \in L^2(0, T; V) \) where \( V \) is a Hilbert space such that \( V' \subset X \subset V \), with continuous injections.

3.2 Characterizations
We have
\[
\text{Im}(K_C^\infty) \subset \text{Im}(R_C^\infty)
\]  
(21)
because for \( f = -Bu \), we have \( R_C^\infty \hat{f} = -K_C^\infty u \).

For the characterization of the exact remediability, we have the following proposition.

Proposition 3.2  
The following properties are equivalent

(i) \((3)+(4)\) is exactly remediable asymptotically.
(ii)
\[
\text{Im}(R_C^\infty) \subset \text{Im}(K_C^\infty)
\]
(22)
(iii)
\[
\exists \gamma > 0 \text{ such that } \forall \theta \in Y'
\]
\[
\| S^*(\cdot)C^*\theta \|_{L^2(0, +\infty; \mathcal{E}') \text{}} \leq \gamma \| B^*S^*(\cdot)C^*\theta \|_{L^2(0, +\infty; \mathcal{U}') \text{}}
\]
(23)

Proof
(i) \( \iff \) (ii) Derives from the definition.
(ii) \( \iff \) (iii) Derives from the following result [7].
Lemma 3.3 Let $X, Y, Z$ be a reflexive Banach spaces and $F \in \mathcal{L}(X, Z)$, $G \in \mathcal{L}(Y, Z)$; then the following properties are equivalent

(i) $\text{Im}(F) \subset \text{Im}(G)$

(ii) 

\[ \exists \gamma > 0 \text{ such that } \| F^* z^* \|_X \leq \gamma \| G^* z^* \|_Y \; \forall z^* \in Z^* \]  

(24)

For the weak remediability characterization, we have the following proposition.

Proposition 3.4 There is equivalence between

(i) $(3)+(4)$ is weakly remediable asymptotically.

(ii)

\[ \text{Im}(R_C^\infty) \subset \overline{\text{Im}(K_C^\infty)} \]  

(25)

(iii)

\[ \ker[\mathcal{B}^*(R_C^\infty)^*] = \ker((R_C^\infty)^*) \]  

(26)

Proof

(i) $\iff$ (ii) Derives from the definition.

(ii) $\iff$ (iii) by considering the orthogonal and using (21).

\[ \square \]

In finite dimensional case or an output given by a finite number of sensors, there is equivalence between weak and exact asymptotic remediability.

In the case of $p$ actuator(s) $(\Omega_i, g_i)_{i=1,p}$, the exact asymptotic remediability characterization is given by the proposition 3.2 using the fact that $S^*(t) = S(-t)$, i.e.

\[ S^*(t) \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \right. \]

\[ \sum_{n \geq 1} \sum_{j=1}^{r_n} (z_1, \varphi_{nj})_{\Omega} \cos(\sqrt{-\lambda_n}t) - \frac{1}{\sqrt{-\lambda_n}} \left< z_2, \varphi_{nj} \right>_{\Omega} \sin(\sqrt{-\lambda_n}t) \]  

\[ + \sum_{n \geq 1} \sum_{j=1}^{r_n} \sqrt{-\lambda_n} (z_1, \varphi_{nj})_{\Omega} \sin(\sqrt{-\lambda_n}t) + (z_2, \varphi_{nj})_{\Omega} \cos(\sqrt{-\lambda_n}t) \varphi_{nj} \]  

then $S^*(t)C^* \left( \begin{array}{c} \theta_1 \\ \theta_2 \end{array} \right)$ is obtained by replacing $z_1$ and $z_2$ respectively by $C_1 \theta_1$ and $C_2 \theta_2$. 

\[ \]
Now, if the output of the system is given by \((q_1, q_2)\) sensors \((D_i, h_i)_{i=1,q}\) and \((D'_i, k_i)_{i=1,q_2}\), we have

\[
C^*_1 \theta_1 = \sum_{i=1}^{q_1} \theta^i_1 h_i \quad \text{and} \quad C^*_2 \theta_2 = \sum_{i=1}^{q_2} \theta^i_2 k_i
\]

for \(\theta_1 = (\theta^i_1)_{i=1,q_1} \in \mathbb{R}^{q_1}\) and \(\theta_2 = (\theta^i_2)_{i=1,q_2} \in \mathbb{R}^{q_2}\).

### 3.2.1 Asymptotically efficient actuators:

We introduce hereafter the notion of asymptotically efficient actuators.

**Definition 3.5** Actuators ensuring the asymptotic weak remediability of the system (3)+(4), are said to be asymptotically efficient actuators (or just efficient actuators).

In the case of \(p\) actuators, we have the following characterization.

**Proposition 3.6** The actuators \((\Omega_i, q_i)_{i=1,p}\) are efficient, if and only if

\[
\ker(C^*) = \bigcap_{n \geq 1} \ker(M_n f_n) \tag{27}
\]

where, for \(n \geq 1\)

\[
f_n : Y' \longrightarrow \mathbb{R}^{r_n} \times \mathbb{R}^{r_n} \quad \theta \longmapsto f_n(\theta) = (f_{n,1}(\theta_1), f_{n,2}(\theta_2)) \tag{28}
\]

with

\[
f_{n,l}(\theta_l) = (\langle C^*_1 \theta_1, \varphi_n \rangle_\Omega, \ldots, \langle C^*_1 \theta_1, \varphi_n \rangle_\Omega \rangle_{n \geq 1}^{tr}
\]

for \(l = 1, 2\).

