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An Associative and Noncommutative Product for the Low Energy Effective Theory of a D-Brane in Curved Backgrounds and Bi-Local Fields

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Abstract

We point out that when a D-brane is placed in an NS-NS B field background with non-vanishing field strength ($H = dB$) along the D-brane worldvolume, the coordinate of one end of the open string does not commute with that of the other in the low energy limit. The degrees of the freedom associated with both ends are not decoupled and accordingly, the effective action must be quite different from that of the ordinary noncommutative gauge theory for a constant B background. We construct an associative and noncommutative product $\star$ which operates on the coordinates of both ends of the string and propose a new type of noncommutative gauge action for the low energy effective theory of a Dp-brane. This effective theory is bi-local and lives in twice as large dimensions ($2D = 2(p+1)$) as in the $H = 0$ case. When viewed as a theory in the D-dimensional space, this theory is non-local and we must force the two ends of the string to coincide. We will then propose a prescription for reducing this bi-local effective action to that in D dimensions and obtaining a local effective action.

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1 Introduction

Noncommutative field theories[1][2] are obtained in an $\alpha' \to 0$ limit of the open string theory. This noncommutativity is inherent in the open string theory[3]. By considering an open string in an NS-NS B field background and taking a specific field theory limit the effects of the B field are encoded into a special multiplication rule of functions, the $*$ product or Moyal product. By studying such field theories we expect that open string physics may be understood in the framework of field theory. Until recently such investigations are mostly restricted to constant B backgrounds[4][6][5][7]. The algebra of the coordinates of the end points of the open string was studied, the connection of the commutative and noncommutative descriptions of the gauge theory was elucidated, and so on.

When the B field is constant, the background metric $g_{\mu\nu}$ is flat. Attempts to extend the analysis to the curved background, *i.e.*, the nonvanishing B field strength $H = dB$, appeared in the context of open string theory in WZW models[8]. In this approach it was found that the algebra of functions on the D-brane worldvolume is given by the q-deformation of the Lie algebra. The general framework for general backgrounds was, however, not yet obtained.

More recently a noncommutative field theory for a weak field strength H and a weakly curved background was investigated[9]. Several correlation functions of the string theory are obtained in the $\alpha' \to 0$ limit and it was concluded that the algebra of functions on the D-brane worldvolume is non-associative and noncommutative. Various correlation functions are expressed in terms of a non-associative product $\bullet$. In [10] the commutation relations of the coordinate x of the end point of the open string were studied by using the approximation of a very short and slowly moving string. It was found that the commutators of the coordinate x contain a momentum p and it was argued that the space on which the D-brane lives is not the usual noncommutative space which we are getting used to. In [11] a new product $\diamond$ was defined in order to make the product $\bullet$ associative on the functions of x and p and an attempt to define a gauge transformation in this 'x-p space' was presented. An explicit construction of the gauge theory action, however, was not obtained yet.

The purpose of the present paper is to understand how the low energy effective theory can be formulated as an associative and noncommutative gauge theory, when the background space is curved and the field strength $H = dB$ along the direction of the D-brane worldvolume is nonvanishing. Our analysis will be restricted to the bosonic string. We assume that the field strength is weak and perform analysis in perturbation series in H up to $\mathcal{O}(H^1)$. We will first obtain the commutation relations of the coordinates x^μ, y^μ of the two ends (at $\sigma = 0$ and π) of the open string in the low energy limit.(sec 2) This will be performed by discarding the oscillators of the string variable $X^\mu(\tau,\sigma)$, but without further approximation. The result is striking. x^μ and y^μ do not commute with each other in contrast to the $H=0$ case. This means that when we take the $\alpha' \to 0$ limit, the gauge and scalar degrees of freedom at both ends of the string are not decoupled in the low energy effective theory. We are forced to take into account the degrees of freedom at both ends in constructing the field theory description and the numbers of gauge and scalar fields are doubled. Furthermore the noncommutative product $\star$ constructed according to the algebra of x and y is *associative*, but operates on both coordinates.(sec 3) We are thus lead to consider a bi-local field theory. All fields are functions of x and y. The gauge theory must be formulated in the $2D$ dimensional (x, y) space instead of the ordinary D dimensional x space. This bi-local field theory will, however, be acausal and must be regarded as an intermediate step toward construction of the local effective field theory, which will be discussed later.

Construction of this bi-local gauge theory can be performed in a standard way. The derivatives are generalized in such a way that they satisfy Leibnitz rule with respect to the product $\star$.(sec 4) The commutator of the gauge covariant derivatives defined in terms of these new derivatives gives the gauge field strength.(sec 5) The trace or the integral is defined in such a way that the cyclic property is respected. An obstruction in this prescription is the curved D-brane worldvolume. The metric of the effective gauge theory is the open string metric[7], which is curved already at $\mathcal{O}(H^1)$[9]. Because the gauge symmetry of the noncommutative gauge theory is realized by differential operators on the fields, if the action contains such a non-constant metric, one anticipates that the gauge symmetry may be broken. Remarkably, we found that there exists a coordinate transformation in the (x^μ, y^μ) space (not in the x^μ space) which effectively makes $H=0$. In other words in a suitably chosen frame $(x'^\mu(x,y), y'^\mu(x,y))$ the coordinates satisfy the commutation relations of the $H=0$ theory. Therefore it is natural to further assume that the metric is flat in this frame. Moreover the derivatives with respect to x' and y' coincide with the new derivatives defined above to satisfy Leibnitz rule. By writing down the action for the noncommutative gauge theory with $H = dB = 0$ in the flat (x', y') frame and by going to the (x, y) frame by means of the inverse coordinate transformation, we obtain the action for a bi-local, noncommutative gauge theory in curved backgrounds.

As mentioned above, the above bi-local theory is non-local and will be acausal. Next we will present a prescription to reduce our bi-local theory to a 'local theory'.(sec 6) Actually, in order to obtain a massless string we must consider a limit $y \to x$. This is not automatically achieved in the $\alpha' \to 0$ limit. We will set $y = x$ by hand in the integrand of the effective action and integrate this over x with a suitable weight function $\omega(x)$. We will show cyclic property of this reduced effective action which relates various correlation functions of the fields at x and y, *i.e.*, at both ends of the string. This reduced effective action is still invariant under a part of the gauge transformation. We propose this as the low energy effective action of the open string in the $H \neq 0$ background. This reduced effective action, however, contains $\star$ in the integrand and the manipulation such as differentiation with respect to x and/or y must be performed before setting $y = x$. In this sense this reduced effective action is not the ordinary action in the x space.

Finally in sec 7 we will speculate on the relation of our noncommutative gauge theory to the Matrix model in curved backgrounds.

2 Noncommutativity of the two end-point coordinates of an open string

We will quantize an open string theory with a Dp-brane in the presence of an NS-NS B field along the Dp-brane, whose worldvolume spans a subspace $\mu = 0, 1, 2, \ldots, p$. The field strength of the B field, $H = dB$, is assumed to be small but non-vanishing in the direction of the Dp-brane worldvolume.

In this case we can perform a perturbative analysis in H. Such an analysis was first attempted in [9]. We will simplify the problem by taking $H_{\mu\nu\lambda}$ constant. By the consistency of the background due to Weyl invarinace the metric tensor $g_{\mu\nu}$ must be a constant up to $\mathcal{O}(H^1)$ [9]. We will choose a gauge

$$B_{\mu\nu}(X) = b_{\mu\nu} + \frac{1}{3} H_{\mu\nu\lambda} X^{\lambda}, \tag{1}$$

where $b_{\mu\nu}$ is a constant. We assume that $H_{\mu\nu\lambda}$ is nonvanishing for $\mu, \nu, \ldots = 1, 2, \ldots, D(\leq p)$. When $H_{\mu\nu\lambda} \neq 0$, $b_{\mu\nu}$ is also in general non-vanishing because we can always shift X^μ by a constant. We also assume that B is block diagonal, *i.e.*, $B_{\mu i} = B_{i\mu} = 0$ for $\mu \leq D,\ i > D$ and D is even and $\det(b_{\mu\nu})_{\mu,\nu=1,2,\ldots,D} \neq 0$. In what follows we will restrict our attention only to this D-dimensional subspace.

The worldsheet action of the open string in the conformal gauge is given by

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \{ g_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu + B_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \}. \tag{2}$$

The coordinate X^μ must satisfy the following boundary conditions at $\sigma = 0, \pi$.

$$g_{\mu\nu} \partial_\sigma X^\nu - B_{\mu\nu} \partial_\tau X^\nu = 0 \tag{3}$$

Inside the worldsheet it must satisfy the equations of motion.

$$g_{\lambda\rho}(\partial^\alpha \partial_\alpha X^\rho + \Gamma_{\mu\nu}{}^\rho \partial_\alpha X^\mu \partial^\alpha X^\nu) - H_{\mu\nu\lambda} \partial_\tau X^\mu \partial_\sigma X^\nu = 0 \tag{4}$$

For small H the metric g is constant up to $\mathcal{O}(H^1)$ and eq of motion (4) reads

$$-\ddot{X}^\mu + X''^\mu - g^{\mu\nu} H_{\nu\lambda\rho} \dot{X}^\lambda X'^\rho = 0 + \mathcal{O}(H^2), \tag{5}$$

where the dot and prime stand for $\partial/\partial\tau$ and $\partial/\partial\sigma$, respectively. The boundary condition (3) is now expressed as

$$X'^\mu - J^\mu{}_\nu \dot{X}^\mu - \frac{1}{3} g^{\mu\nu} H_{\nu\lambda\rho} \dot{X}^\lambda X^\rho = 0 \qquad (\text{at} \quad \sigma = 0, \pi), \tag{6}$$

where $J^\mu{}_\nu = g^{\mu\lambda} b_{\lambda\nu}$.

