Dynamic Feedback Linearization of Nonlinear Control Systems on Homogenous Time Scale

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Time scale is a model of time

Definition

A **time scale** $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers.

- $\mathbb{T} = \mathbb{R}$, continuous time
- $\mathbb{T} = \mathbb{Z}$, discrete time
- $\mathbb{T} = \tau \mathbb{Z} := \{\tau k \mid k \in \mathbb{Z}\}$, $\tau > 0$, discrete time
- $\mathbb{T} = \mathbb{P}_{a,b} := \bigcup_{k=0}^{\infty} [k(a + b), k(a + b) + a]$
- $\mathbb{T} = \mathbb{H} := \left\{ 0, \sum_{k=1}^{n} \frac{1}{k} \mid n \in \mathbb{N} \right\}$
The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf \{ \tau \in \mathbb{T} \mid \tau > t \}.$$ 

The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \inf \{ \tau \in \mathbb{T} \mid \tau > t \}.$$ 

The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$ 

A time scale $\mathbb{T}$ is called **homogeneous** if $\mu \equiv \text{const.}$
**Delta derivative** of $f(t) : \mathbb{T} \rightarrow \mathbb{R}$, denoted by $f^\Delta(t)$, can be defined as the extension of standard time-derivative in the continuous-time case.

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Consider a multi-input nonlinear dynamical system, defined on homogeneous time scale \( T \) and described by the state equations

\[ x^\Delta = f(x, u), \]  

(1)

where

- \( x : T \rightarrow X \subset \mathbb{R}^n \) is an \( n \)-dimensional state vector;
- \( u : T \rightarrow U \subset \mathbb{R}^m \) is an \( m \)-dimensional input vector;
- \( f : X \times U \rightarrow X \) is assumed to be real analytic function.
The pair $(\mathcal{K}, \sigma_f)$ is a $\sigma_f$-differential field.

$\mathcal{K}^*$ denotes the inversive closure of $\mathcal{K}$.

Consider the infinite set of symbols $d\mathcal{C}^* = \{d\zeta, \zeta \in \mathcal{C}^*\}$ and define by $\mathcal{E} := \text{span}_{\mathcal{K}^*} d\mathcal{C}^*$ the vector space spanned over the field $\mathcal{K}^*$ with

$$\mathcal{C}^* = \begin{cases} \mathcal{C}, & \text{if } \mu = 0, \\ \mathcal{C} \cup \{z^{(-\ell)} | \ell \geq 1\}, & \text{if } \mu \neq 0. \end{cases}$$

Any element of $\mathcal{E}$ is called differential one-form.
\( K \) is the field of meromorphic functions in a finite number of the independent system variables from the set
\[
\mathcal{C} = \left\{ x_1, \ldots, x_n; \ u_1^{(k)}, \ldots, u_m^{(k)}, k \geq 0 \right\}.
\]

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Any element of $\mathcal{E}$ is called differential one-form.
\[ \mathcal{C} = \left\{ x_1, \ldots, x_n; u_{1}^{(k)}, \ldots, u_{m}^{(k)}, k \geq 0 \right\} \]

\[ \mathcal{K} \quad \text{field of meromorphic functions} \]

\[ \mathcal{E} := \text{span}_{\mathcal{K}^*} d\mathcal{C}^* \quad \text{inversive difference field } \mathcal{K}^* \]

\[ \mathcal{E} \quad \text{vector space of one-forms} \]
A left polynomial can be uniquely written in the form $\pi(z) = \sum_{\ell=0}^{k} \pi_\ell z^\ell$, $\pi_\ell \in K^*$.  

**Definition**  

The **skew polynomial ring**, induced by $\sigma_f$-differential overfield $K^*$, is the non-commutative ring $K^*[z; \sigma_f, \Delta_f]$ of left polynomials in $z$ with usual addition and multiplication satisfying, for any $\zeta \in K^* \subset K^*[z; \sigma_f, \Delta_f]$, the commutation rule 

$$z\zeta := \zeta^{\sigma_f} z + \zeta^{\Delta_f}.$$  

Let $K^*[z; \sigma_f, \Delta_f]^{q \times q}$ denote the set of $q \times q$ polynomial matrices with entries in $K^*[z; \sigma_f, \Delta_f]$.  