**Proof**

We have seen, in the proposition 3.4, that (3)+(4) is weakly remediable asymptotically, if and only if, \(\ker([B^*(R^\infty_C)]^* = \ker([R^\infty_C])^*\). For \(\theta = (\theta_1, \theta_2) \in Y'\), it is easy to show that \((R^\infty_C)^* \theta = 0 \iff \text{ for any } t > 0\)

\[
\begin{align*}
\sum_{n \geq 1} \sum_{j=1}^{r_n} (\langle C^*_1 \theta_1, \varphi_{nj} \rangle_\Omega \cos(\sqrt{-\lambda_n} t) - \\
\frac{1}{\sqrt{-\lambda_n}} C^*_2 \theta_2, \varphi_{nj} \rangle_\Omega \sin(\sqrt{-\lambda_n} t) \varphi_{nj} = 0
\end{align*}
\]

\[
\sum_{n \geq 1} \sum_{j=1}^{r_n} (\sqrt{-\lambda_n} C^*_1 \theta_1, \varphi_{nj} \rangle_\Omega \sin(\sqrt{-\lambda_n} t) + \\
\langle C^*_2 \theta_2, \varphi_{nj} \rangle_\Omega \cos(\sqrt{-\lambda_n} t) \varphi_{nj} = 0
\]

then, for all \( n \geq 1, j = 1, r_n \) and \( t > 0 \), we have

\[
\begin{cases}
< C_1^* \theta_1, \varphi_{nj} >_{\Omega} \cos(\sqrt{-\lambda_n} t) - \frac{1}{\sqrt{-\lambda_n}} < C_2^* \theta_2, \varphi_{nj} >_{\Omega} \sin(\sqrt{-\lambda_n} t) = 0 \\
\sqrt{-\lambda_n} < C_1^* \theta_1, \varphi_{nj} >_{\Omega} \sin(\sqrt{-\lambda_n} t) + < C_2^* \theta_2, \varphi_{nj} >_{\Omega} \cos(\sqrt{-\lambda_n} t) = 0
\end{cases}
\]

henceforth, \( \forall n \geq 1 \) and \( j = 1, r_n \), we have

\[
\begin{cases}
< C_1^* \theta_1, \varphi_{nj} >_{\Omega} = 0 \\
< C_2^* \theta_2, \varphi_{nj} >_{\Omega} = 0
\end{cases}
\]

i.e. \( C^* \theta = 0 \), and consequently

\[
\ker([R^\infty_C]^*) = \ker(C^*)
\]

On the other hand

\[
B^*([R^\infty_C]^*) \theta = 0 \iff \forall \ i = 1, p
\sum_{n \geq 1} \sum_{j=1}^{r_n} \left( \sqrt{-\lambda_n} < C_1^* \theta_1, \varphi_{nj} >_{\Omega} \sin(\sqrt{-\lambda_n} t) + < C_2^* \theta_2, \varphi_{nj} >_{\Omega} \cos(\sqrt{-\lambda_n} t) \right) < g_i, \varphi_{nj} >_{\Omega} = 0
\]

and hence

\[
\begin{cases}
\sum_{n \geq 1} \sin(\sqrt{-\lambda_n} t) \sum_{j=1}^{r_n} \sqrt{-\lambda_n} < C_1^* \theta_1, \varphi_{nj} >_{\Omega} < g_i, \varphi_{nj} >_{\Omega} = 0 \\
\sum_{n \geq 1} \cos(\sqrt{-\lambda_n} t) \sum_{j=1}^{r_n} < C_2^* \theta_2, \varphi_{nj} >_{\Omega} < g_i, \varphi_{nj} >_{\Omega} = 0
\end{cases}
\]

Consequently, for all \( n \geq 1 \)

\[
\begin{cases}
\sum_{j=1}^{r_n} \sqrt{-\lambda_n} < C_1^* \theta_1, \varphi_{nj} >_{\Omega} < g_i, \varphi_{nj} >_{\Omega} = 0 \\
\sum_{j=1}^{r_n} < C_2^* \theta_2, \varphi_{nj} >_{\Omega} < g_i, \varphi_{nj} >_{\Omega} = 0
\end{cases}
\]

then
\[ M_{nf,1}(\theta_1) = 0 \quad \text{and} \quad M_{nf,2}(\theta_2) = 0 \quad \forall \quad n \geq 1 \]

and hence

\[ \ker(C^*) = \bigcap_{n \geq 1} \ker(M_n f_n) \quad (30) \]

then, we have the result. \( \square \)

If the output is given by \((q_1, q_2)\) sensors \((D_i, h_i)_{i=1,q_1}\) and \((D'_i, k_i)_{i=1,q_2}\), the characterization of efficient actuators, is given by the following proposition.

**Proposition 3.7** The actuators \((\Omega_i, g_i)_{i=1,p}\), are efficient, if and only if

\[ \bigcap_{n \geq 1} \ker(M_n G_{n,1}^{tr}) \times \ker(M_n G_{n,2}^{tr}) = \{0\} \quad (31) \]

where

\[ G_{n,1} = \langle h_i, \varphi_{nj} > \Omega \rangle_{i=1,q_1}^{j=1,r_n} \]

and

\[ G_{n,2} = \langle k_i, \varphi_{nj} > \Omega \rangle_{i=1,q_2}^{j=1,r_n} \]

**Proof**

For \( \theta_l = (\theta_1^l, ..., \theta_q^l)^{tr} \in \mathbb{R}^{q_l} \), with \( l = 1, 2 \), we have

\[ C^*_l \theta_l = \sum_{i=1}^{q_l} \theta_i^l h_i \quad \text{and} \quad C^*_2 \theta_2 = \sum_{i=1}^{q_2} \theta_i^2 k_i \]

Since the functions \((h_i)_{i=1,q_1}\) (respectively \((k_i)_{i=1,q_2}\)) are linearly independent because \(D_i \cap D_j = \emptyset\) (respectively \(D'_i \cap D'_j = \emptyset\)) for \( i \neq j \), then, \( \ker(C^*_l) = \{0\} \) for \( l = 1, 2 \), and using (6.29), we have

\[ \ker[(R^{\infty}_C)^*] = \{0\} \quad (32) \]

On the other hand, we show by the same that,

\[ B^*(R^{\infty}_C)^* \theta = 0 \iff \forall \ l = 1, p \ ; \forall \ n \geq 1 \]

\[ \begin{cases} 
\sum_{j=1}^{r_n} \sqrt{-\lambda_n} g_l, \varphi_{nj} > \Omega \sum_{i=1}^{q_1} \theta_1^i h_i, \varphi_{nj} > \Omega = 0 \\
\sum_{j=1}^{r_n} \sqrt{-\lambda_n} g_l, \varphi_{nj} > \Omega \sum_{i=1}^{q_2} \theta_2^i h_i, \varphi_{nj} > \Omega = 0
\end{cases} \]
\[ \iff M_n G_{n,i}^{tr} \theta_i = 0, \; i = 1, 2, \; \forall \; n \geq 1 \]

and then
\[ \text{ker} [B^*(R_C^\infty)^r] = \bigcap_{n \geq 1} \text{ker} (M_n G_{n,1}^{tr}) \times \text{ker} (M_n G_{n,2}^{tr}) \]

using (26), we have the result. \( \square \)

We deduce the following corollary.