We will solve (5) under the condition (6) up to $\mathcal{O}(H^1)$. We set $X^\mu = X^{(0)\mu} + X^{(1)\mu}$. The leading order solution is given by

$$X^{(0)\mu}(\tau,\sigma) = x^\mu + \tau p^\mu + \sigma (Jp)^\mu + (1+J)^\mu{}_\nu f^\nu(\tau+\sigma) + (1-J)^\mu{}_\nu f^\nu(\tau-\sigma), \tag{7}$$

where $(Jp)^\mu$ stands for $J^\mu{}_\nu p^\nu$ and

$$f^\mu(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n} (a_n^\mu e^{-in\sigma} + a_{-n}^\mu e^{in\sigma}). \tag{8}$$

x^μ, p^μ are 'zero modes' and a_n^μ oscillators. We will next plug this in (5) and derive an equation for the $\mathcal{O}(H^1)$ correction $X^{(1)\mu}$. This can easily be solved.

$$\begin{aligned} X^{(1)\mu} \;=\; & -\frac{1}{4}(\tau^2-\sigma^2)\; g^{\mu\nu}\; H_{\nu\lambda\rho}\; p^\lambda\; (Jp)^\rho - \frac{1}{4}(\tau-\sigma)\; g^{\mu\nu}\; H_{\nu\lambda\rho}\; p^\lambda (1+J)^\rho{}_\alpha\; f^\alpha(\tau+\sigma) \\ & +\frac{1}{4}(\tau+\sigma)\; g^{\mu\nu}\; H_{\nu\lambda\rho}\; p^\lambda\; (1-J)^\rho{}_\alpha\; f^\alpha(\tau-\sigma) \\ & +\frac{1}{4}(\tau-\sigma)\; g^{\mu\nu}\; H_{\nu\lambda\rho}\; (Jp)^\lambda\; (1+J)^\rho{}_\alpha\; f^\alpha(\tau+\sigma) \\ & +\frac{1}{4}(\tau+\sigma)\; g^{\mu\nu}\; H_{\nu\lambda\rho}\; (Jp)^\lambda\; (1-J)^\rho{}_\alpha\; f^\alpha(\tau-\sigma) \\ & +\frac{1}{2} g^{\mu\nu}\; H_{\nu\lambda\rho}\; (1+J)^\lambda{}_\alpha\; (1-J)^\rho{}_\beta\; f^\alpha(\tau+\sigma)\; f^\beta(\tau-\sigma) \\ & + g^{\mu\nu}\; h_\nu\; (\tau+\sigma) + g^{\mu\nu}\; k_\nu\; (\tau-\sigma) \end{aligned} \tag{9}$$

Here h_ν and k_ν are some functions to be determined by (6).

The energies of the oscillation modes are proportional to $1/\sqrt{\alpha'}$ and in the $\alpha' \to 0$ limit they can be ignored. In this paper we will restrict attention only to the 'zero modes' and simply set $a_n^\mu = 0$ $(f^\mu = 0)$. One may speculate that x, p and a may mix up due to the interaction and the commutators of x, p may contain a. We have explicitly checked that this is not the case up to $\mathcal{O}(H^1)$. The full analysis including the oscillators is now in progress and will be reported elsewhere[12]. Eq (6) forces h_ν, k_ν to satisfy an equation

$$\begin{aligned} &(1+J)^\nu{}_\mu h'_\nu(\tau+\sigma) - (1-J)^\nu{}_\mu k'_\nu(\tau-\sigma) \\ &\qquad = -\frac{1}{3} H_{\mu\nu\lambda} x^\nu p^\lambda - \frac{1}{6}\sigma H_{\mu\nu\lambda} p^\lambda (Jp)^\nu p^\nu + \frac{1}{2}\tau J^\nu{}_\mu H_{\nu\lambda\rho} p^\lambda (Jp)^\rho \end{aligned} \tag{10}$$

at $\sigma = 0, \pi$. The primes on h_ν, k_ν stand for the derivatives with respect to the arguments. This equation can be solved by subtracting (10) with $\sigma = 0$ from (10) with $\sigma = \pi$ and $\tau \to \tau + \pi$. This determines h_μ and then (10) gives k_μ.

$$
\begin{aligned}
h_\mu(\tau) &= \frac{\tau^2}{24}\left(\frac{1-3J}{1+J}\right)^\nu_{\ \mu} H_{\nu\lambda\rho}(Jp)^\lambda p^\rho - \frac{\tau}{6}\left(\frac{1}{1+J}\right)^\nu_{\ \mu} H_{\nu\lambda\rho}x^\lambda p^\rho \ , \\
k_\mu(\tau) &= \frac{\tau^2}{24}\left(\frac{1+3J}{1-J}\right)^\nu_{\ \mu} H_{\nu\lambda\rho}(Jp)^\lambda p^\rho + \frac{\tau}{6}\left(\frac{1}{1-J}\right)^\nu_{\ \mu} H_{\nu\lambda\rho}x^\lambda p^\rho . \qquad (11)
\end{aligned}
$$

The next step is to quantize this system. We will use the phase-space integral invariant of Poincaré I to derive Poisson brackets[5]. It is defined by

$$
I = \int_0^\pi d\sigma \delta X^\mu(\tau,\sigma) \wedge \delta P_\mu(\tau,\sigma), \qquad (12)
$$

where P_μ is the canonical momentum conjugate to X^μ,

$$
2\pi\alpha' P_\mu = -g_{\mu\nu}\dot{X}^\nu + B_{\mu\nu}X'^\nu . \qquad (13)
$$

δX^μ and δP_μ represent differential 1-forms. The quantity (12) is indepedent of τ and defines the Poisson structure. Substitution of (7), (9), (13) into (12) yields I as a 2-form in the basis $\delta x^\mu \wedge \delta x^\nu$, $\delta x^\mu \wedge \delta p^\nu$, $\delta p^\mu \wedge \delta p^\nu$. The coefficients are a $2D$ by $2D$ matrix and its inverse gives the Poisson brackets. They are then converted to the commutation relations.

$$
\begin{aligned}
[x^\mu, x^\nu] &= 2\pi\alpha'\{i\theta^{\mu\nu} + \frac{i}{3}(\theta^{\mu\lambda}\theta^{\nu\rho} - G^{\mu\lambda}G^{\nu\rho})H_{\lambda\rho\sigma}\ x^\sigma \\
&\quad + \frac{\pi i}{9}(3\theta^{\mu\lambda}\theta^{\nu\rho} - G^{\mu\lambda}G^{\nu\rho})\ H_{\lambda\rho\sigma}\ (Jp)^\sigma - \frac{\pi i}{9}(\theta^{\mu\lambda}G^{\nu\rho} + G^{\mu\lambda}\theta^{\nu\rho})\ H_{\lambda\rho\sigma}\ p^\sigma\}, \\
[x^\mu, p^\nu] &= 2\pi\alpha'\{-\frac{i}{\pi}G^{\mu\nu} + \frac{i}{3\pi}\theta^{\mu\lambda}G^{\nu\rho}\ H_{\lambda\rho\sigma}\ x^\sigma \\
&\quad + \frac{i}{6}(3\theta^{\mu\lambda}\theta^{\nu\rho} - G^{\mu\lambda}G^{\nu\rho})\ H_{\lambda\rho\sigma}\ p^\sigma - \frac{i}{2}(G^{\mu\lambda}\theta^{\nu\rho} - \theta^{\mu\lambda}G^{\nu\rho})\ H_{\lambda\rho\sigma}\ (Jp)^\sigma\}, \\
[p^\mu, p^\nu] &= \frac{4i}{3}\alpha'\ G^{\mu\lambda}G^{\nu\rho}\ H_{\lambda\rho\sigma}(Jp)^\sigma . \qquad (14)
\end{aligned}
$$

Here θ and G are defined by

$$
\theta^{\mu\nu} = \left(\frac{-J}{1-J^2}g^{-1}\right)^{\mu\nu}, \qquad G^{\mu\nu} = \left(\frac{1}{1-J^2}g^{-1}\right)^{\mu\nu}. \qquad (15)
$$

These are the noncommutative parameter[1] and the open string metric for the $H = 0$ background[7].

Similar commutation relations in the $\alpha' \to 0$ limit are obtained in [10] by using some approximation of short and slowly moving string. This $\alpha' \to 0$ limit is a specific one taken with fine tuning[7]. $g_{\mu\nu}$ is adjusted to be $\mathcal{O}(\alpha'^2)$, while $x^\mu \sim \mathcal{O}(\alpha'^0)$, $b_{\mu\nu}, H_{\mu\nu\lambda}, p^\mu \sim \mathcal{O}(\alpha'^1)$.

[1]This θ is different from that used in the literature by a factor $2\pi\alpha'$. When multiplied by this factor, θ remains finite in the $\alpha' \to 0$ limit. We used the definition (15) in order to avoid messy expression for (14).

Hence $J^\mu{}_\nu \sim \mathcal{O}(\alpha'^{-1})$, $\theta^{\mu\nu} \sim \mathcal{O}(\alpha'^{-1})$ and $G^{\mu\nu} \sim \mathcal{O}(\alpha'^{0})$. Defining the total momentum at $\tau = 0$ by $\tilde{p}_\mu \equiv \int_0^\pi d\sigma P_\mu(0,\sigma) \sim \mathcal{O}(\alpha'^0)$, we get in the $\alpha' \to 0$ limit

$$
\begin{aligned}
[x^\mu, x^\nu] &= 2\pi\alpha' \left(i\hat{\theta}^{\mu\nu}(x) + \frac{2\pi i}{3}\ \alpha'\theta^{\mu\lambda}\theta^{\nu\rho}\ H_{\lambda\rho\sigma}\ \theta^{\sigma\alpha}\ \tilde{p}_\alpha \right), \\
[x^\mu, \tilde{p}_\nu] &= i\delta^\mu_\nu + \frac{\pi i}{3}\alpha'\theta^{\mu\lambda}\theta^{\rho\alpha}\ H_{\nu\lambda\rho}\ \tilde{p}_\alpha, \qquad [\tilde{p}_\mu, \tilde{p}_\nu] = 0,
\end{aligned}
\tag{16}
$$

which agrees with the result of [10], [11]. Here $\hat{\theta}(x)$ is θ in (15) with $J = g^{-1}b$ replaced by $g^{-1}B(x)$.

