Algorithmic reducibilities of algebraic structures

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A countable algebraic structure $\mathcal{M}$ is called (\(x\)-) computable, if for some $\mathfrak{N} \cong \mathcal{M}$ we have $|\mathfrak{N}| \subseteq \omega$ and the atomic diagram $D(\mathfrak{N})$ (\(x\)-) is computable.
A countable algebraic structure \( M \) is called \((x-)\) computable, if for some \( \mathcal{N} \cong M \) we have \( |\mathcal{N}| \subseteq \omega \) and the atomic diagram \( D(\mathcal{N}) \) is computable.

A countable algebraic structure \( M \) is called \((x-)\) decidable, if for some \( \mathcal{N} \cong M \) we have \( |\mathcal{N}| \subseteq \omega \) and the complete diagram \( D^*(\mathcal{N}) \) is \((x-)\) computable.
A countable algebraic structure $\mathcal{M}$ is called \( (\mathbf{x-}) \) computable, if for some $\mathcal{N} \cong \mathcal{M}$ we have $|\mathcal{N}| \subseteq \omega$ and the atomic diagram $D(\mathcal{N})$ (\( \mathbf{x-} \)) is computable.

A countable algebraic structure $\mathcal{M}$ is called \( (\mathbf{x-}) \) decidable, if for some $\mathcal{N} \cong \mathcal{M}$ we have $|\mathcal{N}| \subseteq \omega$ and the complete diagram $D^*(\mathcal{N})$ is \( (\mathbf{x-}) \) computable.
The degree spectrum of an algebraic structure $\mathcal{M}$ is the collection $\text{Sp} (\mathcal{M})$ of all Turing degrees $x$ such that $\mathcal{M}$ is $x$-computable.

The strong degree spectrum of an algebraic structure $\mathcal{M}$ is the collection $\text{Ssp} (\mathcal{M})$ of all Turing degrees $x$ such that $\mathcal{M}$ is $x$-decidable.

If the degree spectrum of an algebraic structure $\mathcal{M}$ has a least element $a$ (that is, if $\text{Sp} (\mathcal{M}) = \{x | x \geq a\}$), then we say that $\mathcal{M}$ has the degree $a$. 

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- In fact, for each countable \( M \) and every incomparable \( b, c \in \text{Sp}(M) \) there is a \( a, a' \leq c' \), incomparable with \( b \) and \( c \) s.t. \( a \in \text{Sp}(M) \).
We say that a structure $\mathfrak{A}$ is reducible to a structure $\mathfrak{B}$ ($\mathfrak{A} \leq_r \mathfrak{B}$), if $\text{Sp} (\mathfrak{B}) \subseteq \text{Sp} (\mathfrak{A})$. 
We say that a structure $\mathcal{A}$ is \textbf{reducible} to a structure $\mathcal{B}$ ($\mathcal{A} \leq_r \mathcal{B}$), if $\text{Sp}(\mathcal{B}) \subseteq \text{Sp}(\mathcal{A})$.

We say that a structure $\mathcal{A}$ is \textbf{uniformly reducible} to a structure $\mathcal{B}$ ($\mathcal{A} \leq_{ur} \mathcal{B}$), if there is an uniform procedure which builds a copy of the structure $\mathcal{A}$ given any copy of the structure $\mathcal{B}$. That is, there is a Turing operator $\Phi$ such that for all $\mathcal{N}$, $|\mathcal{N}| \subseteq \omega$,

$$\mathcal{N} \cong \mathcal{B} \implies (\exists \mathcal{M} \cong \mathcal{A})[|\mathcal{M}| \subseteq \omega \& D(\mathcal{M}) = \Phi^{D(\mathcal{N})}]$$.
For $A \subseteq \omega$ define the undirected graph $\text{Enum}(A)$, consisting from disjoint $n + 3$-cycles, where $n \in A$. 
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$\mathcal{Enum}(A)$ has a degree $\iff A \equiv_e \text{graph } (f)$, $f$ is a total function. In this case, the e-degree of the set $A$ is called total.
For $A \subseteq \omega$ define the undirected graph $\mathcal{E}num(A)$, consisting from disjoint $n + 3$-cycles, where $n \in A$.