**Corollary 3.8** If there exists \( n_0 \) such that
\[
\text{rank} (M_{n_0} G_{n_0,1}^{tr}) = q_1 \text{ and rank} (M_{n_0} G_{n_0,2}^{tr}) = q_2 \tag{33}
\]

then the actuators \((\Omega_i, g_i)_{i=1,p}\) are efficient.

**Proof**
Derives from (31) and the fact that \( \text{ker} (M_{n_0} G_{n_0,l}^{tr}) = \{0\} \) for \( l = 1, 2 \) by using (33). \( \square \)

Let us remark that, the condition \( p \geq \sup_n r_n \) is necessary for actuators \((D_i, g_i)_{i=1,p}\) to be strategic, but it is not necessary to them in order to be efficient, and the condition \( q \leq p \) is not necessary for actuators to be efficient.

On the other hand, in the case of pointwise actuators, the characterization of the remediability is given by analogous properties to those seen for zone actuators.

### 4 Asymptotic controllability and asymptotic remediability

In this part, we study the relationship between controllability and remediability, and therefore between strategic and efficient actuators.

**Proposition 4.1** If (8) is exactly controllable asymptotically, then (3)+(4) is exactly remediable asymptotically.

**Proof**
For \( \theta \in Y' \), we have
\[
\| \mathcal{S}^{*}(\cdot)\mathcal{C}^{\ast}\|_{L^{2}(0,\infty;\mathcal{E}')}^{2} = \int_{0}^{\infty} \| \mathcal{S}^{*}(s)\mathcal{C}^{\ast}\|_{\mathcal{E}'}^{2} \, ds \\
\leq \int_{0}^{\infty} \| \mathcal{S}^{*}(s) \|^2 \, ds \| \mathcal{C}^{\ast}\|_{\mathcal{E}'}^{2} \\
\leq M\| \mathcal{C}^{\ast}\|_{\mathcal{E}'}^{2}
\]

with \( M > 0 \). Since (8) is exactly controllable, there exists \( \gamma_1 > 0 \) such that

\[
\| \mathcal{C}^{\ast}\|_{\mathcal{E}'} \leq \gamma_1 \| \mathcal{B}^{*}\mathcal{S}^{*}(\cdot)\mathcal{C}^{\ast}\|_{L^{2}(0,\infty;\mathcal{U}')} 
\]

consequently, there exists \( \gamma = M(\gamma_1)^2 > 0 \) such that

\[
\| \mathcal{S}^{*}(\cdot)\mathcal{C}^{\ast}\|_{L^{2}(0,\infty;\mathcal{E}')}^{2} \leq \gamma \| \mathcal{B}^{*}\mathcal{S}^{*}(\cdot)\mathcal{C}^{\ast}\|_{L^{2}(0,\infty;\mathcal{U}')}^{2} 
\]

and the result derives from proposition 3.2.

**Proposition 4.2** If (8) is weakly controllable asymptotically, then (3)+(4) is weakly remediable asymptotically.

**Proof**

(3)+(4) is weakly remediable asymptotically if and only if

\[
ker[\mathcal{B}^{*}(\mathcal{R}_{C}^{\infty})^{*}] = ker[(\mathcal{R}_{C}^{\infty})^{*}]
\]

this equivalent to

\[
ker[\mathcal{B}^{*}(\mathcal{R}_{C}^{\infty})^{*}] \subset ker[(\mathcal{R}_{C}^{\infty})^{*}]
\]

or

\[
ker[(\mathcal{H}^{\infty})^{*}\mathcal{C}^{*}] \subset ker[(\mathcal{R}_{C}^{\infty})^{*}]
\]

because \((\mathcal{H}^{\infty})^{*}\mathcal{C}^{*} = \mathcal{B}^{*}(\mathcal{R}_{C}^{\infty})^{*}\). Since \( ker[(\mathcal{H}^{\infty})^{*}] = \{0\} \), then for \( \theta \in ker[(\mathcal{H}^{\infty})^{*}\mathcal{C}^{*}] \), we have \( \theta \in ker[\mathcal{C}^{*}] \). The result derives immediately from the inclusion \( ker[\mathcal{C}^{*}] \subset ker[(\mathcal{R}_{C}^{\infty})^{*}] \).

In multi-actuators and multi-sensors cases, we have the following corollary.

**Corollary 4.3** Strategic actuators are necessarily efficient.

The converse is not true.
5 Cheap asymptotic remediability

In this part, we assume that the operators $K_C^\infty$ and $R_C^\infty$ are well defined, and that the considered system is weakly remediable asymptotically. We study the problem of exact asymptotic compensation with minimal energy, first in the case where the observation is exact and then in the case of an observation error.