In [10] it is argued that the appearence of p^μ on the righthand sides of (14) makes the construction of the low energy effective action difficult. In [11] some proposal for a construction of gauge transformations is presented. An explicit form of the action integral in the x-p space is, however, not proposed.

In this paper we will observe the relations (14) from a different view point and construct a noncommutative gauge theory based on this algebra. Crucial point is that the coordinates of the end points of the open string (at $\tau = 0$) are given by

$$
\begin{aligned}
X^\mu(0,0)|_{\text{zero modes}} &= x^\mu, \\
X^\mu(0,\pi)|_{\text{zero modes}} &= x^\mu + \pi(Jp)^\mu + \mathcal{O}(H^1).
\end{aligned}
\tag{17}
$$

Throughout this paper we will use x and y to stand for the coordinates of the two ends of the open string. The difference of the two end point coordinates is thus proportional to p.

$$
y^\mu - x^\mu = \pi(Jp)^\mu + \mathcal{O}(H^1) \tag{18}
$$

Note that this is $\mathcal{O}(\alpha'^0)$ and does not automatically go to zero but is fixed in the $\alpha' \to 0$ limit. For large momenta the length of the open string grows.[14] If $b_{\mu\nu}$ is invertible, which we assume in this paper, so are $J^\mu{}_\nu$ and $\theta^{\mu\nu}$, and we can invert (18).

$$
p^\mu = \frac{1}{\pi}(J^{-1})^\mu{}_\nu\ (y^\nu - x^\nu) + \mathcal{O}(H^1) \tag{19}
$$

Now the algebra (14) can be reexpressed in terms of x^μ and y^μ.[2]

$$
\begin{aligned}
[x^\mu, x^\nu] &= i\theta^{\mu\nu} + \frac{i}{3}(\theta^{\mu\lambda}\theta^{\nu\rho} - G^{\mu\lambda}G^{\nu\rho})H_{\lambda\rho\alpha}x^\alpha + \frac{i}{9}(3\theta^{\mu\lambda}\theta^{\nu\rho} - G^{\mu\lambda}G^{\nu\rho})H_{\lambda\rho\alpha}(y^\alpha - x^\alpha) \\
&\quad -\frac{i}{9}(\theta^{\mu\lambda}G^{\nu\rho} + G^{\mu\lambda}\theta^{\nu\rho})H_{\lambda\rho\alpha}(J^{-1})^\alpha{}_\beta(y^\beta - x^\beta),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
[x^\mu, y^\nu] &= -\frac{i}{18}(3\theta^{\mu\lambda}\theta^{\nu\rho} - G^{\mu\lambda}G^{\nu\rho})H_{\lambda\rho\alpha}(y^\alpha - x^\alpha) \\
&\quad +\frac{i}{18}(\theta^{\mu\lambda}G^{\nu\rho} + G^{\mu\lambda}\theta^{\nu\rho})H_{\lambda\rho\alpha}(J^{-1})^\alpha{}_\beta(y^\beta - x^\beta),
\end{aligned}
\tag{21}
$$

[2] In what follows we will set $2\pi\alpha' = 1$.

$$
\begin{aligned}
[y^\mu, y^\nu] \quad = \quad & -i\theta^{\mu\nu} - \frac{i}{3}(\theta^{\mu\lambda}\theta^{\nu\rho} - G^{\mu\lambda}G^{\nu\rho})H_{\lambda\rho\alpha}y^\alpha + \frac{i}{9}(3\theta^{\mu\lambda}\theta^{\nu\rho} - G^{\mu\lambda}G^{\nu\rho})H_{\lambda\rho\alpha}(y^\alpha - x^\alpha) \\
& -\frac{i}{9}(\theta^{\mu\lambda}G^{\nu\rho} + G^{\mu\lambda}\theta^{\nu\rho})H_{\lambda\rho\alpha}(J^{-1})^\alpha{}_\beta(y^\beta - x^\beta)
\end{aligned} \tag{22}
$$

(20) and (22) shows that the coordinates of the ends of the open string x^μ, y^μ do not commute. The commutator of y's is obtained from that of x's by interchanging $x \leftrightarrow y$ and reversing the overall sign. Furthermore (21) shows that the coordinate of one end x does not commute with that of the other y. This is in contrast to the $H = 0$ case[6][11]. The gauge and scalar degrees of freedom at the $\sigma = 0$ end and those at $\sigma = \pi$ are not decoupled. To construct a low energy effective theory we need to consider both degrees of freedom. This means that if the two ends of the string are separated, the low energy effective field theory will become acausal. To avoid this problem we will take a limit $y \to x$ in sec 5. Neverthelss these two ends must be separated for a while in order to construct an associative product.

We expect that the full string theory including the oscillators is causal. The reason for the non-commutativity of x and y may lie in the fact that we discarded oscillators. It will be interesting to study the commutators of the full string variable $X^\mu(\tau, \sigma)$.

3 An associative and noncommutative product

It can be shown that the commutation relations (20)-(22) satisfy Jacobi identities. We will then construct a product $\star$ which realizes this algebra. The parameter $H_{\mu\nu\lambda}$ is treated as a small purturbation but $\theta^{\mu\nu}$ must be taken into account to the full order. We report only the results here.

$$
\begin{aligned}
& f_1(x,y) \star f_2(x,y) \\
\equiv \quad & f_1(x,y) * f_2(x,y) \\
& -\frac{i}{6}\left(G^{\mu\lambda}G^{\nu\rho} - \theta^{\mu\lambda}\theta^{\nu\rho}\right)H_{\lambda\rho\alpha}\left(x^\alpha \frac{\partial f_1}{\partial x^\mu} * \frac{\partial f_2}{\partial x^\nu} - y^\alpha \frac{\partial f_1}{\partial y^\mu} * \frac{\partial f_2}{\partial y^\nu}\right) \\
& -\frac{i}{18}\left(G^{\mu\lambda}G^{\nu\rho} - 3\theta^{\mu\lambda}\theta^{\nu\rho}\right)H_{\lambda\rho\alpha} \\
& \qquad \times (y^\alpha - x^\alpha)\left(\frac{\partial f_1}{\partial x^\mu} * \frac{\partial f_2}{\partial x^\nu} - \frac{1}{2}\frac{\partial f_1}{\partial x^\mu} * \frac{\partial f_2}{\partial y^\nu} - \frac{1}{2}\frac{\partial f_1}{\partial y^\mu} * \frac{\partial f_2}{\partial x^\nu} + \frac{\partial f_1}{\partial y^\mu} * \frac{\partial f_2}{\partial y^\nu}\right) \\
& +\frac{i}{18}\left(\theta^{\mu\lambda}G^{\nu\rho} + G^{\mu\lambda}\theta^{\nu\rho}\right)H_{\lambda\rho\alpha}\left(G^{-1}\theta^{-1}\right)^\alpha{}_\beta \\
& \qquad \times (y^\beta - x^\beta)\left(\frac{\partial f_1}{\partial x^\mu} * \frac{\partial f_2}{\partial x^\nu} - \frac{1}{2}\frac{\partial f_1}{\partial x^\mu} * \frac{\partial f_2}{\partial y^\nu} - \frac{1}{2}\frac{\partial f_1}{\partial y^\mu} * \frac{\partial f_2}{\partial x^\nu} + \frac{\partial f_1}{\partial y^\mu} * \frac{\partial f_2}{\partial y^\nu}\right) \\
& -\frac{1}{36}G^{\mu\lambda}G^{\nu\rho}\theta^{\beta\alpha}H_{\lambda\rho\alpha} \\
& \quad \times\left(\frac{\partial f_1}{\partial x^\mu} * \frac{\partial^2 f_2}{\partial x^\nu \partial x^\beta} - \frac{\partial^2 f_1}{\partial x^\mu \partial x^\beta} * \frac{\partial f_2}{\partial x^\nu} + \frac{\partial f_1}{\partial y^\mu} * \frac{\partial^2 f_2}{\partial y^\nu \partial y^\beta} - \frac{\partial^2 f_1}{\partial y^\mu \partial y^\beta} * \frac{\partial f_2}{\partial y^\nu}\right)
\end{aligned}
$$

$$+\frac{1}{72}\left(-3\theta^{\mu\lambda}\theta^{\nu\rho}\theta^{\beta\alpha}+G^{\mu\lambda}G^{\nu\rho}\theta^{\beta\alpha}+\theta^{\mu\lambda}G^{\nu\rho}G^{\beta\alpha}+G^{\mu\lambda}\theta^{\nu\rho}G^{\beta\alpha}\right)H_{\lambda\rho\alpha}$$
$$\times\left(\frac{\partial f_1}{\partial x^\mu}*\frac{\partial^2 f_2}{\partial x^\nu\partial y^\beta}-\frac{\partial^2 f_1}{\partial x^\mu\partial y^\beta}*\frac{\partial f_2}{\partial x^\nu}+\frac{\partial f_1}{\partial y^\mu}*\frac{\partial^2 f_2}{\partial y^\nu\partial x^\beta}-\frac{\partial^2 f_1}{\partial y^\mu\partial x^\beta}*\frac{\partial f_2}{\partial y^\nu}\right) \tag{23}$$

Here $*$ is the ordinary noncommutative product for x and y in the $H=0$ case.

$$\begin{aligned} & f_1(x,y)*f_2(x,y) \\ \equiv\ & \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\nu}-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y'^\nu}\right)f_1(x,y)f_2(x',y')\bigg|_{x'=x,y'=y} \end{aligned} \tag{24}$$

The noncommutative parameter is $\theta^{\mu\nu}$ for x and $-\theta^{\mu\nu}$ for y.