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(Knight, Ash) A structure $\mathcal{A}$ has a degree iff there are a finite collection $\vec{a}$ from $\mathcal{A}$ and a total function $f$ such that $\text{Th}_\exists (\mathcal{A}, \vec{a}) \equiv_e \text{graph} (f)$ and $\text{deg}(f) \in \text{Sp} (\mathcal{A})$. 
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Hence, if $\mathfrak{A}$ has a degree and $\mathfrak{B} \leq_r \mathfrak{A}$, then $\mathfrak{B} \leq_{\text{ur}} (\mathfrak{A}, \vec{a})$ for some $\vec{a}$ from $\mathfrak{A}$.
Theorem. (2009). If a structure $\mathcal{A}$ has a jump degree but has not a degree, then there is a structure $\mathcal{B}$ such that $\mathcal{B} \leq_r \mathcal{A}$ and $\mathcal{B} \not\leq_{ur} (\mathcal{A}, \bar{a})$ for every $\bar{a}$ from $\mathcal{A}$. 

The jump degree of a structure $\mathcal{A}$ is the least Turing jump of the elements of $Sp(\mathcal{A})$.

(Do wney, Coles, Slaman, 2000) The structure $Enum(\mathcal{A})$ always has a jump degree.

Corollary. The following conditions are equivalent:  
1) The e-degree of a set $\mathcal{A}$ is total;  
2) $(\forall \mathcal{B})(\mathcal{B} \leq_r Enum(\mathcal{A}) \Rightarrow \mathcal{B} \leq_{ur} Enum(\mathcal{A}))$. 

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Uniformity vs. non-Uniformity

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**Corollary.** The following conditions are equivalent:
1) The e-degree of a set $\mathcal{A}$ is total;
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Let $D_e = 2^\omega / \equiv_e$ be the upper semilattice of e-degrees with the least element $0_e$. 

(Sorbi, 1997) $D_e$ is not a lattice.

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What about the upper semilattices $D_r$ and $D_u$?
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Let a structure \( \mathcal{M} \) is \( X \)-computable for every non-computable \( X \). Must \( \mathcal{M} \) be computable?
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**Theorem.** (Slaman, 1999; Wehner, 1999; Hirschfeldt, 2007). There are structures $M$ such that $\text{Sp}(M) = \{x | x > 0\}$ and
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**Theorem.** (Slaman, 1999; Wehner, 1999; Hirschfeldt, 2007). There are structures $\mathcal{M}$ such that $\text{Sp}(\mathcal{M}) = \{x | x > 0\}$ and $T\text{h}(\mathcal{M})$ has no computable models.

Corollary. Both $\mathcal{D}$ and $\mathcal{D}_r$ contain the least nonzero element.
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**Corollary.** Both $D_r$ and $D_{ur}$ contain the least nonzero element.

**Theorem** (2009). There is a computable structure $\mathcal{M}$ such that $\text{Ssp}(\mathcal{M}) = \{ x | x > 0 \}$. 

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Almost computable structures

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**Question.** Is there an arithmetical degree which computes every almost computable structure?
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**Theorem.** (2008). There is a degree $a \leq 0''$ such that $\text{Sp}(\mathcal{M}) \neq \{\mathcal{x} | \mathcal{x} \not\leq a\}$ for every $\mathcal{M}$.
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**Theorem.** (2008). There is a degree $a \leq 0''$ such that $\text{Sp}(\mathcal{M}) \neq \{x \mid x \not\leq a\}$ for every $\mathcal{M}$.

To find such an $a \leq 0^{(4)}$ we prove that for every incomparable $b$ and $c$ there exists an $a \leq (b \cup c)^{(4)}$ such that for each $\mathcal{M}$

$$\{b, c\} \subseteq \text{Sp}(\mathcal{M}) \implies a \in \text{Sp}(\mathcal{M}).$$
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To make $a \leq 0''$ we prove that for every $c > 0$ there exist $a, b \leq c''$ such that for each $\mathcal{M}$

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The relativized Lempp’s question II

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- Let a structure $\mathcal{M}$ is $X$-computable for every $X \not\leq_T A$. Must $\mathcal{M}$ be $A$-computable?
The relativized Lempp’s question II

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**Theorem.** (2007, 2008). If a degree $a$ is low or c.e. then there is a structure $\mathcal{M}$ such that $\text{Sp}(\mathcal{M}) = \{x | x \not\leq a\}$. 