5.1 Case of an observation without error

In this part, we suppose that for $f \in L^2(0, +\infty; L^2(\Omega))$, there exists a control $u \in L^2(0, +\infty; U)$ such that:

$$R_C^\infty f + K_C^\infty u = 0$$

i.e. that the set $D$ defined by

$$D = \{ u \in L^2(0, +\infty; U) \text{ such that } R_C^\infty f + K_C^\infty u = 0 \}$$

is non empty. Using the observation only, and with an extension of the Hilbert Uniqueness Method, we show hereafter how to find the optimal control ensuring the exact asymptotic compensation of the disturbance $f$, i.e. minimizing on $D$ the following cost function

$$J(u) = \|u\|^2_{L^2(0, +\infty; U)}$$

For $\theta \in Y' \equiv Y$, let

$$\|\theta\|_F = \left( \int_0^{+\infty} \|B^*S^*(t)C^*\theta\|_{U'}^2 dt \right)^{\frac{1}{2}}$$

$\|\cdot\|_F$ is a semi-norm. If $\text{Ker} C^* = \{0\}$, it is a norm if and only if (3)+(4) is weakly remediable asymptotically. Under this condition, we consider the space

$$F = \overline{Y}^{\|\cdot\|_F}$$

$F$ is a Hilbert space with the inner product

$$\forall \ \theta, \tau \in F$$

and the operator $\Lambda_C^\infty$ defined on $Y$ by

$$\Lambda_C^\infty \theta = \int_0^{+\infty} CS(t)BB^*S^*(t)C^*\theta dt = K_C^\infty(K_C^\infty)^* \theta$$

has a unique extension as an isomorphism from $F$ to $F'$ such that:
\begin{align*}
&\langle \Lambda_c^\infty \theta, \tau \rangle_Y = \langle \theta, \tau \rangle_F \quad \forall \theta, \tau \in F \quad \text{and} \quad \|\Lambda_c^\infty \theta\|_{F'} = \|\theta\|_F \quad \forall \theta \text{ in } F
\end{align*}

We have the following result.

**Proposition 5.1 (The optimal control)**
If \( R_c^\infty f \in F' \), then there exists a unique \( \theta_f \) in \( F \) such that

\begin{align*}
\Lambda_c^\infty \theta_f &= -R_c^\infty f \quad (35)
\end{align*}

and the control \( u_{\theta_f} \) defined by

\begin{align*}
u_{\theta_f}(t) &= B^*S^*(t)C^*\theta_f(t) \quad t > 0 \quad (36)
\end{align*}

verifies

\begin{align*}
R_c^\infty f + K_c^\infty u_{\theta_f} &= 0
\end{align*}

and is optimal with

\begin{align*}
\|u_{\theta_f}\|_{L^2(0, +\infty; U)} &= \|\theta_f\|_F
\end{align*}

**Proof:** We have

\begin{align*}
\Lambda_c^\infty \theta_f &= \int_0^{+\infty} CS(t)BB^*S^*(t)C^*\theta_f dt = K_c^\infty u_{\theta_f} = -R_c^\infty f
\end{align*}

The set \( D \) defined by (34) is closed, convex and non empty. Obviously, \( J \) admits a unique minimum in \( u^* \in D \) characterized by

\begin{align*}
\langle u^*, v - u^* \rangle \geq 0 \quad \forall v \in D
\end{align*}

For \( v \in D \), we have

\begin{align*}
\langle u_{\theta_f}, v - u_{\theta_f} \rangle &= \langle (K_c^\infty)^* \theta_f, v - (K_c^\infty)^* \theta_f \rangle \\
&= \langle \theta_f, K_c^\infty v - \Lambda_c^\infty \theta_f \rangle = 0
\end{align*}

Since \( u^* \) is unique, then \( u^* = u_{\theta_f} \) and \( u_{\theta_f} \) is optimal with

\begin{align*}
\|u_{\theta_f}\|^2 &= \|(K_c^\infty)^* \theta_f\|^2 = \langle \theta_f, \Lambda_c^\infty \theta_f \rangle = \|\theta_f\|_F^2
\end{align*}
5.2 Case of an observation error

In this part, we assume that the system (3) is augmented by the following output equation:

\[ w(t) = y(t) + e(t) \quad t \geq 0 \quad (37) \]

where \(y(t)\) is the exact observation given by (4) and \(e(t)\) is an observation error generally unknown but bounded.

In the normal case where \(f = 0\) and \(u = 0\), the 'normal' observation is given by

\[ \theta(t) = e(t) \]

But in the case of a disturbance \(f \neq 0\) and without control \((u = 0)\), the additional term, in the observation, corresponding to the disturbance \(f\) is given by

\[ R_t f \equiv w(t) - \theta(t) \]

Moreover, if a control term \(Bu(t)\) is introduced, the observation becomes

\[
\begin{align*}
    w(t) &= \theta(t) + \int_0^t CS(t-s)f(s)ds + \int_0^t CS(t-s)Bu(s)ds \\
    &= \theta(t) + R_t f + CH_t u \\
\end{align*}
\]

where

\[ CH_t u \equiv \int_0^t CS(t-s)Bu(s)ds \]

The problem of asymptotic compensation is then similar to that considered for an observation without error. It consists to study the existence of an input operator such that:

\[
\forall f \in L^2(0, +\infty; L^2(\Omega)), \exists u \in L^2(0, +\infty; U) \text{ such that the corresponding observation, noted } w_u, \text{ satisfies the asymptotic condition}
\]

\[ w_u(t) - \theta(t) \longrightarrow 0 \text{ when } t \longrightarrow +\infty \]

With the same notations, this problem can be formulated as follows:

For \( f \in L^2(0, +\infty; L^2(\Omega))\), does exists a control \( u \in L^2(0, +\infty; U) \) such that
\[ R^\infty_c f + K^\infty_c u = 0 \]

The results are similar and the optimal control ensuring the asymptotic compensation of a disturbance \( f \) is given by proposition 5.1.

6 Applications

Here \( \Omega \) is an open and bounded subset of \( \mathbb{R}^n \), with a sufficiently regular boundary \( \Gamma = \partial \Omega \). We consider the system described by the following hyperbolic equation

\[
\begin{cases}
\frac{\partial^2 x}{\partial t^2}(\xi,t) = \Delta x(\xi,t) + \sum_{i=1}^{p} g_i(\xi)u_i(t) + f(\xi,t) \\
\text{in } \Omega \times ]0, +\infty[ \\
x(\xi,0) = \frac{\partial x}{\partial t}(\xi,0) = 0 \text{ on } \Omega \\
x(\eta,t) = 0 \text{ on } \partial \Omega \times ]0, +\infty[ 
\end{cases}
\]