The product (23) is associative. The proof of the associativity (up to $\mathcal{O}(H^1)$)

$$(f_1(x,y)\star f_2(x,y))\star f_3(x,y)=f_1(x,y)\star(f_2(x,y)\star f_3(x,y)) \tag{25}$$

is straightforward.

Because the product $\star$ contains both x and y, the product of the functions of x only, $f_1(x)\star f_2(x)$ will also depend on y. Hence we must regard all fields to be functions of both x and y, *i.e.*, bi-local fields.

4 Derivatives

The low energy effective theory on the D-brane is expected to be some kind of noncommutative gauge theory. To construct such a theory we need to define proper derivatives. Because the $\star$ product (23) depends on the coordinates explicitly, this will not satify Leibnitz rule.

$$\Delta\equiv\frac{\partial}{\partial x^\mu}(f_1\star f_2)-\left(\frac{\partial}{\partial x^\mu}f_1\right)\star f_2-f_1\star\left(\frac{\partial}{\partial x^\mu}f_2\right)\neq 0 \tag{26}$$

To circumvent this problem we must modify the derivatives. We define

$$\frac{\nabla}{\nabla x^\mu}f(x,y) \equiv \frac{\partial}{\partial x^\mu}f(x,y)+\left\{{a_\mu}^\nu,\frac{\partial f}{\partial x^\nu}\right\}_*+\left\{{b_\mu}^\nu,\frac{\partial f}{\partial y^\nu}\right\}_*, \tag{27}$$

$$\frac{\nabla}{\nabla y^\mu}f(x,y) \equiv \frac{\partial}{\partial y^\mu}f(x,y)+\left\{{\overline{a}_\mu}^\nu,\frac{\partial f}{\partial y^\nu}\right\}_*+\left\{{\overline{b}_\mu}^\nu,\frac{\partial f}{\partial x^\nu}\right\}_*. \tag{28}$$

Here ${a_\mu}^\nu$, ${b_\mu}^\nu$, ${\overline{a}_\mu}^\nu$ and ${\overline{b}_\mu}^\nu$ are linear functions of x, y and are $\mathcal{O}(H^1)$. $\{\ \cdot\ \}_*$ is the anti-commutator with respect to the product (24). We note that because a, b, $\overline{a}$, $\overline{b}$ are linear functions, $\{a,\partial f/\partial x\}_*$ can be replaced by $2a\partial f/\partial x$ in (27), (28).

These functions must be determined in such a way that the new derivatives (27), (28) satisfy Leibnitz rule. By (27) we obtain

$$\begin{aligned}\frac{\nabla}{\nabla x^\mu}(f_1\star f_2)-\frac{\nabla f_1}{\nabla x^\mu}\star f_2-f_1\star\frac{\nabla f_2}{\nabla x^\mu} &= [{a_\mu}^\nu,f_1]_* * \frac{\partial f_2}{\partial x^\nu}-\frac{\partial f_1}{\partial x^\nu}*[{a_\mu}^\nu,f_2]_* \\ &\quad+[{b_\mu}^\nu,f_1]_* * \frac{\partial f_2}{\partial y^\nu}-\frac{\partial f_1}{\partial y^\nu}*[{b_\mu}^\nu,f_2]_*+\Delta\end{aligned} \tag{29}$$

Here $[\ \cdot\ ]_*$ is the commutator with respect to the product (24). Δ can be computed from the defintions (23), (26). By requiring (29) to vanish we obtain the equations.

$$\theta^{\lambda\rho}\frac{\partial a_\mu{}^\nu}{\partial x^\rho} - \theta^{\nu\rho}\frac{\partial a_\mu{}^\lambda}{\partial x^\rho} = -\frac{1}{9}G^{\lambda\alpha}G^{\nu\beta}H_{\alpha\beta\mu} - \frac{1}{18}(\theta^{\lambda\alpha}G^{\nu\beta} + G^{\lambda\alpha}\theta^{\nu\beta})H_{\alpha\beta\rho}(G^{-1}\theta^{-1})^\rho{}_\mu, \tag{30}$$

$$\begin{aligned}\theta^{\lambda\rho}\frac{\partial a_\mu{}^\nu}{\partial y^\rho} + \theta^{\nu\rho}\frac{\partial b_\mu{}^\lambda}{\partial x^\rho} &= -\frac{1}{36}(G^{\nu\alpha}G^{\lambda\beta} - 3\theta^{\nu\alpha}\theta^{\lambda\beta})H_{\alpha\beta\mu} \\ &\quad + \frac{1}{36}(\theta^{\nu\alpha}G^{\lambda\beta} + G^{\nu\alpha}\theta^{\lambda\beta})H_{\alpha\beta\rho}(G^{-1}\theta^{-1})^\rho{}_\mu,\end{aligned} \tag{31}$$

$$\begin{aligned}\theta^{\lambda\rho}\frac{\partial b_\mu{}^\nu}{\partial y^\rho} - \theta^{\nu\rho}\frac{\partial b_\mu{}^\lambda}{\partial y^\rho} &= \frac{1}{18}(G^{\nu\alpha}G^{\lambda\beta} - 3\theta^{\nu\alpha}\theta^{\lambda\beta})H_{\alpha\beta\mu} \\ &\quad - \frac{1}{18}(\theta^{\nu\alpha}G^{\lambda\beta} + G^{\nu\alpha}\theta^{\lambda\beta})H_{\alpha\beta\rho}(G^{-1}\theta^{-1})^\rho{}_\mu,\end{aligned} \tag{32}$$

General solution to these equations is given by

$$\begin{aligned}a_\mu{}^\nu(x,y) &= -\frac{1}{12}x^\sigma(\theta^{-1})_{\sigma\lambda}(G^{\lambda\alpha}G^{\nu\beta} - \theta^{\lambda\alpha}\theta^{\nu\beta})H_{\alpha\beta\mu} \\ &\quad + \frac{1}{72}(2x^\sigma + y^\sigma)(\theta^{-1})_{\sigma\lambda}(G^{\lambda\alpha}G^{\nu\beta} - 3\theta^{\lambda\alpha}\theta^{\nu\beta})H_{\alpha\beta\mu} \\ &\quad - \frac{1}{72}(2x^\sigma + y^\sigma)(\theta^{-1})_{\sigma\lambda}(\theta^{\lambda\alpha}G^{\nu\beta} + G^{\lambda\alpha}\theta^{\nu\beta})H_{\alpha\beta\rho}(G^{-1}\theta^{-1})^\rho{}_\mu \\ &\quad - x^\sigma(\theta^{-1})_{\sigma\lambda}\ S^{\lambda\nu}{}_\mu - y^\sigma(\theta^{-1})_{\sigma\lambda}\ U^{\lambda\nu}{}_\mu,\end{aligned} \tag{33}$$

$$\begin{aligned}b_\mu{}^\nu(x,y) &= -\frac{1}{72}(x^\sigma + 2y^\sigma)(\theta^{-1})_{\sigma\lambda}(G^{\lambda\alpha}G^{\nu\beta} - 3\theta^{\lambda\alpha}\theta^{\nu\beta})H_{\alpha\beta\mu} \\ &\quad + \frac{1}{72}(x^\sigma + 2y^\sigma)(\theta^{-1})_{\sigma\lambda}(\theta^{\lambda\alpha}G^{\nu\beta} + G^{\lambda\alpha}\theta^{\nu\beta})H_{\alpha\beta\rho}(G^{-1}\theta^{-1})^\rho{}_\mu \\ &\quad - y^\sigma(\theta^{-1})_{\sigma\lambda}\ T^{\lambda\nu}{}_\mu + x^\sigma(\theta^{-1})_{\sigma\lambda}\ U^{\nu\lambda}{}_\mu.\end{aligned} \tag{34}$$

Here $S^{\lambda\nu}{}_\mu(= S^{\nu\lambda}{}_\mu)$, $T^{\lambda\nu}{}_\mu(= T^{\nu\lambda}{}_\mu)$ and $U^{\lambda\nu}{}_\mu$ are some constants to be discussed later. Similarly $\overline{a}_\mu{}^\nu$, $\overline{b}_\mu{}^\nu$ can be determined. By using the $x \leftrightarrow y$ symmetry we find

$$\overline{a}_\mu{}^\nu(x,y) = a_\mu{}^\nu(y,x), \qquad \overline{b}_\mu{}^\nu(x,y) = b_\mu{}^\nu(y,x). \tag{35}$$

The field strength of the gauge fields will be defined as the commutator of the gauge covariant derivatives. For this procedure to be consistent the derivatives (27), (28) must commute. Otherwise the commutator becomes a differential operator. For instance from (27) we obtain

$$\left[\frac{\nabla}{\nabla x^\mu}, \frac{\nabla}{\nabla x^\nu}\right] f = 2\left(\frac{\partial a_\nu{}^\lambda}{\partial x^\mu} - \frac{\partial a_\mu{}^\lambda}{\partial x^\nu}\right)\frac{\partial f}{\partial x^\lambda} + 2\left(\frac{\partial b_\nu{}^\lambda}{\partial x^\mu} - \frac{\partial b_\mu{}^\lambda}{\partial x^\nu}\right)\frac{\partial f}{\partial y^\lambda} \tag{36}$$

and the two coefficients of the derivatives of f must vanish. We obtain 4 more conditions from the other commutators. It turns out the following three out of the six conditions are independent.

$$\frac{\partial a_\nu{}^\lambda}{\partial x^\mu} = \frac{\partial a_\mu{}^\lambda}{\partial x^\nu}, \qquad \frac{\partial b_\nu{}^\lambda}{\partial x^\mu} = \frac{\partial b_\mu{}^\lambda}{\partial x^\nu}, \qquad \frac{\partial \bar{b}_\nu{}^\lambda}{\partial x^\mu} = \frac{\partial a_\mu{}^\lambda}{\partial y^\nu} \tag{37}$$

These equations impose the following conditions on S, T, U.