**Definition**  

A matrix $U(z) \in K^*[z; \sigma_f, \Delta_f]^{q \times q}$ is called **unimodular** if there exists an inverse matrix $U^{-1}(z) \in K^*[z; \sigma_f, \Delta_f]^{q \times q}$. 
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A matrix \( U(z) \in \mathcal{K}^*[z; \sigma_f, \Delta_f]^{q \times q} \) is called unimodular if there exists an inverse matrix \( U^{-1}(z) \in \mathcal{K}^*[z; \sigma_f, \Delta_f]^{q \times q} \).
A sequence of subspaces $\mathcal{H}_0 \supset \cdots \supset \mathcal{H}_k^* \supset \mathcal{H}_{k+1}^* = \mathcal{H}_{k+1}^* = \cdots =: \mathcal{H}_\infty$ of $\mathcal{E}$ is defined by

\[
\mathcal{H}_0 := \text{span}_{\mathcal{K}^*}\{dx, du\},
\]
\[
\mathcal{H}_k := \{\omega \in \mathcal{H}_{k-1} | \omega^{\Delta f} \in \mathcal{H}_{k-1}\}, \quad k \geq 1.
\]

The sequence plays a key role in the analysis of various structural properties of nonlinear systems, including accessibility and feedback linearization.
Consider system (1) and suppose that the output function \( y = h(x) \), \( y \in \mathbb{Y} \subset \mathbb{R}^m \) is given. Define a chain of subspaces \( \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n \) of \( \mathcal{E} \) as

\[
\mathcal{E}_k = \text{span}_{\mathcal{K}^*} \left\{ dx, dy, dy^{(1)}, \ldots, dy^{(k)} \right\}
\]

and associated list of dimensions \( p_k := \text{dim}_{\mathcal{K}^*} \mathcal{E}_k \).

- For \( k = 0, \ldots, n \), \( \varsigma_k := p_k - p_{k-1} \) is the number of zeros at infinity of order less than or equal to \( k \), with the convention \( p_{-1} := n \).
- The rank \( p^* \) of the system is the total number of zeros at infinity, i.e.,
  \[
p^* = \varsigma_n = p_n - p_{n-1}.
\]
- System (1) is said to be invertible if \( p^* = m \).

**Remark**

*The structure at infinity can be expressed in different manners. For instance, the list \( \{n'_1, \ldots, n'_{p^*}\} \) of the orders of the zeros at infinity is the list of integers \( k \) such that \( \varsigma_k - \varsigma_{k-1} \neq 0 \), each one repeated \( \varsigma_k - \varsigma_{k-1} \) times.*
- Static state feedback linearization
- Dynamic state feedback linearization
Definition

The Brunovsky (controller) canonical form of a system (1), defined on time scale, is introduced as

\[
\begin{align*}
\xi_1^\Delta &= \xi_2 & \cdots & \xi_{r_{m-1}+1}^\Delta &= \xi_{r_{m-1}+2} \\
\xi_2^\Delta &= \xi_3 & \cdots & \xi_{r_{m-1}+2}^\Delta &= \xi_{r_{m-1}+3} \\
\vdots & & \cdots & \vdots \\
\xi_{r_{1}-1}^\Delta &= \xi_{r_{1}} & \cdots & \xi_{r_{m}-1}^\Delta &= \xi_{r_{m}} \\
\xi_{r_{1}}^\Delta &= v_1 & \cdots & \xi_{r_{m}}^\Delta &= v_{m}
\end{align*}
\]

with \( r_1 + \cdots + r_m = n \) and \( r_m \leq \cdots \leq r_2 \leq r_1 \).

Note that \( v : \mathbb{T} \to \mathbb{V} \subset \mathbb{R}^m \) is a vector of new inputs.
Theorem

Suppose $\mathcal{H}_\infty = \{0\}$. Then, there exists a list of integers $r_1, \ldots, r_m$ and $m$ one-forms $\omega_1, \ldots, \omega_m \in \mathcal{H}_1$ whose relative degrees are, respectively, $r_1, \ldots, r_m$ such that

- $\text{span}_{\mathcal{K}^*}\{\omega_i^{\Delta_j^f}, \ i = 1, \ldots, m, j = 0, \ldots, r_j - 1\} = \text{span}_{\mathcal{K}^*}\{dx\} = \mathcal{H}_1$;

- $\text{span}_{\mathcal{K}^*}\{\omega_i^{\Delta_j^f}, \ i = 1, \ldots, m, j = 0, \ldots, r_j\} = \text{span}_{\mathcal{K}^*}\{dx, du\} = \mathcal{H}_0$;

- the one-forms $\{\omega_i^{\Delta_j^f}, \ i = 1, \ldots, m, j \geq 0\}$ are linearly independent; in particular

$$\sum_{i=1}^{m} r_i = n.$$

Theorem

System (1) is linearizable by regular\(^a\) static state feedback $u = \psi(x, v)$ iff $\mathcal{H}_\infty = \{0\}$ and $\mathcal{H}_k$, for $k = 1, \ldots, k^*$, are integrable.