The system (38) is equivalent to

\[
\begin{cases}
\frac{\partial z}{\partial t}(\xi,t) = Az(\xi,t) + (0, \sum_{i=1}^{p} g_i(\xi)u_i(t))^{tr} + (0, f(\xi,t))^{tr} \\
\text{in } \Omega \times ]0, +\infty[ \\
z(\xi,0) = 0 \text{ on } \Omega 
\end{cases}
\]

is augmented by the output equation

\[ y(t) = Cz(t) \]

6.1 Case of a one dimension domain

For \( \Omega = ]0, \alpha[, \) the Laplacian operator \( \Delta \) admits an orthonormal basis of eigenfunctions defined by

\[ \psi_n(\xi) = \sqrt{\frac{2}{\alpha}} \sin\left(\frac{n\pi\xi}{\alpha}\right); n \geq 1 \]

The associated eigenvalues are simple \( (r_n = 1 \text{ for } n \geq 1) \) and given by

\[ \lambda_n = -n^2\frac{\pi^2}{\alpha^2}; n \geq 1 \]
In the case of sensors \((D,h)\) and \((D',k)\), with

\[ D = \text{supp}(h) \subset ]0, \alpha[ \]

and

\[ D' = \text{supp}(k) \subset ]0, 1[ \]

For example, if \(g_1 = \varphi_{n_0}\), the operators \(H^\infty\) and \(K^\infty_C\) are well defined. In this case, \((\Omega_1, g_1)\) is not strategic, because the condition:

\[
\int_{\Omega} g_1(\xi) \sin\left(\frac{n\pi \xi}{\alpha}\right) d\xi \neq 0 ; \ n \geq 1
\]

is not verified. On the other hand, for \(h = k = \varphi_{n_0}\), the operator \(R^\infty_C\) is well defined and \((\Omega_1, g_1)\) is efficient.

### 6.2 Case of rectangle

For \(\Omega = ]0, \alpha[ \times ]0, \beta[\), the eigenvectors of \(\Delta\) are defined by:

\[ \varphi_{m,n}(\xi_1, \xi_2) = \frac{2}{\sqrt{\alpha \beta}} \sin\left(\frac{n\pi \xi_1}{\alpha}\right) \sin\left(\frac{n\pi \xi_2}{\beta}\right) \]

for all \(m, n \geq 1\). The associated eigenvalues are

\[ \lambda_{m,n} = -\left(\frac{m^2}{\alpha^2} + \frac{n^2}{\beta^2}\right) \pi^2 ; \ m, n \geq 1 \]

We show that:

1- If \(\frac{\alpha^2}{\beta^2} \notin Q\), then the eigenvalues are simple and a single actuator \((\Omega_1, g_1)\) with \(\Omega_1 = \text{supp}(g_1) \subset \Omega\) may be sufficient to ensure the weak controllability. Indeed, \((\Omega_1, g_1)\) is strategic if and only if \(\langle g_1, \varphi_{m,n} \rangle \neq 0\) for all \(m, n \geq 1\), i.e

\[
\int_{\Omega_1} g_1(\xi_1, \xi_2) \sin\left(\frac{m\pi \xi_1}{\alpha}\right) \sin\left(\frac{n\pi \xi_2}{\beta}\right) d\xi_1 d\xi_2 \neq 0
\]

2- In the case of a square domain with \(\alpha = \beta = 1\), we have:

\[ \lambda_{m,n} = -(m^2 + n^2) \text{ and } \sup_{m,n \geq 1} r_{m,n} = +\infty \]

and then we can not have the weak controllability by a finite number of actuators. On the other hand, \((3)+(4)\) is weakly remediable if there exists \(m_0, n_0 \geq 1\) such that
\[ \text{rank}(M_{n_0} C_{n_0,1}^{tr}) = q_1 \quad \text{and} \quad \text{rank}(M_{n_0} C_{n_0,2}^{tr}) = q_2 \] (41)

Henceforth, in the case of sensors \((D, h)\) and \((D', k)\), an actuator may be efficient for any \(\alpha\) and \(\beta\). Indeed, \((\Omega_1, g_1)\) is efficient if there exists \(m_0, n_0 \geq 1\) such that

\[
\sum_{j=1}^{r_{m_0,n_0}} \langle g_1, \varphi_{m_0,n_0} \rangle_{\Omega} \langle h, \varphi_{m_0,n_0} \rangle_{\Omega} \neq 0
\]

\[
\sum_{j=1}^{r_{m_0,n_0}} \langle g_1, \varphi_{m_0,n_0} \rangle_{\Omega} \langle k, \varphi_{m_0,n_0} \rangle_{\Omega} \neq 0
\]

that is the case if for example \(g_1 = h = k = \varphi_{m_0,n_0}\). The actuator \((\Omega_1, g_1)\) is then efficient but not strategic.

### 6.2.1 Remark

i) In the case where \(\frac{a^2}{\beta^2} \not\in Q\), we have \(r_{m,n} = 1; \forall \geq 1\), the condition (42) becomes

\[ \langle g_1, \varphi_{m_0,n_0} \rangle_{\Omega} \langle h, \varphi_{m_0,n_0} \rangle_{\Omega} \neq 0 \]

i.e.

\[
\int_{\Omega_1} g_1(\xi_1, \xi_2) \sin\left(\frac{m_0 \pi \xi_1}{\alpha}\right) \sin\left(\frac{n_0 \pi \xi_2}{\beta}\right) d\xi_1 d\xi_2 \int_D k(\xi_1, \xi_2) \sin\left(\frac{m \pi \xi_1}{\alpha}\right) \sin\left(\frac{n \pi \xi_2}{\beta}\right) d\xi_1 d\xi_2 \neq 0
\]

with \(D = \text{supp}(h) \subset \Omega\).

ii) In the case of square domain, the condition becomes:

\[
\sum_{j=1}^{r_{m_0,n_0}} \int_{\Omega_1} g_1(\xi_1, \xi_2) \sin(m_0 j \pi \xi_1) \sin(n_0 j \pi \xi_2) d\xi_1 d\xi_2 .
\]

\[
\int_D h(\xi_1, \xi_2) \sin(m_0 j \pi \xi_1) \sin(n_0 j \pi \xi_2) d\xi_1 d\xi_2 \neq 0
\]

and

\[
\sum_{j=1}^{r_{m_0,n_0}} \int_{\Omega_1} g_1(\xi_1, \xi_2) \sin(m_0 j \pi \xi_1) \sin(n_0 j \pi \xi_2) d\xi_1 d\xi_2 .
\]

\[
\int_{D'} k(\xi_1, \xi_2) \sin(m_0 j \pi \xi_1) \sin(n_0 j \pi \xi_2) d\xi_1 d\xi_2 \neq 0
\]
One actuator may be efficient, but a finite number of actuators cannot be strategic. Thus, for example if \( g_1 = h = k = \varphi_{m_0, n_0} \), the actuator \((\Omega_1, g_1)\) is efficient.