$$S^{\mu\lambda}{}_\rho\ \theta^{\rho\nu} - S^{\nu\lambda}{}_\rho\ \theta^{\rho\mu} = \frac{1}{18} G^{\mu\alpha} G^{\nu\beta} \theta^{\lambda\gamma}\ H_{\alpha\beta\gamma} + \frac{1}{36}\left(\theta^{\nu\alpha} G^{\mu\beta} - \theta^{\mu\alpha} G^{\nu\beta}\right) G^{\lambda\gamma}\ H_{\alpha\beta\gamma}, \tag{38}$$

$$\begin{aligned}(\theta^{-1})_{\mu\rho}\ U^{\lambda\rho}{}_\nu - (\theta^{-1})_{\nu\rho}\ U^{\lambda\rho}{}_\mu \ =\ & \frac{1}{36}\left(G^{-1}\theta^{-1}\right)^\sigma{}_\nu\ G^{\lambda\beta} H_{\mu\sigma\beta} - \frac{1}{36}\left(G^{-1}\theta^{-1}\right)^\sigma{}_\mu\ G^{\lambda\beta} H_{\nu\sigma\beta} \\ & -\frac{1}{36}\left(G^{-1}\theta^{-1}\right)^\alpha{}_\mu \left(G^{-1}\theta^{-1}\right)^\beta{}_\nu \theta^{\lambda\sigma} H_{\alpha\beta\sigma} + \frac{1}{12}\theta^{\lambda\beta} H_{\mu\nu\beta},\end{aligned} \tag{39}$$

$$\begin{aligned}U^{\nu\sigma}{}_\lambda \ =\ & \theta^{\nu\alpha}\ (\theta^{-1})_{\lambda\beta} T^{\beta\sigma}{}_\alpha - \frac{1}{72}\left(G^{\nu\alpha} G^{\sigma\beta} - 3\theta^{\nu\alpha}\theta^{\sigma\beta}\right)\ H_{\lambda\alpha\beta} \\ & + \frac{1}{72}\left(\theta^{\nu\alpha} G^{\sigma\beta} + G^{\nu\alpha}\theta^{\sigma\beta}\right)\ \left(G^{-1}\theta^{-1}\right)^\rho{}_\lambda\ H_{\alpha\beta\rho}\end{aligned} \tag{40}$$

If we recall the symmetry $T^{\lambda\nu}{}_\mu = T^{\nu\lambda}{}_\mu$, (39) comes out as a result of (40) and hence is not an independent condition.

The solution to (38) is given by

$$\begin{aligned}S^{\mu\nu}{}_\lambda \ =\ & c\left(\theta^{\mu\alpha} G^{\nu\beta} - G^{\mu\alpha}\theta^{\nu\beta}\right)\ H_{\alpha\beta\rho}\ (\theta G)^\rho{}_\lambda + c\left\{(\theta G\theta)^{\mu\alpha} G^{\nu\beta} + (\theta G\theta)^{\nu\alpha} G^{\mu\beta}\right\}\ H_{\alpha\beta\lambda} \\ & + c\left\{(\theta G\theta)^{\mu\alpha}\theta^{\nu\beta} + (\theta G\theta)^{\nu\alpha}\theta^{\mu\beta}\right\}\ H_{\alpha\beta\rho}\ (G^{-1}\theta^{-1})^\rho{}_\lambda \\ & + \frac{1}{36}(\theta^{\mu\alpha} G^{\nu\beta} - G^{\mu\alpha}\theta^{\nu\beta})\ H_{\alpha\beta\rho}\ (G^{-1}\theta^{-1})^\rho{}_\lambda,\end{aligned} \tag{41}$$

c is a constant. $T^{\mu\nu}{}_\lambda$ and $U^{\mu\nu}{}_\lambda$ are arbitrary as long as (40) is satisfied. Unfortunately, further conditions which may determine these values are not yet available to the authors.

The fact that the derivatives (27), (28) commute suggests the existence of a coordinate transformation $x, y \to x', y'$ by which the derivatives (27), (28) are reexpressed as ordinary derivatives.

$$\frac{\partial}{\partial x'^\mu} = \frac{\nabla}{\nabla x^\mu} = \frac{\partial}{\partial x^\mu} + 2a_\mu{}^\nu\ \frac{\partial}{\partial x^\nu} + 2b_\mu{}^\nu\ \frac{\partial}{\partial y^\nu} \tag{42}$$

(And a similar equation for $\nabla/\nabla y$.) Here we used the property, $\{a, h\}_* = 2ah$ which holds for a linear function a. In this transformation x, y are treated as commuting variables.

Indeed eq (37) guarantees the existence of such a transformation and it is given by

$$
\begin{aligned}
x'^{\nu} &= x^{\nu} + \frac{1}{18} x^{\sigma} x^{\mu} (\theta^{-1})_{\sigma\lambda} G^{\lambda\alpha} G^{\nu\beta}\ H_{\alpha\beta\mu} \\
&+ \frac{1}{36} x^{\sigma} x^{\mu} (\theta^{-1})_{\sigma\lambda} (\theta^{\lambda\alpha} G^{\nu\beta} + G^{\lambda\alpha} \theta^{\nu\beta}) (G^{-1}\theta^{-1})^{\rho}{}_{\mu}\ H_{\alpha\beta\rho} \\
&- \frac{1}{36} y^{\sigma} x^{\mu} (\theta^{-1})_{\sigma\lambda} (G^{\lambda\alpha} G^{\nu\beta} - 3\theta^{\lambda\alpha}\theta^{\nu\beta})\ H_{\alpha\beta\mu} \\
&+ \frac{1}{36} y^{\sigma} x^{\mu} (\theta^{-1})_{\sigma\lambda} (\theta^{\lambda\alpha} G^{\nu\beta} + G^{\lambda\alpha} \theta^{\nu\beta}) (G^{-1}\theta^{-1})^{\rho}{}_{\mu}\ H_{\alpha\beta\rho} \\
&+ \frac{1}{72} y^{\sigma} y^{\mu} (\theta^{-1})_{\sigma\lambda} (G^{\lambda\alpha} G^{\nu\beta} - 3\theta^{\lambda\alpha}\theta^{\nu\beta})\ H_{\alpha\beta\mu} \\
&- \frac{1}{72} y^{\sigma} y^{\mu} (\theta^{-1})_{\sigma\lambda} (\theta^{\lambda\alpha} G^{\nu\beta} + G^{\lambda\alpha} \theta^{\nu\beta}) (G^{-1}\theta^{-1})^{\rho}{}_{\mu}\ H_{\alpha\beta\rho} \\
&+ x^{\sigma} x^{\mu} (\theta^{-1})_{\sigma\lambda} S^{\lambda\nu}{}_{\mu} + 2x^{\mu} y^{\sigma} (\theta^{-1})_{\sigma\lambda} U^{\lambda\nu}{}_{\mu} - y^{\sigma} y^{\mu} (\theta^{-1})_{\sigma\lambda} U^{\nu\lambda}{}_{\mu} \qquad (43)
\end{aligned}
$$

and a similar equation with the replacement $x \leftrightarrow y, \quad x' \leftrightarrow y'$.

By using (20)-(22) it is possible to show that in the primed coordinate system x' and y' commute with each other.

$$
[x'^{\mu}, x'^{\nu}] = i\theta^{\mu\nu}, \qquad [y'^{\mu}, y'^{\nu}] = -i\theta^{\mu\nu}, \qquad [x'^{\mu}, y'^{\nu}] = 0 \qquad (44)
$$

Here the quadratic terms in (43) are assumed to be symmetrized. In other words the effect of the field strength H can be compensated by a 'coordinate transformation'. This fact suggests that the metric tensor in the '(x', y') space' may be also flat and given by $G_{\mu\nu}$. We should note that this is not the ordinary space but a dimensionally-doubled space combining the coordinates of the two ends of the string and that the transformation (43) mixes up x and y as a whole. It is tempting to assume that the existence of such a transformation persists to all orders of H. We do not have a proof at present.

Let us construct a trace operation $Tr f(x, y)$. This must satisfy the cyclic property.

$$
Tr\ f_1 \star f_2 = Tr\ f_2 \star f_1 \qquad (45)
$$

We assume the form

$$
Tr\ f \equiv \int d^D x d^D y\ G\ \Omega(x, y)\ f(x, y) \qquad (46)
$$

and determine the function $\Omega(x, y)$ by the condition

$$
\int d^D x d^D y\ G\ \Omega\ (f_1 \star f_2) = \int d^D x d^D y\ G\ \Omega\ (f_2 \star f_1). \qquad (47)
$$

For $H = 0$ we must also have $\Omega = 1$. Note that on the righthand side of (46) the product is the ordinary one. G is $\det G_{\mu\nu}$. It is not hard to show that this function is given by

$$
\Omega(x, y) = 1 - \frac{1}{6}(x^{\sigma} + y^{\sigma})\ \theta^{\mu\lambda}\ H_{\mu\lambda\sigma} - \frac{1}{6}(x^{\sigma} + y^{\sigma})\ (G^{-1}\theta^{-1}G^{-1})^{\mu\lambda}\ H_{\mu\lambda\sigma}. \qquad (48)
$$

It may be natural to expect that this extra factor Ω in the trace has an origin in the Jacobian from (x, y) to (x', y'). We indeed obtain

$$\det \frac{(\partial x', \partial y')}{(\partial x, \partial y)} = 1 - \frac{1}{12}(x^\sigma + y^\sigma)\ \theta^{\mu\lambda}\ H_{\mu\lambda\sigma} - \frac{1}{12}(x^\sigma + y^\sigma)\ (G^{-1}\theta^{-1}G^{-1})^{\mu\lambda}\ H_{\mu\lambda\sigma} \\ +2(x^\sigma + y^\sigma)(\theta^{-1})_{\sigma\lambda}(S^{\lambda\mu}{}_\mu + U^{\lambda\mu}{}_\mu) \tag{49}$$

and we can show from (38)-(40) that this coincides with Ω.

5 Noncommutative gauge theory for $H \neq 0$

We are now ready to define gauge transformations and write down the gauge theory action. This action will be an integral over x^μ and y^μ and defines a bi-local theory. We will construct this bi-local theory as an intermediate step for obtaining a 'local gauge theory' in the next section.