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**Theorem.** Let $C$ be a uniformly $\Delta^0_2$ family which is closed downwards under $\leq_1$. Then there is a structure $\mathcal{M}$ such that $\text{Sp}(\mathcal{M}) = \{\text{deg}(X) | X' \not\in C\}$.
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**Theorem.** Let \( C \) be a uniformly \( \Delta^0_2 \) family which is closed downwards under \( \leq_1 \). Then there is a structure \( M \) such that \( \text{Sp}(M) = \{ \text{deg}(X) | X' \not\in C \} \).

In particular, \( \text{Sp}(M) \) can consist from the non-superlow degrees.
The idea of the proofs

- $\text{Sp}(\mathcal{M}) = \{x|x > 0\}$: (Wehner, 1999)

  $$S = \{\{n\} \oplus U | U \text{ is finite } \& U \neq W_n\}.$$
The idea of the proofs

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- \( \text{Sp}(M) = \{ x | x \not\leq a \} \): \( a = \deg(A) \) is low
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The idea of the proofs

- \( \text{Sp}(\mathcal{M}) = \{ x | x > 0 \} \): (Wehner, 1999)

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- \( \text{Sp}(\mathcal{M}) = \{ x | x' \not\in C \} \): \( C = \text{rng}(\nu), \nu \in \Delta^0_2 \)

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For the Uniformity vs. non-Uniformity result \( \nu(n) = W_n^{X_n} \).
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  For the Uniformity vs. non-Uniformity result \( \nu(n) = W_{X_n}^n \).

- \( \text{Sp}(M) = \{ x | x \not\preceq a \} \): \( a = \deg(A) \), \( A \) is c.e.

  \[
  S = \{ \{ n \} \oplus U | U \text{ is the image of an increasing p.r.f } \& U \neq W^A_n \}.
  \]
If \( a = b \cap c \) for low degrees \( a, b \) and \( c \), then
\[
\{ x \mid x \not\leq c \} = \{ x \mid x \not\leq a \} \cup \{ x \mid x \not\leq b \}. 
\]
Hence, \( D_r \) possess nontrivial infs.
Properties of $D_r$ and $D_{ur}$

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Properties of $D_r$ and $D_{ur}$

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- If both two structures have degrees which are low and incomparable to each other, then these two structures have no infimum in $D_r$ and $D_{ur}$. Hence, $D_r$ and $D_{ur}$ are not lattices.

- There are nonprincipal ideals in $D_r$ and $D_{ur}$ which have supremum.
For a structure $\mathcal{M}$ and an e-degree $\mathbf{x}$ we write $\mathcal{M} \leq_{e} \mathbf{x}$, if for some $\mathcal{N} \cong \mathcal{M}$, $|\mathcal{N}| \subseteq \omega$ we have $D(\mathcal{N}) \leq_{e} \mathbf{x}$. 

The e-spectrum of algebraic structure $\mathcal{M}$ is the collection $e\text{-Sp}(\mathcal{M})$ of all e-degrees $\mathbf{x}$ such that $\mathcal{M} \leq_{e} \mathbf{x}$.

We say that a structure $\mathcal{A}$ is e-reducible to a structure $\mathcal{B}$ ($\mathcal{A} \leq_{er} \mathcal{B}$), if $e\text{-Sp}(\mathcal{B}) \subseteq e\text{-Sp}(\mathcal{A})$.

We say that a structure $\mathcal{A}$ is uniformly e-reducible to a structure $\mathcal{B}$ ($\mathcal{A} \leq_{uer} \mathcal{B}$), if there is an e-operator $\Phi$ such that for all $\mathcal{N}$, $|\mathcal{N}| \subseteq \omega$, $\mathcal{N} \cong \mathcal{B} \Rightarrow (\exists \mathcal{M} \cong \mathcal{A})[|\mathcal{M}| \subseteq \omega \& D(\mathcal{M}) = \Phi(D(\mathcal{N}))]$. 