7 Approximations and numerical simulations

This section concerns approximations and numerical simulations of the problem of asymptotic compensation. First, we show how to find an approximation of \( \theta_f \) given by (35) as a solution of a finite dimension linear system

\[
Ax = b
\]

and then the optimal control \( u_{\theta_f} \) given by (36) with a comparison between the corresponding observation noted \( y_{(u_{\theta_f}, f)} = \begin{pmatrix} y_{1,(u_{\theta_f}, f)} \\ y_{2,(u_{\theta_f}, f)} \end{pmatrix} \) and the normal case. The observations \( y_{1,(u_{\theta_f}, f)} \) and \( y_{2,(u_{\theta_f}, f)} \) concern respectively the state of the system and its derivative (speed).

7.1 Approximations

- Coefficients of the system:

Let \( Q(.) \) be the operator defined by

\[
Q(.) : \mathbb{R}^p \longrightarrow Y = Y_1 \times Y_2 = \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}
\]

\[
x \longrightarrow \mathcal{C}S(.)Bx
\]

then

\[
\Lambda_{C}^{\infty} \theta_f = \int_{0}^{+\infty} Q(s)Q^*(s)\theta_f ds
\]

and \( a_{ij} = \langle \Lambda_{C}^{\infty} e_j, e_i \rangle_Y \) where \((e_i)_{1 \leq i \leq q_1.q_2}\) is the canonical basis of \( \mathbb{R}^{q_1} \times \mathbb{R}^{q_2} \), we have

\[
Q(s)x = \mathcal{C}S(s)Bx = \begin{pmatrix}
\sum_{m \geq 1} \sum_{l=1}^{r_m} \frac{1}{\sqrt{-\lambda_m}} \langle Bx, \varphi_{ml} \rangle_{L^2(\Omega)} \sin(\sqrt{-\lambda_m s})C_1 \varphi_{ml} \\
\sum_{m \geq 1} \sum_{l=1}^{r_m} \langle Bx, \varphi_{ml} \rangle_{L^2(\Omega)} \cos(\sqrt{-\lambda_m s})C_2 \varphi_{ml}
\end{pmatrix}
\]
and \( Q^*(: Y \rightarrow \mathbb{R}^p \) is defined by

\[
Q^* \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) = \sum_{m \geq 1} \sum_{l=1}^{r_m} \frac{1}{\sqrt{-\lambda_m}} \sin(\sqrt{-\lambda_m}) B^* \varphi_{ml} < \varphi_{ml}, C_1^* y_1 >_{L^2(\Omega)}
\]

\[+ \sum_{m \geq 1} \sum_{l=1}^{r_m} \cos(\sqrt{-\lambda_m}) B^* \varphi_{ml} < \varphi_{ml}, C_2^* y_2 >_{L^2(\Omega)} \]

then

\[
\Lambda^\infty_{\mathcal{C}} \theta_f = \left( \begin{array}{c}
\sum_{m \geq 1} \sum_{l=1}^{r_m} \int_0^{+\infty} \frac{1}{\sqrt{-\lambda_m}} < B Q^*(s) \theta_f, \varphi_{ml} >_{L^2(\Omega)} \sin(\sqrt{-\lambda_m}) C_1 \varphi_{ml} \\
\sum_{m \geq 1} \sum_{l=1}^{r_m} \int_0^{+\infty} < B Q^*(s) \theta_f, \varphi_{ml} >_{L^2(\Omega)} \cos(\sqrt{-\lambda_m}) C_2 \varphi_{ml}
\end{array} \right)
\]

For \( i, j \geq 1 \), we have

\[
a_{ij} = \sum_{m \geq 1} \sum_{l=1}^{r_m} \sum_{n \geq 1} \sum_{k=1}^{r_n} \sum_{r=1}^{r_p} W(m, l, n, k, r)
\]

where

\[
W(m, l, n, k, r) = \begin{cases}
\frac{1}{\sqrt{-\lambda_m}} \frac{1}{\sqrt{-\lambda_n}} \int_0^{+\infty} \sin(\sqrt{-\lambda_m} s) \sin(\sqrt{-\lambda_n} s) ds < g_r, \varphi_{nk} >_{L^2(\Omega)} < h_1^i, \varphi_{nl} >_{L^2(D_1)} \\
+ \frac{1}{\sqrt{-\lambda_m}} \int_0^{+\infty} \sin(\sqrt{-\lambda_m} s) \cos(\sqrt{-\lambda_n} s) ds < g_r, \varphi_{nk} >_{L^2(\Omega)} < h_1^i, \varphi_{nl} >_{L^2(D_1)} \\
+ \frac{1}{\sqrt{-\lambda_n}} \int_0^{+\infty} \cos(\sqrt{-\lambda_m} s) \sin(\sqrt{-\lambda_n} s) ds < g_r, \varphi_{nk} >_{L^2(\Omega)} < h_1^i, \varphi_{nl} >_{L^2(D_1)} \\
+ \int_0^{+\infty} \cos(\sqrt{-\lambda_m} s) \cos(\sqrt{-\lambda_n} s) ds < g_r, \varphi_{nk} >_{L^2(\Omega)} < h_1^i, \varphi_{nl} >_{L^2(D_1)} \\
< h_2^i, \varphi_{nk} >_{L^2(\Omega)} < h_2^i, \varphi_{nl} >_{L^2(D_1)}
\end{cases}
\]