Let us define the gauge covariant derivatives by

$$D_\mu = \partial/\partial x'^\mu + iA'_\mu = \nabla/\nabla x^\mu + iA'_\mu, \\ \overline{D}_\mu = \partial/\partial y'^\mu - i\overline{A}'_\mu = \nabla/\nabla y^\mu - i\overline{A}'_\mu. \tag{50}$$

Here A'_μ, $\overline{A}'_\mu$ are the gauge fields in the (x', y') frame and transformed to those in the (x, y) frame like the derivatives (42),

$$A'_\mu = A_\mu + 2a_\mu{}^\nu A_\nu - 2b_\mu{}^\nu \overline{A}_\nu, \\ \overline{A}'_\mu = \overline{A}_\mu + 2\overline{a}_\mu{}^\nu \overline{A}_\nu - 2\overline{b}_\mu{}^\nu A_\nu. \tag{51}$$

We will use a 'bar' to denote functions associated with the coordinate y. In (50), (51) the signs in front of A'_μ and $\overline{A}'_\mu$ are opposite, because the two ends of the open string have opposite charges. The gauge transformation is defined by

$$D_\mu \rightarrow U\star D_\mu \star U^{-1}, \qquad \overline{D}_\mu \rightarrow U\star \overline{D}_\mu \star U^{-1}, \qquad \Phi \rightarrow U\star\Phi\star U^{-1}, \tag{52}$$

where $U = \exp_\star(i\Lambda)$ is a gauge function. $\exp_\star$ is an exponential defined in terms of $\star$. Φ is a scalar field in the adjoint representation. For an infinitesimal Λ these read

$$\delta A'_\mu = i[\Lambda, A'_\mu]_\star - \frac{\nabla\Lambda}{\nabla x^\mu}, \qquad \delta\overline{A}'_\mu = i[\Lambda, \overline{A}'_\mu]_\star + \frac{\nabla\Lambda}{\nabla y^\mu}, \qquad \delta\Phi = i[\Lambda, \Phi]_\star. \tag{53}$$

Here $[\ \cdot\ ]_\star$ is the commutator with respect to the product (23). An infinitesimal transformation of A_μ, $\overline{A}_\mu$ is consistently obtained.

$$\delta A_\mu = i[\Lambda, A_\mu]_\star - \frac{\partial\Lambda}{\partial x^\mu} + i\{[\Lambda,\ a_\mu{}^\lambda]_\star,\ A_\lambda\}_\star - i\{[\Lambda,\ b_\mu{}^\lambda]_\star,\ \overline{A}_\lambda\}_\star, \tag{54}$$

$$\delta\overline{A}_\mu = i[\Lambda,\overline{A}_\mu]_\star + \frac{\partial\Lambda}{\partial y^\mu} + i\{[\Lambda,\ \overline{a}_\mu{}^\lambda]_\star,\ \overline{A}_\lambda\}_\star - i\{[\Lambda,\ \overline{b}_\mu{}^\lambda]_\star,\ A_\lambda\}_\star \tag{55}$$

Now by using the standard prescription we can write down the action integral for the bi-local noncommutative gauge theory

$$\begin{aligned} S_{\rm bi-local} &= \frac{1}{4g_{\rm YM}^2 V}\int d^Dx d^Dy\ G\ \Omega(x,y) \\ &\qquad \cdot\Big\{\ G^{\mu\nu}G^{\lambda\rho}\ [D_\mu,\ D_\lambda]_\star \star [D_\nu,\ D_\rho]_\star - 2G^{\mu\nu}\ [D_\mu,\ X^m]_\star\star[D_\nu,\ X^m]_\star \\ &\qquad\quad + [X^m,\ X^n]_\star^2\ +\ G^{\mu\nu}G^{\lambda\rho}\ [\overline{D}_\mu,\ \overline{D}_\lambda]_\star \star [\overline{D}_\nu,\ \overline{D}_\rho]_\star \\ &\qquad\quad -2\ G^{\mu\nu}\ [\overline{D}_\mu,\ \overline{X}^m]_\star\star[\overline{D}_\nu,\ \overline{X}^m]_\star +\ [\overline{X}^m,\ \overline{X}^n]_\star^2\ \Big\} \\ &\equiv \int d^Dxd^Dy\ GV^{-1}\ \Omega\ \mathcal{L}(x,y), \end{aligned} \tag{56}$$

where V is the volume $\int d^Dx\ \sqrt{G}$ and $g_{\rm YM}$ the Yang-Mills coupling. We explicitly write only the integration measure relevant to our discussions. X^m and $\overline{X}^m$ are the scalar fields describing the fluctuations of the D-brane. The scalar fields are doubled, because there are two such independent fluctuations at both ends of the open string. This action is invariant under the gauge transformation (52) due to the cyclic property (47) of the trace (46). Because $G^{\mu\nu}$ is constant, this does not break the gauge symmetry. There is no constraint on the gauge function $\Lambda(x,y)$.

By substituting (27), (28), (50), (51) into (56) we will find a non-constant metric in the action. For example we obtain for the commutator of the covariant derivatives

$$\begin{aligned} [\ D_\mu,\ D_\nu\ ]_\star &= i\ \{\mathcal{F}^{(x,x)}_{\mu\nu} + 2a_\mu{}^\lambda\ \mathcal{F}^{(x,x)}_{\lambda\nu} + 2a_\nu{}^\lambda\ \mathcal{F}^{(x,x)}_{\mu\lambda} + 2b_\mu{}^\lambda\ \mathcal{F}^{(y,x)}_{\lambda\nu} \\ &\qquad +2b_\nu{}^\lambda\ \mathcal{F}^{(x,y)}_{\mu\lambda} + \mathcal{R}_{\mu\nu}\}, \end{aligned} \tag{57}$$

where $\mathcal{F}$'s are the field strength in terms of the ordinary derivatives.

$$\begin{aligned} \mathcal{F}^{(x,x)}_{\mu\nu} &= \frac{\partial}{\partial x^\mu}A_\nu - \frac{\partial}{\partial x^\nu}A_\mu + i\ [\ A_\mu,\ A_\nu\ ]_\star, \\ \mathcal{F}^{(y,x)}_{\mu\nu} &= -\frac{\partial}{\partial y^\mu}A_\nu - \frac{\partial}{\partial x^\nu}\overline{A}_\mu + i\ [\ \overline{A}_\mu,\ A_\nu\ ]_\star = \mathcal{F}^{(x,y)}_{\nu\mu}, \\ \mathcal{F}^{(y,y)}_{\mu\nu} &= \frac{\partial}{\partial y^\mu}\overline{A}_\nu - \frac{\partial}{\partial y^\nu}\overline{A}_\mu - i\ [\ \overline{A}_\mu,\ \overline{A}_\nu\ ]_\star \end{aligned} \tag{58}$$

$\mathcal{R}_{\mu\nu}$ is a sum of terms like

$$-\ i\theta^{\rho\sigma}\frac{\partial a_\mu{}^\lambda}{\partial x^\rho}\left\{A_\lambda,\ \frac{\partial A_\nu}{\partial x^\sigma}\right\}_\star. \tag{59}$$

Here we omit its explicit form. By defining a new curved metric

$$\begin{aligned} \tilde{G}^{(x,x)\mu\nu}(x,y) &= G^{\mu\nu} + 2G^{\mu\lambda}\ a_\lambda{}^\nu + 2G^{\nu\lambda}\ a_\lambda{}^\mu, \\ \tilde{G}^{(y,y)\mu\nu}(x,y) &= G^{\mu\nu} + 2G^{\mu\lambda}\ \overline{a}_\lambda{}^\nu + 2G^{\nu\lambda}\ \overline{a}_\lambda{}^\mu, \\ \tilde{G}^{(x,y)\mu\nu}(x,y) &= 4G^{\mu\lambda}\ b_\lambda{}^\nu, \qquad \tilde{G}^{(y,x)\mu\nu}(x,y) = 4G^{\mu\lambda}\ \overline{b}_\lambda{}^\nu, \end{aligned} \tag{60}$$

we obtain the part of the action for the gauge field.

$$
\begin{aligned}
S_{\text{gauge field}} \quad = \quad & \frac{-1}{4g_{\text{YM}}^2 V} \int d^D x d^D y \sqrt{\tilde{G}(x,y)} \\
& \cdot \{ \tilde{G}^{(x,x)\mu\nu} \cdot \tilde{G}^{(x,x)\lambda\rho} \cdot \mathcal{F}^{(x,x)}_{\mu\lambda} \star \mathcal{F}^{(x,x)}_{\nu\rho} + \tilde{G}^{(y,y)\mu\nu} \cdot \tilde{G}^{(y,y)\lambda\rho} \cdot \mathcal{F}^{(y,y)}_{\mu\lambda} \star \mathcal{F}^{(y,y)}_{\nu\rho} \\
& + 2\tilde{G}^{(x,x)\mu\nu} \cdot \tilde{G}^{(x,y)\lambda\rho} \cdot \mathcal{F}^{(x,x)}_{\mu\lambda} \star \mathcal{F}^{(x,y)}_{\nu\rho} + 2\tilde{G}^{(y,y)\mu\nu} \cdot \tilde{G}^{(y,x)\lambda\rho} \cdot \mathcal{F}^{(y,y)}_{\mu\lambda} \star \mathcal{F}^{(y,x)}_{\nu\rho} \\
& + 2G^{\mu\lambda} \cdot G^{\nu\rho} \cdot \mathcal{F}^{(x,x)}_{\mu\nu} \star \mathcal{R}_{\lambda\rho} + 2G^{\mu\lambda} \cdot G^{\nu\rho} \cdot \mathcal{F}^{(y,y)}_{\mu\nu} \star \overline{\mathcal{R}}_{\lambda\rho} \ \} \qquad (61)
\end{aligned}
$$

Here $\tilde{G}(x,y)$ is the determinant of the $2D \times 2D$ matrix ($\tilde{G}^{(x,x)}_{\mu\nu}$, $\tilde{G}^{(x,y)}_{\mu\nu}$ / $\tilde{G}^{(y,x)}_{\mu\nu}$, $\tilde{G}^{(y,y)}_{\mu\nu}$) and coincides with $\{\Omega(x,y)\}^2$. $\overline{\mathcal{R}}_{\mu\nu}$ is a sum of terms similar to those of $\mathcal{R}_{\mu\nu}$. In the above '$\cdot$' is an ordinary product. In this way the noncommutative gauge theory in the curved space is obtained. Although the metric $\tilde{G}^{(\cdot\cdot)\mu\nu}$ depends on the coordinates, the action is gauge invariant. Similarly, an action for the scalars X, $\overline{X}$ can be obtained.