\(^a\)A compensator is called regular, if it is invertible, i.e., $\text{rank} \mathcal{K}^* \frac{\partial \psi}{\partial v} = m$. 
Theorem

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- $\text{span}_\mathcal{K}^* \{ \omega^j_i, \ i = 1, \ldots, m, j = 0, \ldots, r_j - 1 \} = \text{span}_\mathcal{K}^* \{ dx \} = \mathcal{H}_1$;
- $\text{span}_\mathcal{K}^* \{ \omega^j_i, \ i = 1, \ldots, m, j = 0, \ldots, r_j \} = \text{span}_\mathcal{K}^* \{ dx, du \} = \mathcal{H}_0$;
- the one-forms $\{ \omega^j_i, \ i = 1, \ldots, m, j \geq 0 \}$ are linearly independent; in particular $\sum_{i=1}^m r_i = n$.

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Feedback Linearization

- Static state feedback linearization
- Dynamic state feedback linearization
System (1) is said to be linearizable by dynamic state feedback if there exist a regular dynamic compensator of the form

\[ \eta^\Delta = \zeta(x, \eta, v), \]
\[ u = \psi(x, \eta, v) \]

(2)

with \( \eta \in \mathbb{R}^s \), and an extended coordinate transformation \( \xi = \phi(x, \eta) \) such that, in the new coordinates, the compensated system (1) reads as

\[ \xi^\Delta = A\xi + Bv, \]

where \( \xi \in \mathbb{R}^{n+s} \) and the pair \((A, B)\) is in Brunovsky canonical form.
Define the subspaces of $E$ as $X := \text{span}_{K^*}\{dx\}$, $Y := \text{span}_{K^*}\{dy^{(k)} : k \geq 0\}$, $X_\nu := \text{span}_{K^*}\{dx, du, du^{(1)}, \ldots, du^{(\nu-1)}\}$.

**Definition**

A **linearizing output** is an output function $y = h(x, u, u^{(1)}, \ldots, u^{(\nu-1)})$ that satisfies the following properties:

- $y = h(x, u, u^{(1)}, \ldots, u^{(\nu-1)})$ defines an invertible system;
- $\sum_i n'_i = \dim_{K^*}(X \cap Y) = n$. 
Feedback Linearization
Dynamic state feedback linearization

Theorem

Suppose $\mathcal{H}_\infty = \{0\}$, and let $\Omega := [\omega_1 \ldots \omega_m]^T \in \mathcal{E}^m$ be a system of linearizing one-forms for system (1). Then, there exists a system of linearizing outputs iff there exists a unimodular polynomial matrix $U(z) \in \mathcal{K}^*\{z; \sigma_f, \Delta_f\}^{m \times m}$ such that

$$d(U(z)\Omega) = 0.$$

Corollary

Let (1) be a single-input system and suppose $\mathcal{H}_\infty = \{0\}$. Then, the following statements are equivalent:

1. (1) is linearizable by static state feedback;
2. (1) is linearizable by dynamic state feedback;
3. $d\omega_1 \wedge \omega_1 = 0$, where $\omega_1$ is such that $\mathcal{H}_n = \text{span}_{\mathcal{K}^*} \{\omega_1\}$. 

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Feedback Linearization
Dynamic state feedback linearization: Example

Consider the system

\[
\begin{align*}
\Delta x_1 &= x_2 - u_1 \\
\Delta x_2 &= x_4 u_1 \\
\Delta x_3 &= u_1 \\
\Delta x_4 &= u_2.
\end{align*}
\] (3)

The sequence of subspaces \( \mathcal{H}_k, k \geq 0 \) can be calculated as

\[
\begin{align*}
\mathcal{H}_1 &= \text{span}_{\mathcal{K}^*}\{dx_1, dx_2, dx_3, dx_4\}, \\
\mathcal{H}_2 &= \text{span}_{\mathcal{K}^*}\{x_4^{\rho_f} dx_1 + dx_2, dx_1 + dx_3\}, \\
\mathcal{H}_3 &= \cdots = \mathcal{H}_\infty = \{0\}.
\end{align*}
\]