and for \( M, N \) sufficiently large

\[
a_{ij} \simeq \sum_{m=1}^{M} \sum_{l=1}^{r_m} \sum_{n=1}^{N} \sum_{k=1}^{r_n} \sum_{r=1}^{r_p} W(m, l, n, k, r)
\]
and

\[ b_i = -< R_C^\infty \tilde{f}, e_i >_{Y = \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}} \]

\[ = - \sum_{n \geq 1} \sum_{k=1}^{r_n} \left( \int_{0}^{+\infty} \frac{1}{\sqrt{-\lambda_n}} \sin(\sqrt{-\lambda_n} s) < f(s, \cdot), \varphi_{nk} >_{L^2(\Omega)} ds < h^1_i, \varphi_{nk} > + \int_{0}^{+\infty} \cos(\sqrt{-\lambda_n} s) < f(s, \cdot), \varphi_{nk} >_{L^2(\Omega)} ds < h^2_i, \varphi_{nk} > \right) \]

\[ \simeq - \sum_{n=1}^{N} \sum_{k=1}^{r_n} \left( \int_{0}^{+\infty} \frac{1}{\sqrt{-\lambda_n}} \sin(\sqrt{-\lambda_n} s) < f(s, \cdot), \varphi_{nk} >_{L^2(\Omega)} ds < h^1_i, \varphi_{nk} > + \int_{0}^{+\infty} \cos(\sqrt{-\lambda_n} s) < f(s, \cdot), \varphi_{nk} >_{L^2(\Omega)} ds < h^2_i, \varphi_{nk} > \right) \]

-- The optimal control: The optimal control \( u_{\theta f} \) is defined by

\[ u_{\theta f}(s) = B^* S^*(s) C^* \theta_f \]

Its function coordinates \( u_{j,\theta f}(\cdot) \) are given by

\[ u_{j,\theta f}(s) = \sum_{n \geq 1} \sum_{k=1}^{r_n} \sum_{\iota=1}^{q_1} \sqrt{-\lambda_n} \theta^\iota_{1,f} < h^1_i, \varphi_{nk} >_{L^2(\Omega)} \sin(\sqrt{-\lambda_n} s) < g_j, \varphi_{nk} >_{L^2(\Omega_j)} + \sum_{n \geq 1} \sum_{k=1}^{r_n} \sum_{\tau=1}^{q_2} \theta^\tau_{2,f} < h^2_i, \varphi_{nk} >_{L^2(\Omega)} \cos(\sqrt{-\lambda_n} s) \varphi_{nk} < g_j, \varphi_{nk} >_{L^2(\Omega_j)} \]

\[ \simeq \sum_{n=1}^{N} \sum_{k=1}^{r_n} \sum_{\iota=1}^{q_1} \sqrt{-\lambda_n} \theta^\iota_{1,f} < h^1_i, \varphi_{nk} >_{L^2(\Omega)} \sin(\sqrt{-\lambda_n} s) < g_j, \varphi_{nk} >_{L^2(\Omega_j)} + \sum_{n=1}^{N} \sum_{k=1}^{r_n} \sum_{\tau=1}^{q_2} \theta^\tau_{2,f} < h^2_i, \varphi_{nk} >_{L^2(\Omega)} \cos(\sqrt{-\lambda_n} s) \varphi_{nk} < g_j, \varphi_{nk} >_{L^2(\Omega_j)} \]

for \( N \) sufficiently large.

-- Cost: The cost is given by \( \| u_{\theta f} \|_{L^2(0, +\infty, \mathbb{R}^p)} \)

\[ \| u_{\theta f} \|_{L^2(0, +\infty, \mathbb{R}^p)} = \left( \int_{0}^{+\infty} \| B^* S^*(s) C^* \theta_f \|_{\mathbb{R}^p}^2 ds \right)^{\frac{1}{2}} \]
\[
\sum_{j=1}^{p} \left( \sum_{n \geq 1} \sum_{k=1}^{r_n} \sum_{i=1}^{q_1} \sqrt{-\lambda_n} \theta_{i,j} \int_{0}^{+\infty} \sin(\sqrt{-\lambda_n} \theta_j) ds < h^1_j, \varphi_{n,k} > L^2(D^1_j), \int_{0}^{+\infty} \cos(\sqrt{-\lambda_n} \theta_j) ds < h^2_j, \varphi_{n,k} > L^2(D^2_j) \right)^{2} \]
\]

\[
\sum_{j=1}^{p} \left( \sum_{n \geq 1} \sum_{k=1}^{r_n} \sum_{i=1}^{q_2} \sqrt{-\lambda_n} \theta_{i,j} \int_{0}^{+\infty} \sin(\sqrt{-\lambda_n} \theta_j) ds < h^1_j, \varphi_{n,k} > L^2(D^2_j), \int_{0}^{+\infty} \cos(\sqrt{-\lambda_n} \theta_j) ds < h^2_j, \varphi_{n,k} > L^2(D^2_j) \right)^{2} \]

The corresponding observation:

The coordinates \( y_j^{u_{\theta_j,f}}(\cdot) \) of the observation \( y_{u_{\theta_j,f}}(\cdot) \) corresponding to the control \( u_{\theta_j} \) are obtained as follows:

\[
y(u_{\theta_j,f})(t) = \begin{pmatrix} y_{1,u_{\theta_j}}(t) \\ y_{2,u_{\theta_j}}(t) \end{pmatrix} = Cz_{u_{\theta_j}}(t) = C \left( \int_{0}^{t} S(s)B_{u_{\theta_j}}(s) ds + \int_{0}^{t} S(s)B_{f}(s) ds \right)
\]

and

\[
y_{1,u_{\theta_j}}(t) = \sum_{n \geq 1} \sum_{k=1}^{r_n} \sum_{i=1}^{p} \frac{1}{\sqrt{-\lambda_n}} \int_{0}^{t} u_{i,j}(s) \sin(\sqrt{-\lambda_n} \theta_j) ds < h^1_j, \varphi_{n,k} > L^2(D^1_j)
\]