In the special case $H = 0$, x and y commute and the product of functions of x only is still a function of x only. Thus it is natural to set $A_\mu = A_\mu(x)$, $\overline{A}_\mu = \overline{A}_\mu(y)$, $X^m = X^m(x)$, $\overline{X}^m = \overline{X}^m(y)$. The product $\star$ reduces to $*$. Then the action $S_{\text{bi}-\text{local}}$ becomes the sum of two decoupled noncommutative gauge theories with noncommutative parameters θ and $-\theta$, respectively.

In $H = 0$ noncommutative gauge theory, one obtains unitary and causal S-matrix as long as the noncommutative parameter has only space components.[13] If we regard (56) as the action integral in $2D$-dimensional space, we may be able to expect that this theory is also causal, even if there are terms containing the field strength H. If we regard x and y as two distinct points in the same D-dimensional space, however, this theory will be non-local and we cannot expect to obtain causal S-matrix. We must reduce $S_{\text{bi}-\text{local}}$ to a single x-integral and make the theory 'local'.

6 The $y^\mu \to x^\mu$ limit and the reduced effective action

We need to find a prescription to connect this action to the amplitudes of the string theory in the low energy limit. Let us recall that the energy of the open string is proportional to the length of the string divided by α' and given by $M \simeq \frac{1}{\alpha'}\sqrt{G_{\mu\nu}\ p^\mu\ p^\nu} \simeq \frac{1}{\alpha'}\sqrt{G_{\mu\nu}\ (J^{-1})^\mu{}_\lambda\ (y-x)^\lambda\ (J^{-1})^\nu{}_\rho\ (y-x)^\rho}$. In the low energy limit this goes to a finite value. A massive string will be unstable and eventually decay into a massless string. To obtain a massless string we must take the $y \to x$ limit by hand. If one sends y to x directly in the product $\star$, (23), this product becomes non-associative. In this sense the separation $y - x$ works as a kind of regularization to realize the associative product $\star$. At the final step we need to reduce the x, y integral to a single x integral. The integration measure is not just $d^Dx\ \Omega(x,x)$ but must be modified.

As a simple example let us first consider the following integral in the (x,y) space.

$$\int d^Dxd^Dy\ \Omega(x,y)\ \Phi_1(x,y)\star\Phi_2(x,y)\cdots\star\Phi_n(x,y) \tag{62}$$

$\Phi_i(x,y)$'s are scalar fields. We propose that the following reduced integral in the x space should correspond to this integral in the $y\to x$ limit.[3]

$$\int d^Dx\ \omega(x)\ [\ \Phi_1(x,y)\star\Phi_2(x,y)\cdots\star\Phi_n(x,y)\ ]|_{y=x} \tag{63}$$

In (63) we set $y=x$ only after all the algebra associated with $\star$ is finished. Here $\omega(x)$ is a function which is determined by the requirement of cyclic property. Actually if we set

$$\omega(x)=1+\frac{1}{3}x^\lambda\ \theta^{\mu\nu}\ H_{\lambda\mu\nu}, \tag{64}$$

the two point function of the functions of x only,

$$I_2[\Phi_1,\Phi_2]\equiv\int d^Dx\ \omega(x)\ (\Phi_1(x)\star\Phi_2(x))|_{y=x} \tag{65}$$

is given by

$$I_2[\Phi_1,\Phi_2]=\int d^Dx\Phi_1(x)\cdot\Phi_2(x)\cdot(1+\frac{1}{3}x^\lambda\theta^{\mu\nu}\ H_{\mu\nu\lambda}). \tag{66}$$

This is cyclic invariant.

$$I_2[\Phi_1,\Phi_2]=I_2[\Phi_2,\Phi_1] \tag{67}$$

Contrary to our expectation the function $\omega(x)$ is not proportional to $\sqrt{\det\hat{G}_{\mu\nu}(x)}=\sqrt{G}\ (1-\frac{1}{3}x^\lambda\ \theta^{\mu\nu}\ H_{\lambda\mu\nu})$ but to $\sqrt{\det\hat{G}^{\mu\nu}(x)}$, where $\hat{G}_{\mu\nu}(x)=(g\ (1-(g^{-1}B(x))^2))_{\mu\nu}$ is the open string metric in the presence of H.

Furthermore we can check that this correlation function enjoys the following interesting property.

$$\begin{aligned}\int d^Dx\ \omega\ (\Phi_1(x)\star\Phi_2(x))|_{y=x}&=\int d^Dx\ \omega\ (\Phi_1(x)\star\Phi_2(y))|_{y=x}\\ =\quad\int d^Dx\ \omega\ (\Phi_1(y)\star\Phi_2(x))|_{y=x}&=\int d^Dx\ \omega\ (\Phi_2(y)\star\Phi_1(y))|_{y=x}\end{aligned} \tag{68}$$

This property reveals the string theory origin of the $\star$ product. To see this it is helpful to visualize the $\star$ product as an operation adding one rung to a ladder. Let us consider a ladder and name the two legs as x and y, respectively. The $\star$ product adds one unit on top of the ladder. $\star\ \Phi_i(x)$ adds one rung with the x-leg marked, while $\star\ \Phi_i(y)$ one with the y-leg marked. This operation must be performed step by step and is noncommutative. Finally setting $y=x$ and integrating over x join the two legs at the top and bottom, separately.

[3] At $\mathcal{O}(H^1)$ this prescription is similar to the one ($p=0$) adopted in [11] due to the relation (19).

After this joining the legs are merged into a circle and we expect to have the cyclic property as in the string amplitude. This is the case here. Eq (68) is illustrated in Fig.1.

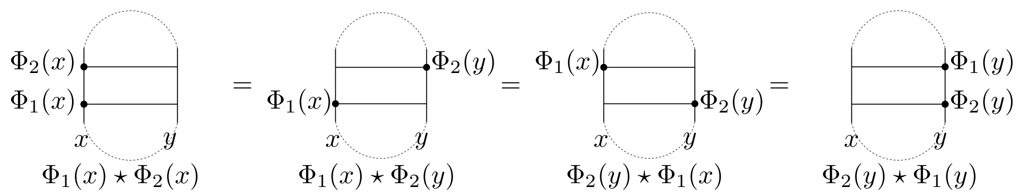

Fig. 1 : Cyclic property on the boundary of a disc

Similarly the three point function

$$I_3[\Phi_1,\Phi_2,\Phi_3] \equiv \int d^D x\ \omega(x)\ (\Phi_1(x)\star\Phi_2(x)\star\Phi_3(x))|_{y=x} \tag{69}$$

is given by

$$\begin{aligned} I_3[\Phi_1,\Phi_2,\Phi_3] \quad = \quad & \int d^D x\{\ \omega(x)\cdot\Phi_1(x)*\Phi_2(x)*\Phi_3(x) - \frac{i}{6}(G^{\mu\lambda}G^{\nu\rho}-\theta^{\mu\lambda}\theta^{\nu\rho})H_{\lambda\rho\sigma}\ x^\sigma \\ & \times(\partial_\mu\Phi_1*\partial_\nu\Phi_2*\Phi_3+\partial_\mu\Phi_1*\Phi_2*\partial_\nu\Phi_3+\Phi_1*\partial_\mu\Phi_2*\partial_\nu\Phi_3) \\ & +\frac{1}{18}(G^{\mu\lambda}\theta^{\nu\alpha}-\theta^{\mu\lambda}G^{\nu\alpha})\ G^{\sigma\rho}\ H_{\lambda\rho\alpha}\ \partial_\mu\Phi_1*\partial_\nu\Phi_2*\partial_\sigma\Phi_3\ \}. \end{aligned} \tag{70}$$

In the $\alpha' \to 0$ limit those terms in (66) and (70) which contain the metric $G^{\mu\nu}$ drop and agree with the results of correlation functions obtained in [9].

It is not difficult to show that the three point function (70) satisfies

$$I_3[\Phi_1,\Phi_2,\Phi_3] = I_3[\Phi_3,\Phi_1,\Phi_2] = I_3[\Phi_2,\Phi_3,\Phi_1]. \tag{71}$$

We can also show the following identities.

$$\begin{aligned} & \int d^D x\ \omega\ (\Phi_1(x)\star\Phi_2(x)\star\Phi_3(x))|_{y=x} = \int d^D x\ \omega\ (\Phi_3(y)\star\Phi_2(y)\star\Phi_1(y))|_{y=x} \\ = & \int d^D x\ \omega\ (\Phi_1(x)\star\Phi_2(x)\star\Phi_3(y))|_{y=x} = \int d^D x\ \omega\ (\Phi_1(x)\star\Phi_3(y)\star\Phi_2(x))|_{y=x} \\ = & \int d^D x\ \omega\ (\Phi_3(y)\star\Phi_1(x)\star\Phi_2(x))|_{y=x} = \int d^D x\ \omega\ (\Phi_1(x)\star\Phi_3(y)\star\Phi_2(y))|_{y=x} \\ = & \int d^D x\ \omega\ (\Phi_3(y)\star\Phi_1(x)\star\Phi_2(y))|_{y=x} = \int d^D x\ \omega\ (\Phi_3(y)\star\Phi_2(y)\star\Phi_1(x))|_{y=x} \end{aligned} \tag{72}$$

The locations of the fields can be shifted from one end x of the string to the other y, and *vice versa*, as long as the ordering on a circle is unchanged. This is natural from the string theory point of view.