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Reducibilities of algebraic structures
For a structure $\mathcal{M}$ and an e-degree $\mathbf{x}$ we write $\mathcal{M} \leq_e \mathbf{x}$, if for some $\mathcal{N} \cong \mathcal{M}$, $|\mathcal{N}| \subseteq \omega$ we have $D(\mathcal{N}) \leq_e \mathbf{x}$.

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For a structure $\mathcal{M}$ and an e-degree $x$ we write $\mathcal{M} \leq_e x$, if for some $\mathcal{N} \cong \mathcal{M}$, $|\mathcal{N}| \subseteq \omega$ we have $D(\mathcal{N}) \leq_e x$.

The e-spectrum of algebraic structure $\mathcal{M}$ is the collection $\text{e-Sp}(\mathcal{M})$ of all e-degrees $x$ such that $\mathcal{M} \leq_e x$.

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Reducibilities of algebraic structures
For a structure \( M \) and an e-degree \( x \) we write \( M \leq_e x \), if for some \( N \cong M \), \( |N| \subseteq \omega \) we have \( D(N) \leq_e x \).

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\[ N \cong B \implies (\exists M \cong A)[|M| \subseteq \omega \land D(M) = \Phi(D(N))]. \]
Theorem. (2009). There is a structure $\mathcal{M}$ such that 
$e\text{-Sp}(\mathcal{M}) = \{x \in D_e | x > 0\}$.
In fact $\mathcal{M}$ codes the family $S = \{\{n\} \oplus U | U \text{ is c.e.} \land U \neq W_n\}$. 

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Reducibilities of algebraic structures
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Corollary. $D_{er}$ contains the least nonzero element.
(Stukachev, 2007).

\[ A \text{ is } \Sigma \text{-definable in } \mathbb{HF}(\mathcal{B}) \text{ without parameters} \]

\[ \Downarrow \]

\[ A \leq_{uer} \mathcal{B} \implies A \leq_{er} \mathcal{B} \]

\[ \Downarrow \]

\[ A \leq_{ur} \mathcal{B} \implies A \leq_{r} \mathcal{B} \]
Theorem.

1. $A \leq_{uer} B$ does not imply that $A$ is $\Sigma$-definable in $\mathbb{HF}(B)$;
2. $A \leq_{ur} B$ does not imply $A \leq_{er} B$;
3. $A \leq_{er} B$ does not imply $A \leq_{ur} B$;
4. $A \leq_{er} B$ and $A \leq_{ur} B$ do not imply $A \leq_{uer} B$;
5. $A \leq_{r} B$ does not imply $A \leq_{er} M$ or $A \leq_{ur} B$.

Everything above is correct up to finite constant enrichments.
Relationships between the reducibilities, III

Are the counterexamples from above are natural?
Are the counterexamples from above are natural?

1. $\mathcal{A} \leq uer \mathcal{B}$ does not imply that $\mathcal{A}$ is $\Sigma$-definable in $\mathbb{HF}(\mathcal{B})$; $\mathcal{A}$ codes the family $\{\{n\} \oplus U \mid U \text{ is c.e.} \& U \neq W_n\}$. $\mathcal{B}$ codes the family of all infinite c.e. sets.
Are the counterexamples from above are natural?

1. \( \mathcal{A} \leq_{uer} \mathcal{B} \) does not imply that \( \mathcal{A} \) is \( \Sigma \)-definable in \( \mathbb{HF}(\mathcal{B}) \);
   \( \mathcal{A} \) codes the family \( \{ \{ n \} \oplus U \mid U \text{ is c.e.} \& \ U \neq W_n \} \).
   \( \mathcal{B} \) codes the family of all infinite c.e. sets.

2. \( \mathcal{A} \leq_{ur} \mathcal{B} \) does not imply \( \mathcal{A} \leq_{er} \mathcal{B} \);
   \( \mathcal{A} \) codes the family of all graphs of computable functions.
   \( \mathcal{B} \) codes the family of all infinite c.e. sets.

3. ?

4. ??

5. ???