\[
+ \sum_{n \geq 1} \sum_{k=1}^{r_n} \frac{1}{\sqrt{-\lambda_n}} \int_{0}^{t} < f(s,), \varphi_{n,k} > L^2(D^2_j) \sin(\sqrt{-\lambda_n} \theta_j) ds < h^1_j, \varphi_{n,k} > L^2(D^1_j)
\]

\[
y_{2,u_{\theta_j}}(t) = \sum_{n \geq 1} \sum_{k=1}^{r_n} \sum_{i=1}^{p} \int_{0}^{t} u_{i,j}(s) \cos(\sqrt{-\lambda_n} \theta_j) ds < h^2_j, \varphi_{n,k} > L^2(D^2_j)
\]

\[
+ \sum_{n \geq 1} \sum_{k=1}^{r_n} \int_{0}^{t} < f(s,), \varphi_{n,k} > L^2(D^2_j) \cos(\sqrt{-\lambda_n} \theta_j) ds < h^2_j, \varphi_{n,k} > L^2(D^2_j)
\]

\[
y_{1,u_{\theta_j}}(t) \simeq \sum_{n \geq 1} \sum_{k=1}^{r_n} \sum_{i=1}^{p} \frac{1}{\sqrt{-\lambda_n}} \int_{0}^{t} u_{i,j}(s) \sin(\sqrt{-\lambda_n} \theta_j) ds < h^1_j, \varphi_{n,k} > L^2(D^1_j)
\]

\[
+ \sum_{n \geq 1} \sum_{k=1}^{r_n} \frac{1}{\sqrt{-\lambda_n}} \int_{0}^{t} < f(s,), \varphi_{n,k} > L^2(D^2_j) \sin(\sqrt{-\lambda_n} \theta_j) ds < h^1_j, \varphi_{n,k} > L^2(D^1_j)
\]

\[
y_{2,u_{\theta_j}}(t) \simeq \sum_{n \geq 1} \sum_{k=1}^{r_n} \sum_{i=1}^{p} \int_{0}^{t} u_{i,j}(s) \cos(\sqrt{-\lambda_n} \theta_j) ds < h^2_j, \varphi_{n,k} > L^2(D^2_j)
\]

\[
+ \sum_{n \geq 1} \sum_{k=1}^{r_n} \int_{0}^{t} < f(s,), \varphi_{n,k} > L^2(D^2_j) \cos(\sqrt{-\lambda_n} \theta_j) ds < h^2_j, \varphi_{n,k} > L^2(D^2_j)
\]

for \( N \) large enough.
7.2 Numerical simulations

7.2.1 Dirichlet case

We consider without loss of generality the diffusion system (38) with \( \Omega = [0, \alpha[ \) and a Dirichlet boundary condition, the functions \( \varphi_n(\cdot) \) are defined by:

\[
\varphi_n(\xi) = \sqrt{\frac{2}{\alpha}} \sin\left(\frac{n\pi \xi}{\alpha}\right); n \geq 1
\]

The associated eigenvalues are simple and given by

\[
\lambda_n = -n^2 \frac{\pi^2}{\alpha^2}; n \geq 1
\]

Then in the case of:

- an initial state: \( z_0(\cdot) \equiv 0 \),
- a sensor: \((D_1, h_1), (D_2, h_2)\), with \( h_1(\xi) = h_2(\xi) = \frac{\sqrt{2}}{2} \) and \( D_1 = D_2 = [0, 1[ \),
- an efficient actuator \((\Omega, g)\) with \( \Omega = [0, 1[ \) and \( g(\xi) = \cos\left(\frac{\pi}{3} \xi\right) \)
- a disturbance function

\[
f(t, \xi) = 240 e^{-\left(\frac{1}{10} t + \xi\right)}; \ t > 0
\]

and for \( M = N = 1 \), we obtain numerical results illustrating the theoretical results established in previous sections. Hence, in figure 1, we give the representation of the observations \( y_{1,(u_0,f)} \equiv y_{1,(u,f)} \), \( y_{1,(0,f)} \) and \( y_{1,(0,0)} \). This figure show that for \( t \) sufficiently large \((t=50)\), we have

\[
y_{(u_0,f)}(t) \simeq y_{(0,0)}(t)
\]
Figure 1: Representation of $y_{1,(u_{\theta_1},f)}$, $y_{1,(0,0)}$ and $y_{1,(0,f)}$ in the Dirichlet case.

In figure 2, we give the representation of the observations $y_{2,(u_{\theta_1},f)} \equiv y_{2,(u,f)}$, $y_{2,(0,f)}$ and $y_{2,(0,0)}$.

Figure 2: Representation of $y_{2,(u_{\theta_1},f)}$, $y_{2,(0,0)}$ and $y_{2,(0,f)}$ in the Dirichlet case.

In figure 3, we give the representation of the optimal control $u_{\theta_1}$ ensuring the asymptotic compensation of the disturbance $f$. 
Figure 2: Representation of the optimal control $u_\theta$, in the Dirichlet case.

The numerical approach is analogous for Neumann boundary conditions and can be extended easily for a two dimension domain (rectangle).

**Conclusion**

In this paper, we have defined and characterized the notions of weak and exact asymptotic controllability and strategic actuators. Then, we have introduced and characterized the weak and exact asymptotic remediability and asymptotically efficient actuators.

We have equally studied the relationship between the asymptotic remediability and the notion of asymptotic controllability. More precisely, we have shown that also in the asymptotic case, the remediability is a weaker notion than the controllability and hence that strategic actuators are efficient. The converse is not true.

Using the observation only, we have also shown how to find the optimal control ensuring the asymptotic compensation of a disturbance in the case where the observation is exact or affected by an error.

As an application, we have considered the wave equation in the case where the domain $\Omega$ is an interval or a rectangle.

In this application, we particularly show that also in the asymptotic case, the remediability is weaker and more flexible than the controllability, and an actuator can be efficient without being strategic.
The obtained results are developed for a class of linear hyperbolic systems, but can be extended to other systems or other situations.

These results are illustrated by those obtained numerically in one dimension case with a Dirichlet boundary condition. The numerical approach can be adapted to other domain and boundary conditions.

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