Higher point functions $I_n[\Phi_1, \cdots, \Phi_n]$ do not have the cyclic property. In [9] it was shown that in order to obtain the four point function in terms of the non-associative product $\bullet$ one must take a specific linear combination of the ordered products of the functions with various positionings of the parenthesis. In our approach the correlation functions will be obtained by functional derivatives of the effective action and various orderings will be automatically taken into account due to the bose symmetry.

The cyclic property (67), (68), (71), (72), and the agreement of (66), (70) with the results of [9] in the $\alpha' \to 0$ limit justifies the prescription (63).

Let us now turn to the gauge theory action (56). The reduced effective action which corresponds to this is given by

$$
\begin{aligned}
S_{\text{reduced effective action}}[A_\mu,\ \overline{A}_\mu,\ X^m,\ \overline{X}^m] \ &= \ \int d^D x \sqrt{G}\ \omega(x)\ [\ \mathcal{L}(x,y)\ ]|_{y=x} \\
= -\frac{1}{4g_{\text{YM}}^2} \int d^D x\ \sqrt{G}\ \omega(x)\ \cdot \{\ & G^{\mu\nu} G^{\lambda\rho}\ [D_\mu,\ D_\lambda]_\star \star [D_\nu,\ D_\rho]_\star \\
& -2G^{\mu\nu}\ [D_\mu,\ X^m]_\star \star [D_\nu,\ X^m]_\star + [X^m,\ X^n]_\star^2 \\
& + G^{\mu\nu} G^{\lambda\rho}\ [\overline{D}_\mu,\ \overline{D}_\lambda]_\star \star [\overline{D}_\nu,\ \overline{D}_\rho]_\star \\
& -2G^{\mu\nu}\ [\overline{D}_\mu,\ \overline{X}^m]_\star \star [\overline{D}_\nu,\ \overline{X}^m]_\star + [\overline{X}^m,\ \overline{X}^n]_\star^2 \} \Big|_{y=x},
\end{aligned}
\tag{73}
$$

where $\mathcal{L}(x,y)$ is the Lagrangian density defined in (56). The restriction $y = x$ here breaks some part of the gauge symmetry. The reduced effective action (73), however, turned out to be still invariant under the gauge transformation with the gauge function

$$
\Lambda(x,y) = \hat{\Lambda}(x) - \hat{\Lambda}(y). \tag{74}
$$

This follows from the identity

$$
\int d^D x\ \omega(x)\ \left\{ [\hat{\Lambda}(x) - \hat{\Lambda}(y), f(x,y)]_\star \right\} \Big|_{y=x} = 0, \tag{75}
$$

which can be proved by expanding $f(x,y)$ as $\sum_n g_n(x) \star h_n(y)$ and using the identities (72). Because $\star$ contains derivatives, the integrand of (75) does not trivially vanish, even if we set $y = x$.

Let us note that in (73) the product inside the bracket is $\star$. Hence the reduced action is not an ordinary action in the D-dimensional space, because the product in the integrand contains both derivatives $\partial/\partial x,\ \partial/\partial y$. The gauge transformation is also defined in the (x,y) space. To write down the gauge invariant action in the ordinary sense we must introduce a set of coordinates (x,y) and consider $2D$ dimensional action integral, as we did in the previous section.

Finally, in order to derive the correlation functions we must further set $A_\mu = \hat{A}_\mu(x)$, $\overline{A}_\mu = \hat{A}_\mu(y)$, $X^m = \hat{X}^m(x)$ and $\overline{X}^m = \hat{X}^m(y)$ before taking the functional derivatives of

(73), because these describe the gauge particles emitted and/or absorbed from the $\sigma = 0$ and π ends of the open string and the fluctuations at the two ends, respectively. The same functions at both ends because of the bose symmetry. Then this reduced action becomes a functional of $\hat{A}_\mu$, $\hat{X}^m$.

We must mention some ambiguity in our prescription. When we perform a partial integration or use the cyclic property (47) in (56), we will obtain different integrands which will, however, lead to the same result. The corresponding reduced integral does not have this property. This is unavoidable, because when we reduce the integral, we insert into the integrand a function proportional to a delta function. The result will depend on where this delta function is inserted and we must specify this. Our proporsal is to reduce the integral in the standard form of the action (56).

Because we have not determined $S^{\mu\nu}{}_\lambda$, $T^{\mu\nu}{}_\lambda$, $U^{\mu\nu}{}_\lambda$ completely, we cannot decide whether (73) reproduces the open string amplitudes in the low energy limit. Moreover such an analysis will meet difficulties, because one cannot generally define free states in an asymptotically non-flat space. We, however, expect that once $S^{\mu\nu}{}_\lambda$, $T^{\mu\nu}{}_\lambda$, $U^{\mu\nu}{}_\lambda$ are determined, the gauge invariance is strong enough to restrict the form of the effective action.

7 Discussions

In this paper we considered an open string theory in an NS-NS B field background with a nonzero constant field strength $H = dB$. The background space is curved although if H is small, the metric $g_{\mu\nu}$ is constant up to $\mathcal{O}(H^1)$. We performed a perturbative analysis of the commutation relations of the string coordinates without oscillators up to $\mathcal{O}(H^1)$ and found that the coordinate x of one end of the string does not commute with the coordinate y of the other. We then constructed an associative and noncommutative product $\star$ which realizes the commutation relations of the coordinates. This product is an operation for functions of both the coordinates x, y and even if two functions of x only are multiplied, the result is a function of both x and y. In this way we are lead to consider bi-local fields which depends on the coordinates of both ends of the string.

We then found that derivatives $\partial/\partial x^\mu$, $\partial/\partial y^\mu$ do not satisfy Leibnitz rule with the $\star$ product and we are forced to modify the derivatives. At present the modified derivatives are not uniquely determined, but remarkably, the new derivatives $\nabla/\nabla x^\mu$, $\nabla/\nabla y^\mu$ turned out to be rewritten as $\partial/\partial x'^\mu$, $\partial/\partial y'^\mu$, respectively, by some 'coordinate transformation' $x,\ y \to x',\ y'$. The commutation relations of the primed variables coincide with those of the unprimed ones with $H = 0$. Hence if we combine the x and y coordinates and consider $2D$ dimensional space (x, y), the primed coordinate system appears to be flat. This provides us with a clue to write down the gauge theory action. The open string metric $\hat{G}_{\mu\nu}(x) = (g\{1 - (g^{-1}B(x))^2\})_{\mu\nu}$ is curved and it may seem difficult to write down the action without spoiling the gauge invariance due to the coordinate dependent metric. Our proporsal is to write down the action in the flat (x', y') frame and then transform it to the (x, y) frame. In this way we obtained an action integral for the noncommutative gauge theory in curved backgrounds. This gauge theory lives in a double-dimensional space, *i.e.*, the space with coordinates (x^μ, y^μ), $\mu = 1, 2, \ldots, D$. This action is invariant under gauge transformations with x, y dependent gauge parameters. We then took a limit $y \to x$ and proposed a prescription to reduce the action integral to that in D-dimensional space with a coordinate x. This reduced action still has a part of the gauge symmetry. This will provide a 'local' low energy effective action for string amplitudes.

Let us now turn to the application of our results to the Matrix model. The formulation of the Matrix model for curved backgrounds has been investigated [15] but is not yet well established. As mentioned above, the backgound space is curved for nonvanishing H. The closed string metric $g_{\mu\nu}$ is flat only up to $\mathcal{O}(H^1)$ and the open string metric $\hat{G}_{\mu\nu}(x)$ is curved already at $\mathcal{O}(H^1)$[9]. On the other hand in the action (56) the metric $G_{\mu\nu}$ is flat and has the ordinary form of the gauge theory action in the flat space. This is based on the fact that we can deform the algebra of the coordinates x, y to that of the flat background by some coordinate transformation $(x, y) \to (x', y')$. Although we do not have a proof that this persists to all orders in H, let us assume this here. Then this observation suggests the following general formulation of the Matrix model for curved backgrounds.

- Double the number of the matrices of the ordinary matrix model. Denote these as X^μ and $\overline{X}^\mu$. This is natural because the open string has two ends. In this paper we considered only the bosonic string. In the case of the superstring the fermion fields on the D-brane must be also doubled.

- The action of the Matrix model is given by

$$S_{\rm Matrix} = \frac{1}{4g_{\rm YM}^2}\int dt\ Tr\{\ 2(D_0\ X^m)^2 + 2(\overline{D}_0\ \overline{X}^m)^2 +\ [X^m,\ X^n]^2\ + [\overline{X}^m,\ \overline{X}^n]^2 + \text{fermions}\ \}. \tag{76}$$

- The action (76) is background independent. The space-time metric is Minkowskian. We can introduce backgrounds as follows. By a separate linear coordinate transformation of X^μ, $\overline{X}^\mu$ the constant metric $G_{\mu\nu}$ can be put in. The matrix multiplication is associative and noncommutative and there should be an isomorphism between matrices X^μ, $\overline{X}^\mu$ and the bi-local fields whose multiplication rule obeys the product $\star$. By the replacement $X^\mu \to iD_\mu$, $\overline{X}^\mu \to i\overline{D}_\mu$ ($\mu = 1, 2, \ldots, D$) and $Tr \to \int d^D x d^D y V^{-1}\ G\ \Omega(x, y)$, we will obtain the action (56). The product $\star$ and the gauge covariant derivatives D_μ, $\overline{D}_\mu$ will introduce the background dependence.

For the above formulation to be valid we must obtain the classical solution for the Matrix model action (76) and then find a way to systematically determine the star product $\star$ and the covariant derivatives D, $\overline{D}$ and demonstrate that the gauge theory action (56) can be obtained. We hope to report on the analysis in the future.

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