For this example both linearizing one-forms can be chosen from \( \mathcal{H}_2 \), i.e., \( \Omega := \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix}^T \), where \( \omega_1 = x_4^{\rho_f} dx_1 + dx_2 \) and \( \omega_2 = dx_1 + dx_3 \). Though \( \mathcal{H}_\infty = \{0\} \), the system is not linearizable by static state feedback, since \( d\omega_1 \wedge \omega_1 \wedge \omega_2 = -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4^{\rho_f} \neq 0 \).
However, the system is linearizable by dynamic state feedback. Indeed, take

$$U(z) = \begin{bmatrix} \frac{1}{x_4^{\rho_f}} & -\frac{1}{x_4^{\rho_f}} z \\ 0 & 1 \end{bmatrix}$$

for which the inverse matrix can be found as

$$U^{-1}(z) = \begin{bmatrix} x_4^{\rho_f} & z \\ 0 & 1 \end{bmatrix}$$
Next, verify that

$$U(z)\Omega = \begin{bmatrix} dx_1 \\ d(x_1 + x_3) \end{bmatrix}.$$ 

Hence, the linearizing outputs are \(y_1 = x_1\) and \(y_2 = x_1 + x_3\). Next, compute the sequence of subspaces \(E_k\) for \(k = 0, \ldots, 4\) as

\[
E_0 = \text{span}_{K^*}\{dx\}, \\
E_1 = \text{span}_{K^*}\{dx, -d u_1\}, \\
E_2 = \text{span}_{K^*}\{dx, -d u_1, -d u_1^\Delta\}, \\
E_3 = \text{span}_{K^*}\{dx, -d u_1, -d u_1^\Delta, -d u_1^{(2)}\}, \\
E_4 = \text{span}_{K^*}\{dx, -d u_1, -d u_1^\Delta, -d u_1^{(2)}, -d u_1^{(3)}\},
\]

where \(\lambda_1, \lambda_2 \in K^*\). Hence, it follows that \(p = \{4, 5, 6, 8, 10\}\), and therefore, \(\varsigma = \{0, 1, 1, 2, 2\}\). Thus, we may conclude that the system is invertible, since \(p^* = \varsigma_4 = 2\). From computations of the subspaces \(E_k\), we know that

\[
y_1^\Delta = x_2 - u_1 \\
y_2^{(3)} = (x_4 + \mu u_2) (x_4 u_1 - y_1^{(2)}) + u_2 (x_2 - y_1^\Delta).
\]
Take $\eta = y_1^\Delta$, $\eta^\Delta = v_1$, and $y_2^{(3)} = v_2$ then the dynamic feedback compensator has the form

$$\eta^\Delta = v_1$$

$$u_1 = x_2 - \eta$$

$$u_2 = \frac{v_2 - x_4(x_2 - \eta) - v_1}{\mu(x_4(x_2 - \eta) - v_1) + x_2 - \eta}. \quad (4)$$

Now, relying on the inversion algorithm we can calculate dimension of the extended state equations according to the formula $s = \sum_{i=1}^{m}(\epsilon_i - \gamma_i)$ as $s = (2 - 1) + (3 - 3) = 1$. The application of (4) to system (3) yields the extended state equations

$$x_1^\Delta = \eta$$

$$x_2^\Delta = x_4(x_2 - \eta)$$

$$x_3^\Delta = x_2 - \eta$$

$$x_4^\Delta = \frac{v_2 - x_4(x_2 - \eta) - v_1}{\mu(x_4(x_2 - \eta) - v_1) + x_2 - \eta}$$

$$\zeta^\Delta = v_1 \quad (5)$$
Then we define the coordinate transformation as

\[
\begin{align*}
\xi_1 &:= y_1 = x_1 \\
\xi_2 &:= y_1^\Delta = x_1^\Delta = x_2 - u_1 \\
\xi_3 &:= y_2 = x_1 + x_3 \\
\xi_4 &:= y_2^\Delta = x_2 - u_1 + u_1 = x_2 \\
\xi_5 &:= y_2^{2} = x_4 u_1.
\end{align*}
\]

In the new coordinates the extended system has the linear form

\[
\begin{align*}
\xi_1^\Delta &= \xi_2 \\
\xi_3^\Delta &= \nu_1 \\
\xi_4^\Delta &= \xi_5 \\
\xi_5^\Delta &= \nu_2.
\end{align*}
\]
Thank you very much for your attention!

Any questions?